KRULL DIMENSION OF THE ENVELOPING ALGEBRA
OF A SEMISIMPLE LIE ALGEBRA

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Abstract. Let $g$ be a complex semisimple Lie algebra and $U(g)$ be its enveloping algebra. We deduce from the work of R. Bezrukavnikov, A. Braverman and L. Positselskii that the Krull-Gabriel-Rentschler dimension of $U(g)$ is equal to the dimension of a Borel subalgebra of $g$.

1. Introduction

The Krull(-Gabriel-Rentschler) dimension of a ring $R$ was introduced in [3] and is denoted by $\text{Kdim } R$. Let $g$ be a semisimple complex Lie algebra and $U(g)$ be its enveloping algebra. It has been conjectured that $\text{Kdim } U(g)$ is equal to $\text{dim } b$ where $b$ is a Borel subalgebra of $g$. It is easy to see that $\text{Kdim } U(g) \geq \text{dim } b$; indeed, this follows from the fact $U(g)$ is a free (left) module over $U(b)$ and that $\text{Kdim } U(b) = \text{dim } b$ (see [2]). The opposite inequality is therefore the hard part of the conjecture.

P. Smith [10] proved the conjecture for $g = \mathfrak{sl}(2, \mathbb{C})$. Let $G$ be a simply connected semisimple complex algebraic group with Lie algebra $g$, $U$ be a maximal unipotent subgroup of $G$ and set $X = G/U$ (the “basic affine space”). In [7] it was shown that the conjecture would follow from $\text{Kdim } D(X) \leq \text{dim } X$, where $D(X)$ is the ring of globally defined differential operators on $X$ (in the sense of [5]). This result was established in [7] when $g$ is a direct sum of copies of $\mathfrak{sl}(2, \mathbb{C})$, and in [8] when $g = \mathfrak{sl}(3, \mathbb{C})$. Up to now, these were the only cases known and no progress was made on the conjecture.

The difficulty in the study of $D(X)$ comes from the fact that $D(X) = D(\overline{X})$ for some singular variety $\overline{X}$. Recently R. Bezrukavnikov, A. Braverman and L. Positselskii were able to prove, among other things, that $D(X)$ is a Noetherian ring. This is deduced from the existence of a finite set $\{F_w\}_{w \in W}$ ($W$ being the Weyl group of $g$) of automorphisms of $D(X)$ such that: for every $D(X)$-module $M \neq 0$, there exists a twist $F_{w_0}$ of $M$ such that the localization $\mathcal{O}_X \otimes_{\mathcal{O}(X)} M_{F_{w_0}}$ is nonzero. In this note we want to explain how this result easily implies that $\text{Kdim } D(X) \leq \text{dim } X$, and, consequently, $\text{Kdim } U(g) = \text{dim } b$. 

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## 2. Krull dimension

The definitions and general results related to Krull dimension can be found in [9, Chapter 6] and we will simply quote a few facts that we need.

Recall that the deviation of a partially ordered set (poset) \((A, \preceq)\) is defined (when it exists) as follows:

- \(\text{dev} \emptyset = -\infty\);
- \(\text{dev} A = 0\) if and only if \(A\) satisfies the descending chain condition;
- \(\text{dev} A = \alpha\) (some ordinal) if \(\text{dev} A \neq \beta\) for \(\beta < \alpha\), and if \((a_i)_{i \in \mathbb{N}}\) is a descending chain in \(A\), then there exists \(i_0\) such that \(\text{dev}\{x \in A : a_i \succ x \succ a_{i+1}\} < \alpha\) for all \(i \geq i_0\).

For the proof of the next lemma, see [9, 6.1.5, 6.1.6].

**Lemma 2.1.** (a) Let \(B \hookrightarrow A\) be a strictly increasing map of posets. Then, \(\text{dev} B \leq \text{dev} A\) when \(\text{dev} A\) exists.

(b) If \(A\) satisfies the ascending chain condition, then \(\text{dev} A\) exists.

