

BIG CELLS AND LU FACTORIZATION IN REDUCTIVE MONOIDS

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ABSTRACT. It is well known that an invertible matrix admits a factorization as a product of a lower triangular matrix L and an upper triangular matrix U if and only if all the principal minors of the matrix are non-zero. The corresponding problem for singular matrices is much more subtle. We study this problem in the general setting of a reductive monoid and obtain a solution in terms of the Bruhat-Chevalley order. In the process we obtain a decomposition of the big cell $\overline{B^-B}$ of a reductive monoid, where B and B^- are opposite Borel subgroups of the unit group.

1. PRELIMINARIES

Let k be an algebraically closed field. By a *reductive monoid* M we will mean an irreducible linear algebraic monoid M defined over k such that the unit group G is reductive; cf. [4], [9]. The multiplicative monoid $M_n(k)$ of all $n \times n$ matrices over k is an example. More generally the Zariski closure in $M_n(k)$, of a reductive group in $GL_n(k)$, is a reductive monoid. Let T be a maximal torus contained in opposite Borel subgroups B, B^- of G . Let $W = N_G(T)/T$ denote the Weyl group of G and let S denote the generating set of simple reflections of W . By the Bruhat decomposition,

$$(1) \quad G = \bigsqcup_{w \in W} BwB.$$

The Bruhat-Chevalley order on W is defined as:

$$(2) \quad x \leq y \quad \text{if } BxB \subseteq \overline{ByB}.$$

As is well known, this is equivalent to x being a subword of a reduced expression $y = s_1 \cdots s_m$, $s_1, \dots, s_m \in S$. The length $\ell(y)$ is defined to be m . It is well known that for $x, y \in W$,

$$(3) \quad x \leq y \Leftrightarrow xBy^{-1} \cap B^-B \neq \emptyset \Leftrightarrow yB^-x^{-1} \cap B^-B \neq \emptyset.$$

By (1), (3),

$$(4) \quad xB^- \subseteq \bigcup_{x' \leq x} B^-x'B.$$

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Also by induction on $\ell(y)$,

$$(5) \quad xBy \subseteq \bigcup_{y' \leq y} B^-xy'B.$$

By (4), (5)

$$(6) \quad xB^-By \subseteq \bigcup_{\substack{x' \leq x \\ y' \leq y}} B^-x'y'B.$$

For $w_1, \dots, w_n \in W$, let

$$w_1 * \dots * w_n = \begin{cases} w_1 \cdots w_n & \text{if } \ell(w_1 \cdots w_n) = \ell(w_1) + \dots + \ell(w_n), \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Then for $x, y, z \in W$,

$$(7) \quad z \leq xy \Rightarrow z = x' * y' \text{ for some } x' \leq x, y' \leq y.$$

It is also easily seen that

$$(8) \quad x * z \leq y * z \Rightarrow x \leq y.$$

For $I \subseteq S$, let W_I denote the parabolic subgroup of W generated by I and let

$$(9) \quad D_I = \{x \in W \mid xw = x * w \text{ for all } w \in W_I\}.$$

If w_0, v_0 are respectively the longest elements of W and W_I , then

$$(10) \quad w_0 D_I v_0 = D_I.$$

$P = BW_I B$ and $P^- = B^-W_I B^-$ are opposite parabolic subgroups of G with common Levi subgroup $L = P \cap P^-$. $B_L = B \cap L$ and $B_L^- = B^- \cap L$ are opposite Borel subgroups of L . If $x \in W$, then by [1, Chapter 2],

$$(11) \quad x \in D_I \Leftrightarrow xB_L x^{-1} \subseteq B \Leftrightarrow xB_L^- x^{-1} \subseteq B^-.$$

Now for monoids. The idempotent set $E(\overline{T})$ of \overline{T} is a finite lattice isomorphic to the face lattice of a rational polytope. The *cross-section lattice* Λ of M is defined as:

$$\Lambda = \{e \in E(\overline{T}) \mid Be = eBe\} = \{e \in E(\overline{T}) \mid eB^- = eB^-e\}.$$

By [3], Λ is a cross-section of the $G \times G$ -orbits of M such that for all $e, f \in \Lambda$,

$$e \leq f \Leftrightarrow GeG \subseteq \overline{GfG} \Leftrightarrow e \in MfM.$$

