COHOMOLOGICAL DIMENSION
OF CERTAIN ALGEBRAIC VARIETIES

K. DIVAANI-AAZAR, R. NAGHIPOUR, AND M. TOUSI

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ABSTRACT. Let \( a \) be an ideal of a commutative Noetherian ring \( R \). For finitely generated \( R \)-modules \( M \) and \( N \) with \( \text{Supp} \ N \subseteq \text{Supp} \ M \), it is shown that \( \text{cd}(a,N) \leq \text{cd}(a,M) \). Let \( N \) be a finitely generated module over a local ring \( (R, m) \) such that \( \text{Min}_R \hat{N} = \text{Assh}_R \hat{N} \). Using the above result and the notion of connectedness dimension, it is proved that \( \text{cd}(a,N) \geq \dim N - c(N/aN) - 1 \). Here \( c(N) \) denotes the connectedness dimension of the topological space \( \text{Supp} N \). Finally, as a consequence of this inequality, two previously known generalizations of Faltings’ connectedness theorem are improved.

1. Introduction

Throughout, let \( R \) denote a commutative Noetherian ring (with identity) and \( a \) an ideal of \( R \). The study of the cohomological dimension and connectedness of algebraic varieties has produced some interesting results and problems in local algebra. For an \( R \)-module \( M \), the cohomological dimension of \( M \) with respect to \( a \) is defined as

\[
\text{cd}(a,M) := \max\{i \in \mathbb{Z} : H^i_a(M) \neq 0\}.
\]

The cohomological dimension has been studied by several authors; see, for example, Faltings [7], Hartshorne [9] and Huneke–Lyubeznik [11]. In particular in [7] and [11], several upper bounds for cohomological dimension were obtained. The main aim of this article is to establish lower bounds for cohomological dimension of finitely generated modules over a local ring. This is done by using the notion of connectedness dimension. For a Noetherian topological space \( X \), the subdimension and connectedness dimension of \( X \) are defined respectively as

\[
s\text{dim} X := \min\{\dim Z : Z \text{ is an irreducible component of } X\}, \quad c(X) := \min\{\dim Z : Z \subseteq X, Z \text{ is closed and } X \setminus Z \text{ is disconnected}\}.
\]

For more details about these notions, we refer the reader to [3] Ch. 19. In particular, if \( M \) is an \( R \)-module and \( \text{Supp} M \) is considered as a subspace of \( \text{Spec} R \) equipped with Zariski topology, we denote \( c(\text{Supp} M) \) and \( s\text{dim}(\text{Supp} M) \) by \( c(M) \) and \( s\text{dim}(M) \) respectively.
and \( s \dim M \) respectively. It is clear from the definition that a Noetherian topological space \( X \) is connected if and only if \( c(X) \geq 0 \). Recall that the dimension of the empty space is defined to be \(-1\).

We shall prove:

**Theorem 1.1.** Let \((R, \mathfrak{m})\) be a local ring and \( N \) a finitely generated \( R \)-module.

(i) If \( R \) is complete, then \( \text{cd}(a, N) \geq \min\{c(N), s \dim N - 1\} - c(N/aN) \).

(ii) If \( \text{Min}_R \hat{N} = \text{Ass}_{\hat{R}} \hat{N} \), then \( \text{cd}(a, N) \geq \dim N - c(N/aN) - 1 \).

One of our tools for proving Theorem 1.1 is the following, which plays a key rôle in this paper.

**Theorem 1.2.** Let \( M \) and \( N \) be finitely generated \( R \)-modules with \( \text{Supp} N \subseteq \text{Supp} M \). Then \( \text{cd}(a, N) \leq \text{cd}(a, M) \). In particular, \( \text{cd}(a, N) = \text{cd}(a, M) \) whenever \( \text{Supp} N = \text{Supp} M \).

In [10], M. Hochster and C. Huneke generalized Faltings’ connectedness theorem [6]. Also in [5], P. Schenzel and the first author have proved two generalizations of Faltings’ connectedness theorem. As a consequence of Theorem 1.1(ii), we remove the indecomposability condition in [10, Theorem 3.3] and [5, Theorem 4.3].

Our terminology follows that of [5]. Moreover for an \( R \)-module \( M \), the set of minimal elements of \( \text{Ass}_R M \) is denoted by \( \text{Min}_R M \) and \( \{ p \in \text{Ass} M : \dim R/p = \dim M \} \) by \( \text{Ass}_R M \).