If \(R\) is a ring we denote by \(R\text{-mod}\) the category of finitely generated left \(R\)-modules. Let \(M \in R\text{-mod}\) and \(\mathcal{L}(M)\) be the lattice of submodules of \(M\). Then \((\mathcal{L}(M), \subseteq)\) is a poset; we say that the Krull dimension of \(M\) exists if \(\mathcal{L}(M)\) has a deviation, in which case we set \(K\text{dim }R M = \text{dev} \mathcal{L}(M)\). By Lemma 2.1, \(K\text{dim }M\) exists if \(R\) is (left) Noetherian and one has \(K\text{dim }R M \leq K\text{dim }R (\text{[9], 6.2.18})\).

**Examples.**
1. Let \(m\) be a finite dimensional complex Lie algebra and \(l \subset m\) be a subalgebra. Then, \(K\text{dim }U(l) \leq K\text{dim }U(m) \leq \text{dim }m\). When \(m\) is solvable an easy induction on \(\text{dim }m\) (using Lie’s Theorem) shows that \(K\text{dim }U(m) = \text{dim }m\).

2. Let \(D(Z)\) be the ring of differential operators on a smooth affine complex algebraic variety \(Z\). Then \(D(Z)\) is Noetherian and \(K\text{dim }D(Z) = \text{dim }Z\); see [9, 15.1.20, 15.3.7].

We will use the following easy result:

**Lemma 2.2.** Let \(R_j, j = 1, \ldots, s,\) be some rings and \(M_j \in R_j\text{-mod}\). Then, if \(K\text{dim }M_j\) exists for all \(j\), we have

\[
K\text{dim} \bigoplus_{j=1}^s R_j M_j = \max \{K\text{dim }M_j : j = 1, \ldots, s\}.
\]

**Proof.** The claim follows from the identification of \(\mathcal{L}(\bigoplus_{j=1}^s M_j)\) with \(\mathcal{L}(M_1) \times \cdots \times \mathcal{L}(M_s)\).

### 3. Rings of differential operators

If \(Z\) is a complex algebraic variety we denote by \(O_Z\) its structural sheaf and by \(D_Z\) the sheaf of differential operators on \(Z\), as defined in [5]. By taking global sections we get the following \(C\)-algebras:

\[
O(Z) = O_Z(Z), \quad D(Z) = D_Z(Z).
\]

Assume that \(Z\) is smooth and denote by \(D_Z\text{-coh}\) the category of coherent left \(D_Z\)-modules (see [2] for a definition). Recall [2] that when \(Z\) is affine, the functor \(\mathcal{M} \to \Gamma(Z, \mathcal{M})\) yields an equivalence of categories \(D_Z\text{-coh} \cong D(Z)\text{-mod}\).
**Notation.** Let $\overline{X}$ be an irreducible affine variety and $X$ be a nonempty (dense) open subset of smooth points in $\overline{X}$. We will work under the following hypothesis:

$\overline{X}$ is normal and $\text{codim}_{\overline{X}}(\overline{X} \setminus X) \geq 2$.

In this situation one has $\mathcal{O}(X) = \mathcal{O}(\overline{X})$ and it is easy to show that this implies $\mathcal{D}(X) = \mathcal{D}(\overline{X})$; see, e.g., [6, II.2, Proposition 2]. Since $X$ is quasi-compact and open in $\overline{X}$, we can write $X = \bigcup_{i=1}^{s} U_i$, where each $U_i$ is a principal affine open subset of $\overline{X}$, i.e., $U_i = \{x \in \overline{X} : f_i(x) \neq 0\}$ for some $f_i \in \mathcal{O}(X)$. Recall that $\{f_i^{k}\}_{k \in \mathbb{N}}$ is an Ore subset in $\mathcal{D}(\overline{X})$ and that

$$\mathcal{D}(U_i) = \mathcal{D}(\overline{X})[f_i^{-1}] = \mathcal{O}(X)[f_i^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{D}(X).$$

Therefore, if $\mathcal{M} \in \mathcal{D}_X$-coh, each restriction $\mathcal{M}_{|U_i} \in \mathcal{D}_{U_i}$-coh is determined by $\mathcal{M}(U_i) = \Gamma(U_i, \mathcal{M}) \in \mathcal{D}(U_i)$-mod.

The next lemma is well known, and we include a proof for completeness.