Here as usual, $e \leq f$ means $ef = e = fe$. For $e \in \Lambda$,

$$P = \{x \in G \mid xe = exe\}, \quad P^- = \{x \in G \mid ex = exe\}$$

are opposite parabolic subgroups. If $L = P \cap P^-$, $B_L = B \cap L$, $B_L^- = B^- \cap L$, then

$$(12) \quad Be = B_L e = eB_L, \quad eB^- = eB_L^- = B_L^- e.$$

By [6] the Bruhat decomposition (1) extends to M as:

$$(13) \quad M = \bigsqcup_{\sigma \in R} B\sigma B,$$

where $R = \overline{N_G(T)}/T$ is the Renner monoid. W is the unit group of R and

$$R = \bigsqcup_{e \in \Lambda} WeW.$$

Let $e \in \Lambda$. Then

$$(14) \quad W(e) = \{w \in W \mid we = ew\}, \quad W_e = \{w \in W \mid we = e = ew\}$$

are parabolic subgroups of W . Let $W(e) = W_I, W_e = W_K, I, K \subseteq S$. Then

$$(15) \quad W(e) = W_e \times \widetilde{W}(e),$$

where $\widetilde{W}(e) = W_{I \setminus K}$. Let

$$(16) \quad D(e) = D_I, \quad D_e = D_K.$$

Then

$$(17) \quad D_e \cap W(e) = \widetilde{W}(e), \quad D_e = D(e)\widetilde{W}(e).$$

So if $\sigma \in R$, then

$$(18) \quad \sigma = xwey \quad \text{for unique } e \in \Lambda, x \in D(e), w \in \widetilde{W}(e), y \in D(e)^{-1}.$$

We call this the *standard form* of σ . The order (3) on W extends naturally to R if we define:

$$(19) \quad \sigma \leq \sigma' \quad \text{if } B\sigma B \subseteq \overline{B\sigma'B}.$$

If $\sigma = xwey, \sigma' = x'w'e'y'$ are in standard form, then by [2],

$$(20) \quad \sigma \leq \sigma' \Leftrightarrow e \leq e', xw \leq x'w'u, u^{-1}y' \leq y \text{ for some } u \in W(e')W_e.$$

If $\sigma = xwey, \sigma' = x'w'e'y'$ in standard form, then (20) simplifies to:

$$(21) \quad \sigma \leq \sigma' \Leftrightarrow xw \leq x'w'u, u^{-1}y' \leq y \text{ for some } u \in W(e).$$

For the combinatorial properties of this order on WeW , see [5]. By [7], [8], the length $\ell(\sigma)$ is defined as

$$(22) \quad \ell(\sigma) = \ell(xw) - \ell(v_0yw_0),$$

where $\sigma = xwey$ in standard form and v_0, w_0 are as in (10). This agrees with the length function given by the order of WeW .

2. TRIANGULAR MONOIDS AND LU DECOMPOSITION

In this section we study the triangular monoids \overline{B} and \overline{B}^- . As noted in [6], the decomposition (13) leads to the decomposition:

$$(23) \quad \overline{B} = \bigsqcup_{\sigma \in R^+} B\sigma B$$

for a suitable submonoid R^+ of R . Similarly,

$$(24) \quad \overline{B}^- = \bigsqcup_{\sigma \in R^-} B^-\sigma B^-$$

for a suitable submonoid R^- of R .

Example 2.1. For $M_3(k)$, the poset (R^+, \leq) is given in Figure 1. It corrects an error in [6, Figure 3].

Theorem 2.2. *Let $\sigma = xwey \in R^+$ be in standard form. Then*

- (i) $\sigma \in R^+$ if and only if $xw \leq y^{-1}$.
- (ii) $\sigma \in R^-$ if and only if $wy \leq x^{-1}$.
- (iii) $\sigma \in R^-R^+$ if and only if $w = w_1w_2$ with $w_1 \leq x^{-1}$ and $w_2 \leq y^{-1}$.