## 2. Cohomological dimension

First of all, we collect the well known properties of the notion of cohomological dimension in a lemma. Before stating the lemma, recall that the height of an ideal \( a \) with respect to an \( R \)-module \( M \) is defined as \( \text{ht}_M a = \min\{\dim M_p : p \supseteq a\} \).

**Lemma 2.1.** Let \( a \) denote an ideal of \( R \). Then:

(i) for an \( R \)-module \( M \), \( \text{ht}_M a \leq \text{cd}(a, M) \leq \dim M \),

(ii) \( \text{cd}(a, R) = \max\{\text{cd}(a, N) : N \text{ is an } R\text{-module} \}

\quad = \max\{i \in \mathbb{Z} : H^i_a(N) \neq 0 \text{ for some } R\text{-module } N\} \),

(iii) \( \text{cd}(a, R) \leq \text{ara}(a) \), where \( \text{ara}(a) \) denotes the arithmetic rank of \( a \), and

(iv) if \( f : R \to R' \) is a homomorphism of commutative Noetherian rings, then \( \text{cd}(aR', R') \leq \text{cd}(a, R) \) and, also for any \( R' \)-module \( M \), \( \text{cd}(a, M) = \text{cd}(aR', M) \).

Furthermore if \( f \) is faithfully flat, then \( \text{cd}(aR', R') = \text{cd}(a, R) \).

The following is one of the main results of this paper.

**Theorem 2.2.** Let \( a \) denote a proper ideal of \( R \) and \( M, N \) two finitely generated \( R \)-modules such that \( \text{Supp} N \subseteq \text{Supp} M \). Then \( \text{cd}(a, N) \leq \text{cd}(a, M) \).

**Proof.** It is enough to show that \( H^i_a(N) = 0 \) for all \( i \) with \( \text{cd}(a, M) < i \leq \dim M + 1 \), and all finitely generated \( R \)-module \( N \) with \( \text{Supp} N \subseteq \text{Supp} M \). We argue this by descending induction on \( i \). The assertion is clear for \( i = \dim M + 1 \) by Grothendieck vanishing theorem. Now, suppose \( i \leq \dim M \). Since \( \text{Supp} N \subseteq \text{Supp} M \), by Gruson’s theorem [12, Theorem 4.1], there is a chain

\[ 0 = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_k = N, \]

such that the factors \( N_j/N_{j-1} \) are homomorphic images of a direct sum of finitely many copies of \( M \). By using short exact sequences, we may reduce the situation to
the case $k = 1$. Then there is an exact sequence
\[ 0 \to L \to M^n \to N \to 0 \]
for some $n \in \mathbb{N}$ and some finitely generated $R$-module $L$. This induces a long exact sequence of local cohomology modules
\[ \cdots \to H^i_a(L) \to H^i_a(M^n) \to H^i_a(N) \to H^{i+1}_a(L) \to \cdots, \]
so that, by the inductive hypothesis, $H^{i+1}_a(L) = 0$. Hence $H^i_a(N) = 0$. Thus the argument is complete by induction.

**Corollary 2.3.** (i) Let $0 \to L \to M \to N \to 0$ be an exact sequence of finitely generated $R$-modules. Then $\text{cd}(a, M) = \max\{\text{cd}(a, L), \text{cd}(a, N)\}$.

(ii) Let $f : R \to S$ be a monomorphism of commutative Noetherian rings such that $S$ is finitely generated as an $R$-module. Then for any proper ideal $a$ of $R$, $\text{cd}(a, R) = \text{cd}(aS, S)$.

(iii) If $M$ is a finitely generated faithful $R$-module, then $\text{cd}(a, M) = \text{cd}(a, R)$.

**Proof.** (i) From the long exact sequence
\[ \cdots \to H^i_a(L) \to H^i_a(M) \to H^i_a(N) \to H^{i+1}_a(L) \to \cdots, \]
we deduce $\text{cd}(a, M) \leq \max\{\text{cd}(a, L), \text{cd}(a, N)\}$, while Theorem 2.2 implies $\max\{\text{cd}(a, L), \text{cd}(a, N)\} \leq \text{cd}(a, M)$. Therefore (i) holds.

(ii) follows by Lemma 2.1(iv) and Theorem 2.2.

(iii) Clearly $\text{Supp}(M) = \text{Spec } R$, and so the result follows by Theorem 2.2.

**Remark 2.4.** (i) One can deduce Lemma 2.1(ii) from Theorem 2.2 easily, because $H^i_a(\cdot)$ commutes with direct limits.

