ON THE AVERAGE CURVATURE OF A CONVEX CURVE IN A SURFACE OF NONPOSITIVE GAUSSIAN CURVATURE

JIN LU

(Communicated by Ronald A. Fintushel)

Abstract. In this paper, the upper bound of the average curvature of a convex curve in a simply connected surface of nonpositive Gaussian curvature is obtained.

1. Introduction

It is well known that if the geodesic curvature of a curve in the hyperbolic plane $H^2$ is less than or equal to one, then the curve is embedded in $H^2$. In [1], M. Bridgeman considered the inverse question, defined the average curvature of a curve, and gave an upper bound of average curvature of a convex curve embedded in $H^2$. He also proved that the average curvature of the bi-infinite convex curve in $H^2$ is bounded above by one. It is natural to ask the following question: What is the upper bound of the average curvature of a convex curve embedded in a surface of nonpositive Gaussian curvature? In this paper we establish this upper bound.

In this paper a surface means a 2-dimensional complete Riemannian manifold. A curve $\alpha$ in a surface is called convex if any geodesic joining two points of $\alpha$ intersects $\alpha$ only at those two points. According to [1], the average curvature $K(\alpha)$ of a finite length curve $\alpha$ is defined by

$$K(\alpha) = \frac{\int_{\alpha} k_g ds}{\int_{\alpha} |\alpha'| ds} = \frac{\text{Total curvature along } \alpha}{\text{Length of } \alpha},$$

where $k_g$ is the geodesic curvature of $\alpha$, and $s$ is the arc-length along $\alpha$. If $\alpha$ is an infinite length curve, the average curvature $K(\alpha)$ is defined by

$$K(\alpha) = \limsup_{L \to \infty} \{K(\bar{\alpha}) \mid \bar{\alpha} \text{ is a subarc of } \alpha \text{ of length } L\}.$$

The main result of this paper is:

Theorem 1. Let $M$ be a simply connected surface whose Gaussian curvature $G$ satisfies

$$-1 \leq G \leq -a^2 < 0,$$

Received by the editors October 13, 2000 and, in revised form, July 11, 2001.

2000 Mathematics Subject Classification. Primary 53C45, 58A05; Secondary 52A10, 52A40.

Key words and phrases. Average curvature, curve, surface.

This work was supported by the Chinese NNSF.

©2002 American Mathematical Society
and let $\alpha$ be a convex curve in $M$. If the length of $\alpha$ is $L$, then the average curvature $K(\alpha)$ of $\alpha$ satisfies

$$K(\alpha) \leq \frac{\sinh(L)}{\sinh(aL)} \sqrt{1 + \left(\frac{\pi \sinh(aL)}{L \sinh(L)}\right)^2} + \frac{\pi}{L}.$$ 

The other results and corollaries will be discussed in Section 3 of this paper.

2. Notations and Lemmas

Let $M$ be a simply connected surface of nonpositive Gaussian curvature, and let $\alpha$ be a convex curve in $M$ with endpoints $x, y$. Join $x$ and $y$ by a unit-speed geodesic $\gamma$ such that $\gamma(0) = x$. Denote by $\theta_0$ the interior angle formed by $\alpha$ and $\gamma$ at $x$, and by $\Omega$ the region bounded by $\alpha$ and $\gamma$ (see Figure 1). Let $T_xM$ be the tangent space of $M$ at $x$, let $(\rho, \theta)$ be the polar coordinate of $T_xM$, and let the metric in $T_xM$ be taken as $ds^2 = d\rho^2 + \rho^2 d\theta^2$. Since the Gaussian curvature of $M$ is nonpositive, we can express $\alpha$ and $\Omega$ in the following way: By the convexity of the curve $\alpha$, if the orthonormal basis $\{e_1, e_2\}$ of $T_xM$ is taken to be suitable, $\alpha$ can be written as

$$\alpha : \exp_x(\rho(\theta) \cos \theta e_1 + \rho(\theta) \sin \theta e_2), \quad 0 \leq \theta \leq \theta_0,$$

and $\Omega$ can be written as

$$\Omega = \{\exp_x(\rho \cos \theta e_1 + \rho \sin \theta e_2) \mid 0 \leq \rho \leq \rho(\theta), \quad 0 \leq \theta \leq \theta_0\},$$

where $\rho(\theta)$, $0 \leq \theta \leq \theta_0$, is a function of $\theta$ satisfying $\rho(\theta_0) = 0$. Let $L$ be the length of $\alpha$; it is easy to see that $\rho(\theta) \leq L$, $0 \leq \theta \leq \theta_0$.