**Lemma 3.1.** Let $\mathcal{L}(\mathcal{M})$ be the lattice of $\mathcal{D}_X$-submodules of $\mathcal{M} \in \mathcal{D}_X$-coh. Then $\mathcal{L}(\mathcal{M})$ satisfies the ascending chain condition.

**Proof.** Let $(\mathcal{M}_j)_{j \in \mathbb{N}}$ be an ascending chain of $\mathcal{D}_X$-submodules of $\mathcal{M}_0 = \mathcal{M}$. Set $\mathcal{M}_{j,i} = \mathcal{M}_j(U_i)$ for $i = 1, \ldots, s$ and $j \in \mathbb{N}$. Since the functor $\Gamma(U_i, -)$ is left exact, $(\mathcal{M}_{j,i})_{j \in \mathbb{N}}$ is an ascending chain of submodules in the finitely generated $\mathcal{D}(U_i)$-module $\mathcal{M}(U_i)$. Therefore, there exists $j(i) \in \mathbb{N}$ such that $\mathcal{M}_{j,i} = \mathcal{M}_{j(i),i}$ for all $j \geq j(i)$. Set $j_0 = \max\{j(i) : i = 1, \ldots, s\}$; then, since $X = \bigcup_{i=1}^{s} U_i$, we get that $\mathcal{M}_j = \mathcal{M}_{j_0}$ for all $j \geq j_0$.

The previous lemma and [22] enable us to define the Krull dimension of $\mathcal{M} \in \mathcal{D}_X$-coh by

$$\text{Kdim} \mathcal{M} = \text{dev} \mathcal{L}(\mathcal{M}).$$

**Proposition 3.2.** Let $\mathcal{M} \in \mathcal{D}_X$-coh. Then,

$$\text{Kdim} \mathcal{M} \leq \max\{\text{Kdim}_{\mathcal{D}(U_i)} \mathcal{M}(U_i) : i = 1, \ldots, s\} \leq \dim X.$$  

**Proof.** Observe that $\mathcal{M} = \bigoplus_{i=1}^{s} \mathcal{M}(U_i)$ is a finitely generated module over the ring $R = \bigoplus_{i=1}^{s} \mathcal{D}(U_i)$. As $\Gamma(U_i, -)$ is left exact and $X = \bigcup_{i=1}^{s} U_i$, the map $\mathcal{N} \rightarrow \bigoplus_{i=1}^{s} \mathcal{N}(U_i)$ yields a strictly increasing map from $\mathcal{L}(\mathcal{M})$ to $\mathcal{L}(\mathcal{M})$. Thus, by definition and Lemma 3.1, we obtain

$$\text{Kdim} \mathcal{M} \leq \text{Kdim} M = \max\{\text{Kdim}_{\mathcal{D}(U_i)} \mathcal{M}(U_i) : i = 1, \ldots, s\}.$$  

Since $\text{Kdim} \mathcal{D}(U_i) = \dim U_i = \dim X$ for all $i$ (cf. [22] Example 2), the assertion is proved.

Recall that we have a localization functor $L : \mathcal{D}(X)$-mod $\rightarrow \mathcal{D}_X$-coh defined by

$$L(\mathcal{M}) = \mathcal{D}_X \otimes_{\mathcal{D}(X)} \mathcal{M}.$$  

**Lemma 3.3.** The functor $L$ is exact.

**Proof.** Let $V = \overline{X}_f = \{x \in \overline{X} : f(x) \neq 0\}$, $f \in \mathcal{O}(X)$, be a principal open subset of $\overline{X}$ contained in $X$. We have already noticed that $\mathcal{D}_X(V) = \mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{D}(X)$, hence

$$\Gamma(V, L(\mathcal{M})) = \mathcal{D}_X(V) \otimes_{\mathcal{D}(X)} \mathcal{M} = \mathcal{O}(X)[f^{-1}] \otimes_{\mathcal{O}(X)} \mathcal{M}.$$
The lemma then follows from the exactness of the localization functor $M \to \mathcal{O}_X \otimes_{\mathcal{O}(X)} M$ on the category $\mathcal{O}(X)$-mod.