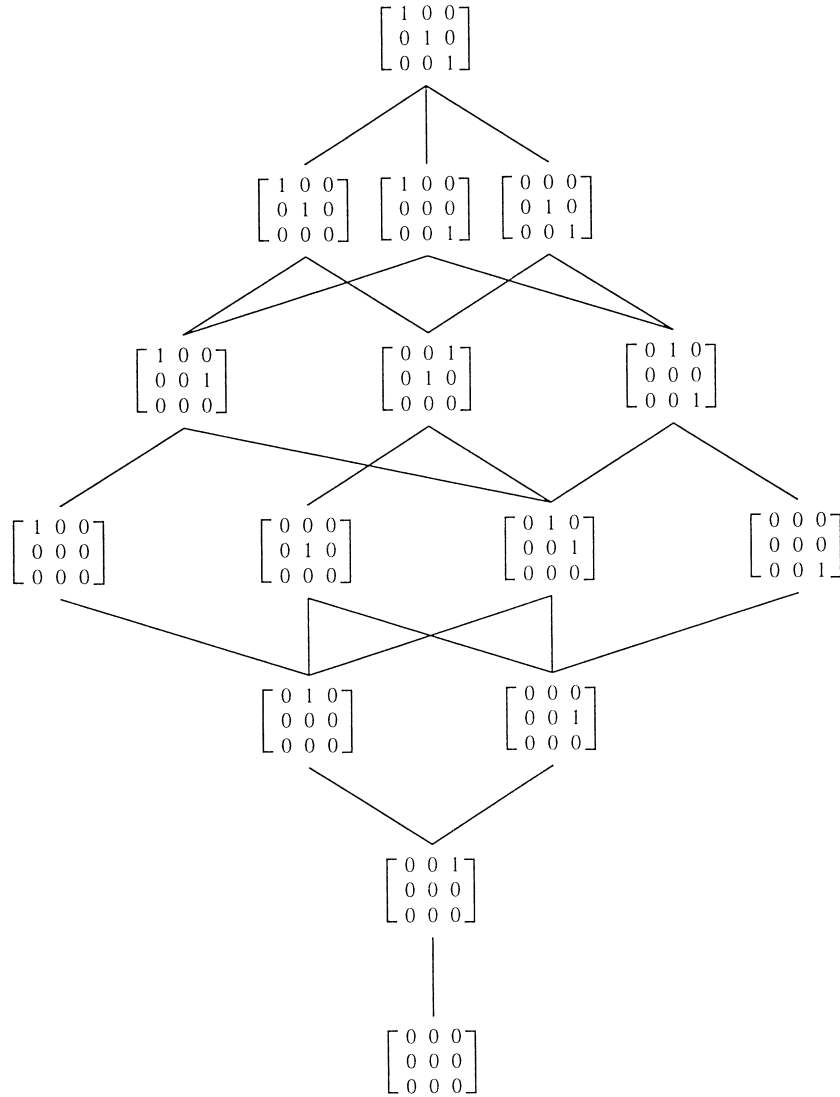


FIGURE 1.

Proof. (i) This follows from (20) since $\sigma \in R^+$ if and only if $\sigma \leq 1$.

(ii) Clearly $R^- = w_0R^+w_0$. Now by (10),

$$w_0\sigma w_0 = w_0xv_0 \cdot v_0wv_0 \cdot e \cdot v_0yw_0, \quad w_0xv_0 \in D(e), \quad v_0yw_0 \in D(e)^{-1}.$$

By (i), $\sigma \in R^-$ if and only if

$$w_0xwv_0 = w_0xv_0 \cdot v_0wv_0 \leq w_0y^{-1}v_0.$$

This is equivalent to $y^{-1}v_0 \leq xv_0$. This is in turn equivalent by (10) to $y^{-1}w^{-1} \leq x$ since $y^{-1}v_0 = y^{-1} * w^{-1} * wv_0$ and $xwv_0 = x * wv_0$.

(iii) Suppose first that $w = w_1w_2$ with $w_1, w_2 \in \widetilde{W}(e)$, $w_1 \leq x^{-1}$ and $w_2 \leq y^{-1}$. Then by (i), (ii), $xw_1e \in R^-$, $w_2ey \in R^+$. So $\sigma = (xw_1e)(w_2ey) \in R^-R^+$. Assume

conversely that $\sigma \in R^-R^+$. Then $\sigma = \sigma^-\sigma^+$ for some $\sigma^- \in R^-$ and $\sigma^+ \in R^+$. Since $xex^{-1}\sigma = \sigma = \sigma y^{-1}ey$,

$$\sigma = (xex^{-1}\sigma^-)(\sigma^+y^{-1}ey).$$

Hence we can assume without loss of generality that $\sigma^+, \sigma^- \in WeW$. Let $\sigma^- = x_1w_1ey_1$, $\sigma^+ = x_2w_2ey_2$ in standard form. Since $\sigma^-\sigma^+ = \sigma \in WeW$, $ey_1x_2e \in WeW$. So $u = y_1x_2 \in W(e)$. Hence $x_2 = y_1^{-1}u$. Since $x_2, y_1^{-1} \in D(e)$, $x_2 = y_1^{-1}$. Hence $\sigma = \sigma^-\sigma^+ = x_1w_1w_2ey_2$. So $x = x_1$, $w = w_1w_2$, $y = y_2$. Since $\sigma^- \in R^-$, $w_1 \leq w_1y_1 \leq x^{-1}$. Since $\sigma^+ \in R^+$, $w_2 \leq x_2w_2 \leq y^{-1}$. \square

Next we obtain a decomposition of $\overline{B^-B}$, the monoid analogue of Chevalley's big cell B^-B of G . We note that for $M_n(k)$, this consists of (possibly singular) matrices admitting a factorization into a product of a lower triangular matrix and an upper triangular matrix.