(ii) Part (ii) of Corollary 2.3 is proved in [3, Proposition 8.1.2] by using methods of algebraic geometry.

(iii) Let $M$ and $N$ be two finitely generated $R$-modules such that $M \neq aM$ and that $\text{Supp}(N) \subseteq \text{Supp}(M)$. Then $\text{cd}(a, N) \leq \text{cd}(a, M)$.

(iv) Let $M$ and $N$ be two finitely generated $R$-modules. For each $i \in \mathbb{N}_0$,
\[ \max\{\text{cd}(a, \text{Ext}_i^R(M, N)), \text{cd}(a, \text{Tor}_i^R(M, N))\} \leq \min\{\text{cd}(a, M), \text{cd}(a, N)\}. \]

(v) In view of Corollary 2.3(iii) results concerning cohomological dimension of $R$ with respect to an ideal $a$ can be extended to $\text{cd}(a, M)$ for any finitely generated faithful $R$-module $M$. See for example [3, Theorem 2 and Remark].

We shall use the following result in the proof of Theorem 2.7.

**Lemma 2.5.** Let the situation be as in Lemma 2.1, and let $x \in R$. Then for an $R$-module $M$,
\[ \text{cd}(a + Rx, M) \leq \text{cd}(a, M) + 1. \]

**Proof.** Let $b = a + Rx$ and $\text{cd}(a, M) = r$. By [3, Proposition 8.1.2], there is a long exact sequence
\[ \cdots \to H^i_b(M) \to H^i_a(M) \to H^i_a(M_x) \to H^{i+1}_b(M) \to H^{i+1}_a(M) \to \cdots \]
where $M_x$ is the localization of $M$ with respect to the multiplicatively closed subset $\{x^i : i \in \mathbb{N}_0\}$ of $R$. Since $H^i_a(M) = 0$ for all $i > r$, it turns out that $H^i_b(M_x) \cong H^{i+1}_a(M)$ for all $i > r$. Thus each element of $H^i_b(M_x)$ is annihilated by some power of $b$. By applying the functor $H^i_a(\cdot)$ on the isomorphism $M_x \cong M_x$, $n \in \mathbb{N}$, we
deduce that $H^i_a(M_x) \xrightarrow{x^n} H^i_a(M_x)$ is an isomorphism. But each element of $H^i_a(M_x)$ is annihilated by $x^n$ for some $n \in \mathbb{N}$. This yields that $H^i_a(M_x) = 0$ for all $i > r$. Therefore $H^i_b(M) = 0$ for all $i > r + 1$, as required.

We recall some properties of the notions $c(N)$ and $s\dim N$ in the following lemma (see [3, Ch. 19]).

**Lemma 2.6.** Let $N$ be a finitely generated $R$-module. Then the following hold:

(i) $s\dim N = \min\{\dim R/p : p \in \text{Min}_R N\}$,

(ii) $c(N) = \min\{\dim(R/(\bigcap_{p \in A} p + \bigcap_{p \in B} p)) : A \text{ and } B \text{ are non-empty subsets of } \text{Min}_R N \text{ such that } A \cup B = \text{Min}_R N\}$,

(iii) $c(N) \leq s\dim N$, and

(iv) if $(R,m)$ is local, then $c(\text{Supp } N \setminus \{m\}) = c(N) - 1$.

**Theorem 2.7.** Let $a, b$ be two ideals of a local ring $(R,m)$ and $N$ a finitely generated $R$-module such that $\min\{\dim N/aN, \dim N/bN\} > \dim N/(a + b)N$.

(i) If $\text{Min}_R N$ consists of a single prime $p$, then $\text{cd}(a \cap b, N) \geq \dim N - \dim N/(a + b)N - 1$.

(ii) If $R$ is complete, then $\text{cd}(a \cap b, N) \geq \min\{c(N), s\dim N - 1\} - \dim N/(a + b)N$.

**Proof.** Let $R_1 = R/\text{Ann}_R N$. Then $\text{cd}(a \cap b, N) = \text{cd}((a \cap b)R_1, R_1)$, by Lemma 2.1(iv) and Theorem 2.2. On the other hand one can easily check that $s\dim N = s\dim R_1$ and that $c(N) = c(R_1)$. Therefore we may and do assume that $N = R$. Now, by replacing $\text{ara}(a \cap b)$ by $\text{cd}(a \cap b, R)$ and using Lemma 2.5, we can process similar to the proof of [3, Proposition 19.2.7] to deduce (i). Also, in view of Lemma 2.1(i) and 2.1(iv), one can deduce (ii) by similar argument as in [3, Lemma 19.2.8].