**Figure 1.**

**Lemma 1.** If the Gaussian curvature $G$ of $M$ satisfies

$$-1 \leq G \leq -a^2 < 0,$$

then the area $\text{Area}(\Omega)$ of $\Omega$ satisfies

$$\frac{1}{a} \int_0^{\theta_0} (\cosh(a\rho(\theta)) - 1) d\theta \leq \text{Area}(\Omega) \leq \int_0^{\theta_0} (\cosh(\rho(\theta)) - 1) d\theta.$$

**Proof.** Let $\frac{\partial}{\partial \theta}$ be the vector field in $T_xM$ that is orthogonal to radical direction. If $p = \exp_x(\rho \cos \theta e_1 + \rho \sin \theta e_2)$ is a point in $\Omega$, then

$$\Gamma_0(t) = \exp_x t(\cos \theta e_1 + \sin \theta e_2), \quad 0 \leq t \leq \rho,$$

is the unit-speed geodesic connecting $x$ to $p$. From [4] there exists a Jacobi field
$U(t)$ along the geodesic $\Gamma_0(t)$ such that:

1. $U(t)$ is orthogonal to $\Gamma_0'(t)$.
2. $U(0) = 0$, $U'(0) = \frac{1}{\rho} \frac{\partial}{\partial \theta} (\rho, \theta)$. Hence from $|\frac{\partial}{\partial \theta} (\rho, \theta)| = \rho$, we have $|U'(0)| = 1$.
3. $d\exp_\rho \frac{\partial}{\partial \theta} (\rho, \theta) = U(\rho)$.

Since $U(t)$ is a Jacobi field along $\Gamma_0(t)$, it is well known that there exists a function $f(t)$ on $[0, \rho]$ such that $|f(t)| = |U(t)|$ and which satisfies

\[
\begin{cases}
  f''(t) + G(t)f(t) = 0, \\
  f(0) = 0, \quad f'(0) = 1,
\end{cases}
\]

where $G(t)$ is the Gaussian curvature of $M$ at $\Gamma_0(t)$.

Consider the equations

\[
\begin{cases}
  f_1''(t) - a^2 f_1(t) = 0, \\
  f_1(0) = 0, \quad f_1'(0) = 1,
\end{cases} \quad \text{and} \quad \begin{cases}
  f_2''(t) - f_2(t) = 0, \\
  f_2(0) = 0, \quad f_2'(0) = 1.
\end{cases}
\]

Since $-1 \leq G(t) \leq -a^2$, by the Sturm comparison theorem \([3]\) we deduce

\[
\sinh(\alpha t) \leq f(t) \leq \sinh(t), \quad 0 \leq t \leq \rho.
\]

Therefore

\[
\sinh(\alpha \rho) \leq \left| d\exp_\rho \frac{\partial}{\partial \theta} (\rho, \theta) \right| = |U(\rho)| = |f(\rho)| \leq \sinh(\rho).
\]

Since the Gaussian curvature of $M$ is nonpositive, according to \([2]\), the Riemannian metric on $M$ can be written as

\[
ds^2 = d\rho^2 + \left| d\exp_\rho \frac{\partial}{\partial \theta} (\rho, \theta) \right|^2 d\theta^2.
\]

Hence $\text{Area}(\Omega) = \int_0^{\theta_0} d\theta \int_0^{\rho(\theta)} |d\exp_\rho \frac{\partial}{\partial \theta} (\rho, \theta)| d\rho$. From the above inequality we have

\[
\frac{1}{a} \int_0^{\theta_0} (\cosh(\alpha \rho(\theta)) - 1) d\theta = \int_0^{\theta_0} d\theta \int_0^{\rho(\theta)} \sinh(\alpha \rho) d\rho \leq \text{Area}(\Omega)
\]

\[
\leq \int_0^{\theta_0} d\theta \int_0^{\rho(\theta)} \sinh(\rho) d\rho = \int_0^{\theta_0} (\cosh(\rho(\theta)) - 1) d\theta.
\]