Theorem 2.3. $\overline{B^-B} = \bigsqcup_{\sigma \in R^-R^+} B^-\sigma B.$

Proof. Now $B^-R^- \subseteq \overline{B^-}$ and $R^+B \subseteq \overline{B}$. Hence $B^-R^-R^+B \subseteq \overline{B^-B}$. Let $\sigma = xwey \in R$ be in standard form such that $\sigma \in \overline{B^-B}$. Now $xex^{-1}, yey^{-1} \in \overline{T}$ and

$$\sigma = (xex^{-1})\sigma(y^{-1}ey) \in (xex^{-1}\overline{B^-})(\overline{B}y^{-1}ey).$$

By (23), (24), there exists $\sigma_1 \in R^- \cap WeW$ and $\sigma_2 \in R^+ \cap WeW$ such that

$$(25) \quad \sigma \in B^-\sigma_1B^-B\sigma_2B.$$

Let $\sigma_1 = x_1w_1ey_1$ and $\sigma_2 = x_2w_2ey_2$ in standard form. By Theorem 2.2, $w_1y_1 \leq x_1^{-1}$ and $x_2w_2 \leq y_2^{-1}$. By (6),

$$(26) \quad w_1y_1B^-Bx_2w_2 \subseteq \bigcup_{\substack{z_1 \leq w_1y_1 \\ z_2 \leq x_2w_2}} B^-z_1z_2B.$$

By (25), (26), there exists $z_1 \leq w_1y_1$, $z_2 \leq x_2w_2$ such that

$$(27) \quad \sigma \in B^-x_1eB^-z_1z_2Bey_2B.$$

By (12), $Be = eB_L$ and $eB^- = B_L^-e$. Since $x_1, y_2^{-1} \in D(e)$ we see by (11) that $x_1B_L^-x_1^{-1} \subseteq B^-$ and $y_2^{-1}B_Ly_2 \subseteq B$. Thus by (27),

$$\sigma \in B^-x_1ez_1z_2ey_2B.$$

By (13), $\sigma = x_1ez_1z_2ey_2$. Hence $x = x_1$, $y = y_2$, $z_1z_2 \in W(e)$ and $we = z_1z_2e$. By (7), $w = u_1u_2$ with

$$u_1 \leq z_1 \leq w_1y_1 \leq x^{-1}, \quad u_2 \leq z_2 \leq x_2w_2 \leq y^{-1}.$$

By Theorem 2.2, $\sigma \in R^-R^+$. This completes the proof. \square

The Bruhat decomposition (1) and the Bruhat-Renner decomposition (13) lead respectively to the closure orders (2) and (19) on W and R . In the same way the decomposition in Theorem 2.3 of the monoid big cell $\overline{B^-B}$ leads to the order \preceq on R^-R^+ given by

$$\sigma \preceq \sigma' \quad \text{if } B^-\sigma B \subseteq \overline{B^-\sigma'B}.$$

Example 2.4. For $M_3(k)$, the order \preceq on the rank 2 elements R^-R^+ is given in Figure 2.

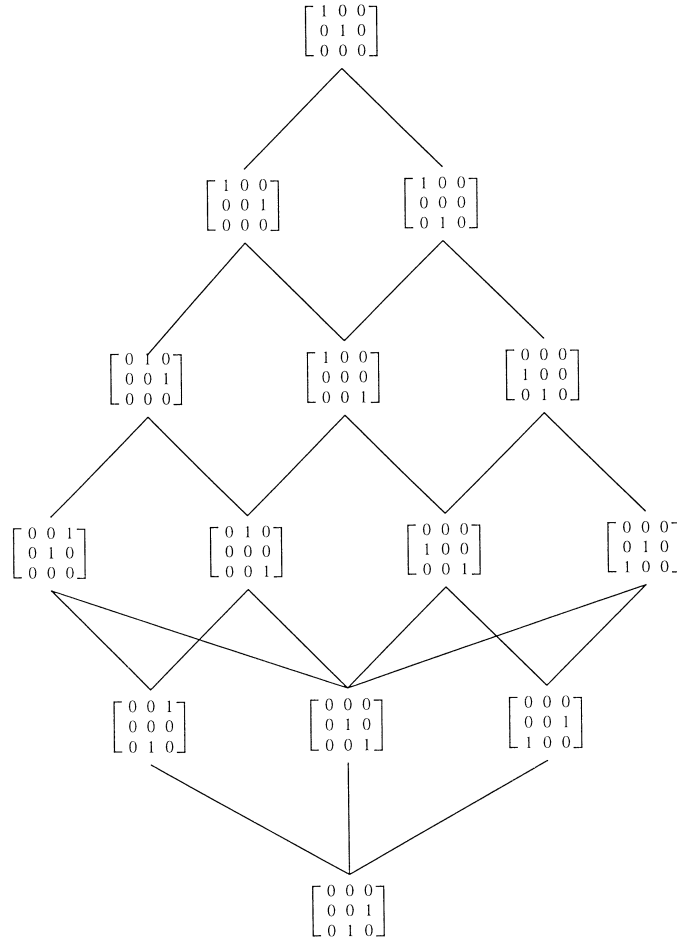


FIGURE 2.

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