

**CODIMENSION OF POLYNOMIAL SUBSPACE IN $L_2(\mathbb{R}, d\mu)$
 FOR DISCRETE INDETERMINATE MEASURE μ**

ANDREW G. BAKAN

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ABSTRACT. A calculation formula is established for the codimension of the polynomial subspace in $L_2(\mathbb{R}, d\mu)$ with discrete indeterminate measure μ . We clarify how much the masspoint of the n -canonical solution of an indeterminate Hamburger moment problem differs from the masspoint of the corresponding N -extremal solution at a given point of the real axis.

1. INTRODUCTION AND MAIN RESULT

Let $\mathcal{M}^*(\mathbb{R})$ be the set of positive Borel measures on \mathbb{R} having moments of every order and infinite support,

$$\mathcal{N} := \{f \in \text{Hol}(\mathbb{C} \setminus \mathbb{R}) \mid \text{Im}f(z) / \text{Im}z > 0 \quad \forall z \in \mathbb{C} \setminus \mathbb{R}\};$$

$\mathfrak{P} := \mathcal{N} \cup \mathbb{R}$, $\mathfrak{P}^* := \mathfrak{P} \cup \{\infty\}$ and $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$. We write

$$\mathcal{N}_2 := \left\{ \begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix} \mid a, d, b, c \in \mathcal{E}; \quad a(z)d(z) - b(z)c(z) \equiv 1; \right.$$

$$\left. \frac{a(z)t + c(z)}{b(z)t + d(z)} \in \mathcal{N} \quad \forall t \in \mathbb{R}^* \right\}$$

for the set of all Nevanlinna matrices, where \mathcal{E} denotes the set of all entire functions real-valued on the real axis.

A measure $\mu \in \mathcal{M}^*(\mathbb{R})$ is said to be indeterminate if the set V_μ of all measures $\nu \in \mathcal{M}^*(\mathbb{R})$ such that

$$\int_{\mathbb{R}} x^n d\mu(x) = \int_{\mathbb{R}} x^n d\nu(x) \quad \forall n \geq 0,$$

contains at least one measure not coincident with μ . In that case the moment problem generated by $\mu \in \mathcal{M}^*(\mathbb{R})$ (or, more precisely, generated by moments of μ) is called an indeterminate Hamburger moment problem, and all measures from V_μ are referred to as its solutions (see [1, II, §1]). If $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$ (see, for example, [1]) denote, corresponding to this indeterminate moment problem,

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sequences of polynomials of the first and of the second kind, respectively, then by the Nevanlinna theorem, one can construct, using the formulas

$$\begin{aligned} A(z) &= z \sum_{k=0}^{\infty} Q_k(0)Q_k(z), & C(z) &= 1 + z \sum_{k=0}^{\infty} P_k(0)Q_k(z), \\ B(z) &= -1 + z \sum_{k=0}^{\infty} Q_k(0)P_k(z), & D(z) &= z \sum_{k=0}^{\infty} P_k(0)P_k(z), \end{aligned}$$

the Nevanlinna matrix $\begin{pmatrix} -A(z) & C(z) \\ B(z) & -D(z) \end{pmatrix} \in \mathcal{N}_2$ such that the known Nevanlinna formula

$$(1.1) \quad \int_{\mathbb{R}} \frac{d\nu_{\varphi}(t)}{t-z} = - \frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)} \quad \forall z \in \mathbb{C} \setminus \mathbb{R}$$

establishes the homeomorphism $\mathfrak{P}^* \ni \varphi \rightarrow \nu_{\varphi} \in V(\mu)$ of \mathfrak{P}^* onto $V(\mu)$.

The special solutions in (1.1) corresponding to $\varphi \in \mathfrak{P}^*$ being a real constant or ∞ are called *N-extremal*. All of them are discrete measures. It is known that for each $x \in \mathbb{R}$

$$(1.2) \quad \max_{\nu \in V(\mu)} \nu(\{x\}) = \rho(x) := \left(\sum_{n=0}^{\infty} P_n(x)^2 \right)^{-1},$$

and this maximum is attained on only one *N-extremal* measure ν , depending on $x \in \mathbb{R}$ (see [1, Th.3.4.1]). More precisely, every *N-extremal* measure at any growth point x has a maximal mass $\rho(x)$ in the sense of (1.2), and the function $\rho(x)$ defined in (1.2) is called a *maximal weight function* of the moment problem generated by the measure μ . It is also known that the following equality holds (see, for example, [6, (2.3)]):

$$(1.3) \quad B'(x)D(x) - D'(x)B(x) = \frac{1}{\rho(x)} \quad \forall x \in \mathbb{R}.$$

N-extremal measures were characterized by M. Riesz in 1923. Denote by $\mathcal{P}[\mathbb{C}]$ the set of all algebraic polynomials with arbitrary complex coefficients.