Now, we are ready to state the next main theorem of this section, namely the connectedness bound for a finitely generated module over a complete local ring which is a generalization and refinement of Grothendieck’s connectedness theorem (see [3, Exposé XIII, Théorème 2.1]).

**Theorem 2.8.** Let $a$ be a proper ideal of a complete local ring $(R,m)$, and let $N$ be a finitely generated $R$-module. Then $\text{cd}(a, N) \geq \min\{c(N), s\dim N - 1\} - c(N/aN)$.

**Proof.** Let $\text{Min}_R(N/aN) = \{p_1, \ldots, p_n\}$ and $c := c(N/aN)$. If $n = 1$, we have $c = \dim R/p_1$ (see Lemma 2.6(ii)). Let $p \in \text{Min}_R N$ be such that $p \subseteq p_1$. Then as $\text{Supp } R/p \subseteq \text{Supp } N$, by virtue of Lemmas 2.1(i), 2.1(iv) and Theorem 2.2,

$$\text{ht } p_1/p \leq \text{cd}(p_1/p, R/p) = \text{cd}(p_1, R/p) \leq \text{cd}(p_1, N).$$

Because $\text{Rad}(a + \text{Ann}_R N) = p_1$, it turns out that $\text{cd}(p_1, N) = \text{cd}(a, N)$.

Next, since $R/p$ is catenary, we deduce that $c = \dim R/p - \text{ht } p_1/p \geq s\dim N - \text{cd}(a, N)$, as desired. Accordingly, we may assume that $n > 1$. Then there exist two non-empty subsets $A, B$ of $\text{Min}_R N/aN$ for which $A \cup B = \text{Min}_R N/aN$, and

$c = \dim(R/(\bigcap_{p \in A} p + (\bigcap_{p \in B} p)).$
Moreover, we may assume that $A \cap B = \emptyset$. Put $b := \bigcap_{p \in A} p$ and $c := \bigcap_{p \in B} p$. Then dim $N/bN > c$, dim $N/cN > c$ and $b \cap c = \text{Rad}(a + \text{Ann}_R N)$. Therefore the proof finishes by Theorem 2.7(ii).

**Corollary 2.9.** Let the situation be as in Theorem 2.8. Then $\text{cd}(a, N) \geq c(N) - c(N/aN) - 1$. Moreover if $|\text{Min}_R N| > 1$, then the inequality is strict.

**Proof.** The assertion is clear by Theorem 2.8, because, by Lemma 2.6(iii), $c(N) \leq s\text{dim } N$, with strict inequality if $|\text{Min}_R N| > 1$.

### 3. Connectedness Theorem

In [10], M. Hochster and C. Huneke have extended Faltings’ original connectedness theorem [9] as follows. Let $(R, \mathfrak{m})$ be an equidimensional complete local ring of dimension $d$, and $a$ a proper ideal of $R$. If $H^d_{\mathfrak{m}}(R)$ is indecomposable, then the punctured spectrum of $R/a$ is connected provided $\text{cd}(a, R) \leq d - 2$. Next this result is generalized to finitely generated modules in [5]. In this section, our objective is to remove the indecomposability assumption. To this end, we give a refinement of Theorem 2.8 in Theorem 3.4. Before we do this, we bring some definitions and lemmas.

**Definition.** Let $(R, \mathfrak{m})$ be a $d$-dimensional local ring. A finitely generated $R$-module $K$ is called the *canonical module of $R$*, if $K \otimes_R \hat{R} \cong \text{Hom}_R(H^d_{\mathfrak{m}}(R), E(R/\mathfrak{m}))$.

**Proposition 3.1.** Let $p_1, \ldots, p_n$ be prime ideals of a finite dimensional Noetherian ring $R$ such that $p_i \not\subset p_j$ for all $1 \leq i \neq j \leq n$. Suppose that $R$ is $(S_2)$ and that $R_p$ possesses a canonical module for all $p \in \text{Spec } R$. Also, assume that for each prime ideal $p$ of $R$, dim $R = \dim R/p + \text{ht } p$. Set $a := \bigcap_{i=1}^m p_i$ and $b = \bigcap_{i=m+1}^n p_i$, for some $1 \leq m < n$. Then

$$\text{cd}(a \cap b, R) \geq \dim R - \dim R/(a + b) - 1.$$ 

**Proof.** Let $q$ be a prime ideal of $R$ containing $a + b$ such that $\dim R/(a + b) = \dim R/q$. Our assumption on $p_i$’s implies that the ideals $aR_q$ and $bR_q$ are not $qR_q$-primary. Now the claim follows immediately from Lemma 2.1(iv) and the following lemma.