**Lemma 2.** If the Gaussian curvature $G$ of $M$ satisfies $G \leq -a^2 < 0$, then the length of $\alpha$ satisfies

\[
\int_0^{\theta_0} \sqrt{\rho^2(\theta) + \sinh^2(\rho(\theta))} d\theta \leq L \frac{\sinh(L)}{\sinh(aL)}.
\]
Proof. From the proof of Lemma 1 we have $|d \exp_x \frac{\partial}{\partial \theta}(\rho, \theta)| \geq \sinh(\rho \theta)$. Notice that the function $\frac{\sinh(\alpha t)}{\sinh(t)}$ is monotonically decreasing when $t \geq 0$, and $\rho(\theta) \leq L$. From the equation of $\alpha$ and the representation of the metric $ds^2$ in Lemma 1 we deduce that

$$L = \int_{0}^{\theta_0} \sqrt{\rho'(\theta)^2 + \left| d \exp_x \frac{\partial}{\partial \theta}(\rho, \theta) \right|^2} \, d\theta$$

$$\geq \int_{0}^{\theta_0} \sqrt{\rho^2(\theta) + \sinh^2(\rho \theta)} \, d\theta$$

$$= \int_{0}^{\theta_0} \sqrt{\rho^2(\theta) + \frac{\sinh^2(\rho \theta)}{\sinh^2(\rho \theta)}} \sinh^2(\rho \theta) \, d\theta$$

$$\geq \int_{0}^{\theta_0} \sqrt{\rho^2(\theta) + \frac{\sinh^2(\alpha L)}{\sinh^2(L)}} \sinh^2(\rho \theta) \, d\theta$$

$$\geq \frac{\sinh(\alpha L)}{\sinh(L)} \int_{0}^{\theta_0} \sqrt{\rho^2(\theta) + \sinh^2(\rho \theta)} \, d\theta.$$ 

\[\square\]

The following lemma is an application of the classical elementary Lobachevskii (plane hyperbolic) geometry; we omit the proof.

**Lemma 3.** Let $C$ be a circle in a surface of constant Gaussian curvature $-b^2 < 0$. If the circumference of $C$ is $L$, then the area $A(L)$ of the disk bounded by $C$ is

$$A(L) = \frac{L}{b} \sqrt{1 + \left(\frac{2\pi}{L}\right)^2} - \frac{2\pi}{b}.$$ 

Hence $A(L)$ is a monotonically increasing function of $L$.

3. Proofs of the theorems

We denote by $\text{Area}(\cdot)$ the area of a region, and by $\text{Length}(\cdot)$ the length of a curve.

**Proof of Theorem 1.** Take a fixed point $O$ in the hyperbolic plane $H^2$, and an orthonormal basis $\{\overline{e_1}, \overline{e_2}\}$ of $T_O H^2$. Let $\alpha_H$ be the curve in $H^2$,

$$\alpha_H : \exp_O(\rho(\theta) \cos \theta \overline{e_1} + \rho(\theta) \sin \theta \overline{e_2}), \quad 0 \leq \theta \leq \theta_0,$$

where $\theta_0$ and $\rho(\theta)$ were defined in Section 2. Set

$$\Omega_H = \{ \exp_O(\rho \cos \theta \overline{e_1} + \rho \sin \theta \overline{e_2}) \mid 0 \leq \rho \leq \rho(\theta), \ 0 \leq \theta \leq \theta_0 \}$$

and

$$\gamma_H : \exp_O(\rho \overline{e_1}), \quad 0 \leq \rho \leq \rho(0),$$

so $\gamma_H$ is the unit-speed geodesic connecting the endpoints of $\alpha_H$, and $\Omega_H$ is the region bounded by $\alpha_H$ and $\gamma_H$.