Riesz’s Theorem ([6]). *Let $\mu \in \mathcal{M}^*(\mathbb{R})$.*

1. *If μ is an indeterminate measure and $\nu \in V_{\mu}$, then $\mathcal{P}[\mathbb{C}]$ is dense in $L_2(\mathbb{R}, d\nu)$ if and only if ν is an *N-extremal* measure.*

2. *If μ is a determinate measure (i.e., $V_{\mu} = \{\mu\}$), then $\mathcal{P}[\mathbb{C}]$ is dense in $L_2(\mathbb{R}, d\mu)$.*

If in (1.1) $\varphi \in \mathfrak{P}^*$ is a rational function of degree n , i.e., $\varphi = \frac{p}{q}$, where p and q are polynomials without common zeros and the maximum of the degrees of p and q is equal to n , then ν_{φ} is called the *n-canonical measure*, and, according to (1.1),

$$(1.4) \quad \int_{\mathbb{R}} \frac{d\nu_{\varphi}(t)}{t-z} = - \frac{A(z)p(z) - C(z)q(z)}{B(z)p(z) - D(z)q(z)} \quad \forall z \in \mathbb{C} \setminus \mathbb{R}, \quad \varphi = \frac{p}{q}.$$

That is why any *n-canonical* measure is also discrete with some masspoints at zeros of $B(z)p(z) - D(z)q(z)$, i.e.,

$$d\nu_{\varphi}(x) = \sum_{\lambda \in \Lambda_{Bp-Dq}} \nu_{\lambda}^{\varphi} \cdot \delta_{\lambda}(x),$$

where Λ_f denotes the set of all zeros of some entire function f , δ_λ is Dirac's measure at the point λ , and the masses ν_λ^φ are given by the corresponding residues. It is clear that if $n \geq 1$, then, according to (1.2),

$$(1.5) \quad 0 < \nu_\lambda^\varphi < \rho(\lambda) \quad \forall \lambda \in \Lambda_{Bp-Dq}.$$

The 0-canonical solutions and ν_∞ are the same as the N -extremal measures.

It is well-known that $\nu \in V_\mu$ is n -canonical if and only if the measure

$$(1 + x^2)^{-n} d\nu(x)$$

is N -extremal (see [1, Th.3.4.3]). Another characterization of n -canonical measures is given in the following result (1984) of Cassier, which generalizes Riesz's theorem.

Cassier's Theorem ([3], [2]). *Let $\mu \in \mathcal{M}^*(\mathbb{R})$ be an indeterminate measure. The measure μ is n -canonical if and only if the closure of the algebraic polynomials $\mathcal{P}[\mathbb{C}]$ in the space $L_2(\mathbb{R}, d\mu)$ is of codimension n .*

In this paper we partially answer the natural question as how much ν_λ^φ (from (1.5)) is less than $\rho(\lambda)$. Besides that, we also calculate the codimension of the closure of $\mathcal{P}[\mathbb{C}]$ in the space $L_2(\mathbb{R}, d\mu)$ for any indeterminate discrete measure μ .

Theorem 1. *Let*

$$d\mu(x) = \sum_{k \geq 1} \mu_k \cdot \delta_{\lambda_k}(x)$$

be any discrete indeterminate measure from the class $\mathcal{M}^(\mathbb{R})$. Then the following statements hold.*

(A) *If ρ is a maximal weight function of the indeterminate moment problem generated by μ , then*

$$(1.6) \quad \sum_{k \geq 1} \left(1 - \frac{\mu_k}{\rho(\lambda_k)} \right) = \text{codim}_{L_2(\mathbb{R}, d\mu)} \mathcal{P}[\mathbb{C}],$$

where $\text{codim}_{L_2(\mathbb{R}, d\mu)} \mathcal{P}[\mathbb{C}] \in \{0, 1, 2, \dots\} \cup \{+\infty\}$ denotes the codimension of the closure of the algebraic polynomials $\mathcal{P}[\mathbb{C}]$ in the space $L_2(\mathbb{R}, d\mu)$.