**Lemma 3.2.** Let $(R, \mathfrak{m})$ be a $(S_2)$ local ring which possesses a canonical module. Let $a$ and $b$ be two non-$\mathfrak{m}$-primary ideals of $R$ such that $a + b$ is $\mathfrak{m}$-primary. Then

$$\text{cd}(a \cap b, R) \geq \dim R - 1.$$ 

**Proof.** Assume that the contrary is true. Then the Mayer-Vietoris sequence (see e.g. [3] 3.2.3) yields the isomorphism

$$H^d_{\mathfrak{m}}(R) = H^d_{a+b}(R) \cong H^d_a(R) \oplus H^d_b(R).$$

The module $H^d_{\mathfrak{m}}(R)$ is indecomposable by [2] Remark 1.4 and so either $H^d_a(R) = 0$ or $H^d_b(R) = 0$. Suppose $H^d_a(R) = 0$; then $H^d_{\mathfrak{m}}(R) \cong H^d_b(R)$. It follows from [2] Proposition 1.2 and Lemma 1.1 that $\text{Assh } \hat{R} = \text{Ass } \hat{R}$. By virtue of [3] Ex. 8.2.6, once applied to $\mathfrak{m}$ and a second time applied to $a$, it follows that $\dim R/aR + p = 0$ for all $p \in \text{Ass } \hat{R}$. This leads that $a$ is $\mathfrak{m}$-primary, which is a contradiction. 

□
Lemma 3.3. Let $R$ be a Noetherian ring such that $R$ is $(S_2)$ and that $R_p$ has a canonical module for all $p \in \text{Spec } R$. Assume that $\dim R$ is finite and that for each $p \in \text{Spec } R$, $\dim R = \dim R/p + \text{ht } p$. Then for each proper ideal $\mathfrak{a}$ of $R$,

$$\text{cd}(\mathfrak{a}, R) \geq \dim R - c(R/\mathfrak{a}) - 1.$$ 

Proof. Without loss of generality we can and do assume that $\mathfrak{a} = \text{Rad}(\mathfrak{a})$. Let $p_1, \ldots, p_n$ be the distinct minimal primes of $\mathfrak{a}$, and let $c := c(R/\mathfrak{a})$. If $n = 1$, we have $\mathfrak{a} = p_1$ and $c = \dim R/p_1$. Hence

$$c = \dim R - \text{ht } p_1 \geq \dim R - \text{cd}(p_1, R).$$

Consider now the case where $n > 1$. By Lemma 2.6(ii), there exist two disjoint non-empty subsets $A, B$ of $\{1, \ldots, n\}$ for which $A \cup B = \{1, \ldots, n\}$ and $c = \dim(R/(\bigcap_{i \in A} p_i) + (\bigcap_{j \in B} p_j))$. Set $b = \bigcap_{i \in A} p_i$ and $c = \bigcap_{j \in B} p_j$. Then $p_i \not\subseteq p_j$ for all $1 \leq i, j \leq n$, and $b \cap c = \mathfrak{a}$. We can now use Proposition 3.1 to complete the proof.

Theorem 3.4. Let $\mathfrak{a}$ be a proper ideal of a local ring $(R, m)$ and let $N$ be a finitely generated $R$-module such that $\text{Min}_R N = \text{Assh}_R N$. Then

$$\text{cd}(\mathfrak{a}, N) \geq \dim N - c(N/\mathfrak{a}N) - 1.$$ 

Proof. Let $R_1 = R/\text{Ann}_R N$. Then $c(N/\mathfrak{a}N) = c(R_1/\mathfrak{a}R_1)$ and $\text{cd}(\mathfrak{a}, N) = \text{cd}(\mathfrak{a}R_1, R_1)$ by Lemma 2.1(iv) and Theorem 2.2. On the other hand $\text{Min}_R R_1 = \text{Assh}_R R_1$. Thus it is sufficient to prove the claim for the ring $R$ itself. Since $c(R/\mathfrak{a}) \geq c(\tilde{R}/\tilde{\mathfrak{a}}\tilde{R})$ by [1] Lemma 19.3.1, we can assume that $R$ is complete. Since $\dim R = \dim R$, in view of Theorem 2.8 it is enough to show that $c(R) \geq \dim R - 1$. Let $J = \bigcap q$, where $q$ runs through all the primary components of the zero ideal of $R$ such that $\dim R/q = \dim R$. It is clear that $\dim R/J = \dim R$. Also, since $\text{Min } R = \text{Assh } R$, it follows from Lemma 2.6(ii) that $c(R/J) = c(R)$. Thus by replacing $R$ with $R/J$, we may assume that $\text{Assh } R = \text{Ass } R$. By [1] 1.11 and Theorem 3.2], there exists a commutative Noetherian semi-local ring $S$ and a monomorphism $\varphi: R \rightarrow S$ such that:

(i) $S$ is finitely generated as an $R$-module,
(ii) $S$ is $(S_2),$
(iii) $S_p$ has a canonical module for all $p \in \text{Spec } S$, and
(iv) every maximal chain of prime ideals in $S$ is of length $\dim S$.

Let $p_1, \ldots, p_n$ be the distinct minimal prime ideals of $R$. Then there exist two non-empty subsets $A, B$ of $\{1, \ldots, n\}$ for which $A \cup B = \{1, \ldots, n\}$ and

$$c(R) = \dim(R/(\bigcap_{i \in A} p_i) + (\bigcap_{j \in B} p_j)).$$

Since by condition (i), $S$ is integral over $R$, it follows that $\dim R = \dim S$ and that for each $1 \leq i \leq n$ there exists $q_i \in \text{Spec } S$ such that $\varphi^{-1}(q_i) = p_i$. For a given prime ideal $q$ of $S$, we show that $q \in \text{Min } S$ if and only if $p = \varphi^{-1}(q) \in \text{Min } R$. To this end, first note that the ring $S/q$ is integral over the ring $R/p$, and so $\dim S/q = \dim R/p$. Since $\text{Ass } R = \text{Assh } R$, it turns out that $p \in \text{Min } R$ if and only if $\dim R/p = \dim R$. On the other hand (iv) implies that $q \in \text{Min } S$ if and only if
\[ \dim S/q = \dim S. \] Therefore the claim is immediate. Put

\[ A' = \{ q \in \operatorname{Min} S : \varphi^{-1}(q) = p_i \text{ for some } i \in A \} \]

and \[ B' = \{ q \in \operatorname{Min} S : \varphi^{-1}(q) = p_j \text{ for some } j \in B \}. \] So, we have

\[ c(R) \geq \dim S/\left( \bigcap_{q \in A'} q + \bigcap_{q \in B'} q \right) \]

Therefore the result follows by Lemma 3.3. Note that by (iv), for each prime ideal \( p \) of \( S \), \( \dim S = \dim S/p + \operatorname{ht} p \).

Now we are prepared to present the main result of this section which is a generalization of [10, Theorem 3.3] and of [5, Corollary 4.2 and Theorem 4.3].

**Corollary 3.5.** Let \( a \) be a proper ideal of a local ring \((R, \mathfrak{m})\). Let \( N \) be a \( d \)-dimensional finitely generated \( R \)-module such that \( \operatorname{Ass} \hat{R} \hat{N} = \operatorname{Min} \hat{R} \hat{N} \). Then \( \operatorname{Supp} N/aN \setminus \{ \mathfrak{m} \} \) is connected provided \( \text{cd}(a, N) \leq d - 2 \).

**Proof.** By Lemma 2.6(iv), \( c(\operatorname{Supp}(N/aN) \setminus \{ \mathfrak{m} \}) = c(N/aN) - 1 \). Hence by Theorem 3.4, \( c(\operatorname{Supp}(N/aN) \setminus \{ \mathfrak{m} \}) \geq \dim N - \text{cd}(a, N) - 2 \). Thus

\[ c(\operatorname{Supp}(N/aN) \setminus \{ \mathfrak{m} \}) \geq 0, \]

and so \( \operatorname{Supp}(N/aN) \setminus \{ \mathfrak{m} \} \) is connected, as desired. \( \square \)

**References**


Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran – and – Department of Mathematics, Az-Zahra University, Tehran, Iran

E-mail address: kdivaani@ipm.ir

Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran – and – Department of Mathematics, University of Tabriz, Tabriz, Iran

E-mail address: naghipour@tabrizu.ac.ir

Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran – and – Department of Mathematics, Shahid Beheshti University, Tehran, Iran

E-mail address: mtousi@vax.ipm.ac.ir