Suppose that $\tau \in \text{Aut} \mathcal{D}(X)$ is an automorphism of the algebra $\mathcal{D}(X)$. If $M \in \mathcal{D}(X)$-mod we denote by $M^\tau \in \mathcal{D}(X)$-mod the module defined by: $M^\tau = M$ as an abelian group and $a \cdot v = \tau(a)v$ for all $a \in \mathcal{D}(X)$, $v \in M$. We now make the supplementary hypothesis:

(\mathcal{H}) There exist $\tau_1, \ldots, \tau_p \in \text{Aut} \mathcal{D}(X)$ such that, for every $0 \neq M \in \mathcal{D}(X)$-mod, $L(M^{\tau_j}) \neq 0$ for some $j \in \{1, \ldots, p\}$.

We then define $\lambda(M) \in \mathcal{D}_X$-coh for $M \in \mathcal{D}(X)$-mod by setting

$$\lambda(M) = \bigoplus_{j=1}^p L(M^{\tau_j}).$$

**Theorem 3.4.** One has $\text{Kdim} M \leq \text{Kdim} \lambda(M)$ for all $M \in \mathcal{D}(X)$-mod. In particular,

$$\text{Kdim} \mathcal{D}(X) \leq \dim X.$$  

**Proof.** The hypothesis (\mathcal{H}) ensures that $N \to \lambda(N)$ is a strictly increasing map from $\mathcal{L}(M)$ to $\mathcal{L}(\lambda(M))$. Thus, using Proposition 3.2,

$$\text{Kdim} M = \text{dev} \mathcal{L}(M) \leq \text{dev} \mathcal{L}(\lambda(M)) = \text{Kdim} \lambda(M) \leq \dim X,$$

as required.

The properties of the map $\lambda : \mathcal{L}(M) \to \mathcal{L}(\lambda(M))$ imply that $M \in \mathcal{D}(X)$-mod is Noetherian; cf. [11, Theorem 1.3].

4. The Krull Dimension of $U(\mathfrak{g})$

Let $G$ be a simply connected semisimple complex algebraic group with Lie algebra $\mathfrak{g}$. Let $U$ be a maximal unipotent subgroup of $G$ and set $X = G/U$.

**Theorem 4.1.** The quasi-affine variety $X$ satisfies the hypotheses of (\mathcal{H}) (in particular the hypothesis (\mathcal{H})).

**Proof.** It is a classical fact that $X$ can be embedded in a normal affine variety $\overline{X}$ such that $\text{codim}(\overline{X} \setminus X) \geq 2$. This can be shown as follows. Let $\varpi_1, \ldots, \varpi_\ell$ be the fundamental dominant weights of $\mathfrak{g}$; denote by $E(\varpi_j)$, $j = 1, \ldots, \ell$, a simple $G$-module with highest weight $\varpi_j$ and set $E = \bigoplus_{j=1}^\ell E(\varpi_j)$. If $v_j \in E(\varpi_j)$ is a highest weight vector, the orbit $G.(v_1 \oplus \cdots \oplus v_\ell) \subset E$ is isomorphic to $X$ and its closure $\overline{X}$ (in $E$) has the required properties, see [4] and [11].

Thanks to [4], each element $w$ of the Weyl group of $\mathfrak{g}$ yields an automorphism $F_w \in \text{Aut} \mathcal{D}(X)$. By [11, Theorem 3.8], for every nonzero $M \in \mathcal{D}(X)$-mod there exists $w$ such that $L(M^{F_w}) \neq 0$. Thus $X$ satisfies the hypothesis (\mathcal{H}).

Observe that $\dim X$ is the dimension of a Borel subalgebra of $\mathfrak{g}$.

**Corollary 4.2.** One has

$$\text{Kdim} U(\mathfrak{g}) = \text{Kdim} \mathcal{D}(X) = \dim X.$$  

**Proof.** By Theorem 3.4 we have $\text{Kdim} \mathcal{D}(X) \leq \dim X$. From [7, Proposition 3.2] we know that $\text{Kdim} U(\mathfrak{g}) \leq \text{Kdim} \mathcal{D}(X)$, thus $\text{Kdim} U(\mathfrak{g}) \leq \text{Kdim} \mathcal{D}(X) \leq \dim X$. The result then follows from $\dim X \leq \text{Kdim} U(\mathfrak{g})$ (see [2, Example 1]).
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