(B) *If μ is an n -canonical measure for some nonnegative integer n , then there exist numbers $\theta_k \in [0, 1)$, $k \geq 1$, such that*

$$\begin{cases} \mu_k = (1 - \theta_k) \rho(\lambda_k) & \forall k \geq 1; \\ \sum_{k \geq 1} \theta_k = n. \end{cases}$$

2. AUXILIARY LEMMA

It has been proved in [1, III, 1.1.] that $f(z) \in \mathcal{N}$ and

$$(2.1) \quad \sup_{|y| \geq 1} |yf(iy)| < \infty$$

if and only if there exists a nondecreasing function $\sigma(x)$ of bounded variation on the whole real axis such that

$$f(z) = \int_{\mathbb{R}} \frac{d\sigma(u)}{u - z} \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

In Lemma 1 below we establish a useful corollary of this statement.

Let $\varphi \in \mathcal{N}$ be a meromorphic function with the set of all its zeros $\{b_k\}_{k \geq 1} \subset \mathbb{R}$. Denote by $\{a_k\}_{k \geq 1} \subset \mathbb{R}$ all its nonzero poles. Then by a known theorem (see [5, VII, Th.2]), there exist nonnegative numbers $A_*(\varphi), A_{-1}(\varphi), A_k(\varphi), k \geq 1$, and a real number $A_0(\varphi)$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$

$$(2.2) \quad \varphi(z) = A_{-1}(\varphi)z + A_0(\varphi) - \frac{A_*(\varphi)}{z} + \sum_{k \geq 1} A_k(\varphi) \left(\frac{1}{a_k - z} - \frac{1}{a_k} \right),$$

where

$$(2.3) \quad \sum_{k \geq 1} \frac{A_k(\varphi)}{1 + a_k^2} < \infty.$$

Lemma 1. *Let $\varphi \in \mathcal{N}$ be a meromorphic function with zeros $\{b_k\}_{k \geq 1} \subset \mathbb{R}$, and assume the coefficient $A_{-1}(\varphi)$ in its representation (2.2) is positive. Then for all $z \in \mathbb{C} \setminus \mathbb{R}$ corresponding to (2.2), the representation of the function $-1/\varphi \in \mathcal{N}$ has the following specific form:*

$$(2.4) \quad -\frac{1}{\varphi(z)} = \sum_{k \geq 1} \frac{A_k(-\frac{1}{\varphi})}{b_k - z}, \quad \sum_{k \geq 1} A_k \left(-\frac{1}{\varphi} \right) < \infty,$$

where $A_k(-\frac{1}{\varphi}) \geq 0 \quad \forall k \geq 1$.

Proof. It is easy to verify that (2.3) implies

$$(2.5) \quad \sum_{k \geq 1} A_k(\varphi) \left(\frac{1}{a_k - z} - \frac{1}{a_k} \right) = \sum_{k \geq 1} A_k(\varphi) \frac{z}{a_k(a_k - z)} = \bar{\sigma}(y),$$

where $z = iy$ and $|y| \rightarrow \infty$. Since $A_{-1}(\varphi) > 0$, then (2.2) and (2.5) yield

$$\varphi(iy) = A_{-1}(\varphi)iy + \bar{\sigma}(|y|), \quad |y| \rightarrow \infty,$$

and hence as $|y| \rightarrow \infty$ we have

$$(2.6) \quad -\frac{1}{\varphi(iy)} = -\frac{1}{A_{-1}(\varphi)iy + \bar{\sigma}(|y|)} = \frac{i}{yA_{-1}(\varphi)} (1 + \bar{\sigma}(1)).$$

The asymptotic representation (2.6) indicates that the function $-1/\varphi \in \mathcal{N}$ satisfies condition (2.1): $\sup_{|y| \geq 1} |y| |-1/\varphi(iy)| < \infty$. Applying the fact from [1, III, 1.1] mentioned at the beginning of this section, we obtain the existence of nonnegative numbers $A_k(-\frac{1}{\varphi}), k \geq 1$, such that for all $z \in \mathbb{C} \setminus \mathbb{R}$ the relations (2.4) are true. Lemma 1 is proved. □