Furthermore, set

$$\tilde{\alpha}_H : \exp_O(\rho(|\theta|) \cos \theta \overline{e_1} + \rho(|\theta|) \sin \theta \overline{e_2}), \quad -\theta_0 \leq \theta \leq 0,$$
ON THE AVERAGE CURVATURE

\[ \Omega_H = \{ \exp_O (\rho \cos \theta \vec{e}_1 + \rho \sin \theta \vec{e}_2) \mid 0 \leq \rho \leq \rho(|\theta|), \ -\theta_0 \leq \theta \leq 0 \} \]

Graphically, the curve \( \bar{\alpha}_H \) and region \( \bar{\Omega}_H \) are the reflection of \( \alpha \) and \( \Omega \) about \( \gamma_H \) in \( H^2 \) respectively. Notice that since \( |d \exp_O \frac{\partial}{\partial \theta} (\rho, \theta)| = \sinh \rho \) in \( H^2 \), we have

\[ \text{Area}(\Omega_H) = \text{Area}(\bar{\Omega}_H) = \int_{\theta_0}^{\theta_0} d\theta \int_0^{\rho(\theta)} \sinh \rho d\rho = \int_0^{\theta_0} (\cosh(\rho(\theta)) - 1) d\theta \]

and

\[ \text{Length}(\alpha_H) = \text{Length}(\bar{\alpha}_H) = \int_{\theta_0}^{\theta_0} \sqrt{\rho'^2(\theta) + \sinh^2(\rho(\theta))} d\theta. \]

Let the interior angle formed by \( \alpha \) and \( \gamma \) at \( y \) be \( \theta_1 \) (see Figure 1). Then by the Gauss-Bonnet theorem we have

\[ \int_{\Omega} G dV + \int_{\alpha \cup \gamma} k_g ds + \pi - \theta_0 + \pi - \theta_1 = 2\pi \chi(\Omega). \]

Since \( \gamma \) is a geodesic, \( k_g = 0 \) on \( \gamma \). Obviously \( \chi(\Omega) = 1 \), hence

\[ \int_{\Omega} G dV + \int_{\alpha} k_g ds = \theta_0 + \theta_1. \]

By the assumption, \( G \geq -1 \), so

\[ \text{Area}(\Omega) \geq \int_{\alpha} k_g ds - (\theta_0 + \theta_1); \]

hence by Lemma 1 we have

\[ \int_0^{\theta_0} (\cosh(\rho(\theta)) - 1) d\theta \geq \int_{\alpha} k_g ds - (\theta_0 + \theta_1). \]

Assume that \( K(\alpha) > \frac{\sinh(L)}{\sinh(aL)} \sqrt{1 + \left( \frac{\pi \sinh(aL)}{L \sinh(L)} \right)^2} + \frac{\pi}{\pi}. \) Since \( L \cdot K(\alpha) = \int_{\alpha} k_g ds \), we have

\[ \text{Area}(\Omega_H \cup \bar{\Omega}_H) = 2 \int_{\theta_0}^{\theta_0} (\cosh(\rho(\theta)) - 1) d\theta \geq 2 \left( \int_{\alpha} k_g ds - (\theta_0 + \theta_1) \right) \]

\[ = 2 (L \cdot K(\alpha) - (\theta_0 + \theta_1)) \]

\[ > 2L \frac{\sinh(L)}{\sinh(aL)} \sqrt{1 + \left( \frac{\pi \sinh(aL)}{L \sinh(L)} \right)^2} + 2\pi - 2(\theta_0 + \theta_1). \]

Denote by \( A(S) \) the area of the disk bounded by a circle of circumference \( S \) in \( H^2 \). Since \( \theta_0, \theta_1 \leq 2\pi \), from Lemma 3 we have

\[ \text{Area}(\Omega_H \cup \bar{\Omega}_H) > A \left( 2L \frac{\sinh(L)}{\sinh(aL)} \right). \]

By Lemma 2,

\[ \text{Length}(\alpha_H \cup \bar{\alpha}_H) = 2 \text{Length}(\alpha_H) \]

\[ \leq 2 \int_{\theta_0}^{\theta_0} \sqrt{\rho'^2(\theta) + \sinh^2(\rho(\theta))} d\theta \]

\[ \leq 2L \frac{\sinh(L)}{\sinh(aL)}. \]
From the isoperimetric inequality, \( \text{Area}(\Omega_H \cup \bar{\Omega}_H) \) is less than or equal to
\[
A(\text{Length}(\alpha_H \cup \bar{\alpha}_H)),
\]
and from (4) we have \( A(\text{Length}(\alpha_H \cup \bar{\alpha}_H)) < A \left( \frac{2L \sinh(L)}{\sinh(aL)} \right) \). Hence
\[
\text{Area}(\Omega_H \cup \bar{\Omega}_H) < A \left( \frac{2L \sinh(L)}{\sinh(aL)} \right),
\]
which contradicts (3).