3. PROOF OF THEOREM 1

3.1. Let us consider some positive integer $n \geq 1$ and any n -canonical measure $\mu \in \mathcal{M}^*(\mathbb{R})$. According to (1.4), for the indeterminate moment problem generated by μ , we have two polynomials $p, q, \max \{\deg p, \deg q\} = n$, and two entire functions $U(z), V(z)$ such that

$$(3.1) \quad \int_{\mathbb{R}} \frac{d\mu(t)}{t - z} = -\frac{A(z)p(z) - C(z)q(z)}{B(z)p(z) - D(z)q(z)} =: \frac{U(z)}{V(z)} \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

But then, for some $\alpha \in [0, 2\pi]$,

$$(3.2) \quad \begin{pmatrix} U(z) \\ V(z) \end{pmatrix} = \begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix} \begin{pmatrix} g(z) \\ h(z) \end{pmatrix},$$

where

$$(3.3) \quad \begin{pmatrix} a(z) & c(z) \\ b(z) & d(z) \end{pmatrix} = \begin{pmatrix} -A(z) & C(z) \\ B(z) & -D(z) \end{pmatrix} \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix},$$

$$\begin{pmatrix} g(z) \\ h(z) \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix}.$$

It follows from known properties of the class \mathcal{N} and Nevanlinna matrices (see [5, VII, Th.2], p. 412, (1); p. 414, Theorem) that

$$(3.4) \quad \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathcal{N}_2, \quad \frac{g}{h} \in \mathcal{N}.$$

Moreover, one can easily derive that for almost all $\alpha \in [0, 2\pi]$, the following relations hold:

$$(3.5) \quad \begin{cases} \deg h = \deg(p(z)\sin \alpha + q(z)\cos \alpha) = \max\{\deg p, \deg q\} = n; \\ 0 \notin \Lambda_\varphi, \Lambda_{\varphi_1} \cap \Lambda_{\varphi_2} = \emptyset \quad \forall \varphi, \varphi_1 \neq \varphi_2 \in \{U, V, a, b, c, d, g, h\}. \end{cases}$$

Equalities (1.3) and (3.1) can be rewritten as follows:

$$(3.6) \quad d'(x)b(x) - b'(x)d(x) = \frac{1}{\rho(x)} \quad \forall x \in \mathbb{R},$$

$$(3.7) \quad \frac{U(z)}{V(z)} = \int_{\mathbb{R}} \frac{d\mu(t)}{t-z} = \frac{a(z)g(z) + c(z)h(z)}{b(z)g(z) + d(z)h(z)} \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

Everywhere below we assume that the number $\alpha \in [0, 2\pi]$, introduced in (3.3), satisfies (3.5).

3.2. It is easy to see from (1.2) that, for every $a \in \mathbb{R}$, the maximal weight function corresponding to the shifted measure $d\mu_a(x) := d\mu(x-a)$ equals $\rho(x-a)$ and that $\text{codim}_{L_2(\mathbb{R}, d\mu_a)} \mathcal{P}[\mathbb{C}] = \text{codim}_{L_2(\mathbb{R}, d\mu)} \mathcal{P}[\mathbb{C}]$. That is why, without loss of generality, we assume that $0 \notin \{\lambda_k\}_{k \geq 1}$ and $V(0) = 1$. According to (3.7), $U(z)/V(z) = \sum_{k \geq 1} \mu_k / (\lambda_k - z)$, and therefore

$$(3.8) \quad U(\lambda_k) = -\mu_k V'(\lambda_k) \quad \forall k \geq 1.$$

Besides that, it follows from (3.2) and (3.4) that

$$\begin{cases} g(z) = U(z)d(z) - V(z)c(z), \\ h(z) = V(z)a(z) - U(z)b(z), \end{cases}$$

from which one can easily obtain

$$(3.9) \quad \begin{cases} g(\lambda_k) = U(\lambda_k)d(\lambda_k) \stackrel{(3.8)}{=} -\mu_k d(\lambda_k)V'(\lambda_k), \\ h(\lambda_k) = -U(\lambda_k)b(\lambda_k) \stackrel{(3.8)}{=} \mu_k b(\lambda_k)V'(\lambda_k). \end{cases}$$

But the inclusions (3.4), together with known properties of Nevanlinna matrices (see [7]), mean that

$$\frac{V}{bh} = \frac{bg + dh}{bh} = \frac{g}{h} + \frac{d}{b} \in \mathcal{N},$$

and so,

$$-\frac{bh}{V} = -\frac{1}{\frac{g}{h} + \frac{d}{b}} \in \mathcal{N}.$$

Therefore, by a well-known theorem [5, VII, Th.2], there exist $V_0 \in \mathbb{R}$ and $V_{-1}, V_k \geq 0 \forall k \geq 1$ such that

$$(3.10) \quad -\frac{h(z)b(z)}{V(z)} = V_{-1}z + V_0 + \sum_{k \geq 1} \frac{z}{\lambda_k(\lambda_k - z)} V_k \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

3.3. We prove now that in the expansion (3.10), $V_{-1} = 0$. Assume the contrary: $A_{-1} \left(-\frac{bh}{V}\right) \equiv V_{-1} > 0$ (here and everywhere below we use the notation from (2.2) for all functions $\varphi \in \mathcal{N}$). Then (2.4) gives

$$(3.11) \quad \frac{V}{bh} = \frac{g}{h} + \frac{d}{b} = \sum_{k \geq 1} \frac{A_k \left(\frac{V}{bh}\right)}{\beta_k - z}, \quad \sum_{k \geq 1} A_k \left(\frac{V}{bh}\right) < \infty,$$

where $\{\beta_k\}_{k \geq 1}$ are all the zeros of $b(z)h(z)$. Denoting by $\{c_k\}_{k \geq 1}$ all the zeros of the entire function $b(z)$, we conclude by (3.11) that, if $\beta_k = c_m$ for some positive integers m and k , then

$$A_k \left(\frac{V}{bh}\right) = -\frac{d(c_m)}{b'(c_m)}$$

and, hence,

$$\sum_{k \geq 1} -\frac{d(c_k)}{b'(c_k)} < \infty.$$

But on the other hand, using equality (3.6) we have

$$(3.12) \quad \infty > \sum_{k \geq 1} -\frac{d(c_k)}{b'(c_k)} = \sum_{k \geq 1} \frac{-d(c_k)b'(c_k)}{b'(c_k)^2} = \sum_{k \geq 1} \frac{1}{\rho(c_k)b'(c_k)^2}.$$

Due to (3.4) and (3.7), the function $b(z)$ is an element of the Nevanlinna matrix corresponding to the indeterminate moment problem generated by the measure μ . That's why the measure $\sum_{k \geq 1} \rho(c_k)\delta_{c_k}(x)$ is N -extremal (see (1.1)). But now inequality (3.12) gives a contradiction with the second necessary Hamburger condition for N -extremal measures ([4, p. 516, (8.24)], [1, IV, Addenda and exercises, 2, Th. 1, (7)]). This contradiction proves the required equality $V_{-1} = 0$ in (3.10). Thus, for all $z \in \mathbb{C} \setminus \mathbb{R}$ we can rewrite (3.10) as follows:

$$(3.13) \quad -\frac{h(z)b(z)}{V(z)} = V_0 + \sum_{k \geq 1} \frac{z}{\lambda_k(\lambda_k - z)} V_k, \quad V_0 \in \mathbb{R}, \quad V_k \geq 0 \quad \forall k \geq 1.$$

3.4. Differentiating equality (3.13), we get

$$-\frac{(h(z)b(z))'V(z) - h(z)b(z)V'(z)}{V(z)^2} = -\left(\frac{h(z)b(z)}{V(z)}\right)' = \sum_{k \geq 1} \frac{V_k}{(\lambda_k - z)^2}$$

and

$$(3.14) \quad \sum_{k \geq 1} V_k \left(\frac{V(z)}{\lambda_k - z}\right)^2 = h(z)b(z)V'(z) - h'(z)b(z)V(z) - h(z)b'(z)V(z).$$

Denote by $\eta_1, \eta_2, \dots, \eta_n$ all zeros of the polynomial $h(z)$ and substitute $z = \eta_m$ in (3.14):

$$\sum_{k \geq 1} V_k \left(\frac{V(\eta_m)}{\lambda_k - \eta_m} \right)^2 = -h'(\eta_m)b(\eta_m)V(\eta_m) \quad \forall 1 \leq m \leq n.$$