If \( \alpha \) is a convex curve of length \( L \) in the hyperbolic plane \( H^2 \), then for any \( a \) with \(-1 \leq -a^2 \), Theorem 1 holds. Letting \( a \to 1 \), we deduce that

**Corollary 1.** If \( \alpha \) is a convex curve of length \( L \) in the hyperbolic plane \( H^2 \), then
\[
K(\alpha) \leq \sqrt{1 + \left( \frac{\pi}{L} \right)^2 + \frac{\pi}{L}}.
\]
Furthermore, if \( \alpha \) is a convex curve of infinite length, then \( K(\alpha) \leq 1 \).

This corollary is Theorem 1 and Corollary 1 in \([1]\).

**Theorem 2.** Let \( M \) be a simply connected surface whose Gaussian curvature \( G \) satisfies
\[-1 \leq -b^2 \leq G \leq 0,
\]
and let \( \alpha \) be a convex curve in \( M \). If the length of \( \alpha \) is \( L \), then the geodesic curvature along \( \alpha \) satisfies
\[
\int_{\alpha} k_g ds \leq \sinh(bL) \sqrt{1 + \left( \frac{\pi}{\sinh(bL)} \right)^2 + \pi}.
\]

**Proof.** Since the idea of the proof of this theorem is similar to Theorem 1, we only give a sketch. In this proof, we use the same notations as in Section 2 and this section.

By the same method as Lemma 1 and Lemma 2, we can deduce that the area of \( \Omega \) satisfies
\[
\text{Area}(\Omega) \leq \frac{1}{b} \int_{\theta_0}^{\theta_1} \left( \cosh(bp(\theta)) - 1 \right) d\theta,
\]
and the length of \( \alpha \) satisfies
\[
\int_{0}^{\theta_0} \sqrt{\rho^2(\theta) + \sinh^2(bp(\theta))} d\theta \leq \sinh(bL).
\]

From the equality (1) we can deduce that
\[
b^2 \text{Area}(\Omega) \geq \int_{\alpha} k_g ds - \theta_0 + \theta_1,
\]
so from (5) and \( b^2 < 1 \) we have
\[
\int_{0}^{\theta_0} \left( \cosh(bp(\theta)) - 1 \right) d\theta \geq b \int_{0}^{\theta_0} \left( \cosh(bp(\theta)) - 1 \right) d\theta \geq \int_{\alpha} k_g ds - (\theta_0 + \theta_1);
\]
hence
\[
\frac{1}{b} \int_{0}^{\theta_0} \left( \cosh(bp(\theta)) - 1 \right) d\theta \geq \frac{1}{b} \int_{\alpha} k_g ds - \frac{1}{b}(\theta_0 + \theta_1),
\]
which is similar to (2). By the same discussion as that of Theorem 1, replacing \( H^2 \) by the surface of constant Gaussian curvature \(-b^2\), and using Lemma 3 we can derive the conclusion.

From Theorem 2, letting \( b \to 0 \), we deduce the following interesting corollary.

**Corollary 2.** The curvature \( k \) of the convex curve \( \alpha \) in the Euclidean plane \( \mathbb{R}^2 \) satisfies

\[
\int k ds \leq 2\pi.
\]

**Remark 1.** From the proof of Theorems 1 and 2 it is easy to see that if \( M \) is a simply connected surface whose Gaussian curvature \( G \) satisfies

\[-1 \leq -b^2 \leq G \leq -a^2 < 0,\]

and \( \alpha \) is a convex curve of length \( L \) in \( M \), then

\[
K(\alpha) \leq \frac{\sinh(bL)}{\sinh(aL)} \sqrt{1 + \left( \frac{\pi \sinh(aL)}{L \sinh(bL)} \right)^2} + \frac{\pi}{L}.
\]

**References**