In addition to these equalities, the equality $V = bg + dh$ implies $V(\eta_m) = b(\eta_m)g(\eta_m)$, and, therefore,

$$-h'(\eta_m)b(\eta_m) = \sum_{k \geq 1} V_k \frac{V(\eta_m)}{(\lambda_k - \eta_m)^2} = \left(\sum_{k \geq 1} \frac{V_k}{(\lambda_k - \eta_m)^2} \right) b(\eta_m)g(\eta_m),$$

from which we get

$$(3.15) \quad 1 = \sum_{k \geq 1} V_k \left(-\frac{g(\eta_m)}{h'(\eta_m)} \right) \frac{1}{(\lambda_k - \eta_m)^2} \quad \forall 1 \leq m \leq n.$$

Under our condition on the number α , differentiation of the obvious equality

$$\frac{g(z)}{h(z)} = C_0 + \sum_{m=1}^n \frac{g(\eta_m)}{h'(\eta_m)} \frac{1}{(z - \eta_m)}$$

gives

$$(3.16) \quad \left(\frac{g(z)}{h(z)} \right)' = \sum_{m=1}^n \left(-\frac{g(\eta_m)}{h'(\eta_m)} \right) \frac{1}{(z - \eta_m)^2}.$$

Thus, summing (3.15) over all m and taking (3.16) into account, we have

$$(3.17) \quad n = \sum_{k \geq 1} V_k \left(\frac{g}{h} \right)'(\lambda_k).$$

To finish the proof of our theorem, it remains only to recount the terms in the right side of (3.17).

3.5. Equality (3.13) shows that

$$V_k = \frac{h(\lambda_k)b(\lambda_k)}{V'(\lambda_k)} \quad \forall k \geq 1,$$

which, together with (3.9), indicates that

$$V_k = \frac{h(\lambda_k)^2}{\mu_k V'(\lambda_k)^2} \quad \forall k \geq 1,$$

and therefore

$$(3.18) \quad V_k \left(\frac{g}{h} \right)'(\lambda_k) = \frac{1}{\mu_k V'(\lambda_k)^2} (g'(\lambda_k)h(\lambda_k) - g(\lambda_k)h'(\lambda_k)) \quad \forall k \geq 1.$$

For the sake of convenience, we denote

$$\left\| \frac{F}{G} \right\| (z) := F'(z)G(z) - F(z)G'(z)$$

for any two entire functions $F(z), G(z)$. That is why equalities (3.18) can be rewritten as follows:

$$(3.19) \quad V_k \left(\frac{g}{h} \right)'(\lambda_k) = \frac{\left\| \frac{g}{h} \right\|(\lambda_k)}{\mu_k V'(\lambda_k)^2} \quad \forall k \geq 1.$$

3.6. Now we will find an acceptable expression for $\left\|\frac{g}{h}\right\|(\lambda_k)$ from (3.19). Differentiating the equality

$$\frac{V}{bh} = \frac{g}{h} + \frac{d}{b},$$

we get

$$V'(z)b(z)h(z) - V(z)(bh)'(z) = b(z)^2 \left\|\frac{g}{h}\right\|(z) + h(z)^2 \left\|\frac{d}{b}\right\|(z).$$

Setting $z = \lambda_k$ here, we obtain

$$(3.20) \quad V'(\lambda_k)b(\lambda_k)h(\lambda_k) = b(\lambda_k)^2 \left\|\frac{g}{h}\right\|(\lambda_k) + h(\lambda_k)^2 \left\|\frac{d}{b}\right\|(\lambda_k).$$

Replacement of $h(\lambda_k)$ here by its expression from (3.9) gives

$$\mu_k V'(\lambda_k)^2 b(\lambda_k)^2 = b(\lambda_k)^2 \left\|\frac{g}{h}\right\|(\lambda_k) + \mu_k^2 V'(\lambda_k)^2 b(\lambda_k)^2 \left\|\frac{d}{b}\right\|(\lambda_k),$$

or

$$\mu_k V'(\lambda_k)^2 = \left\|\frac{g}{h}\right\|(\lambda_k) + \mu_k^2 V'(\lambda_k)^2 \left\|\frac{d}{b}\right\|(\lambda_k).$$

That is why

$$(3.21) \quad \left\|\frac{g}{h}\right\|(\lambda_k) = \mu_k V'(\lambda_k)^2 \left(1 - \mu_k \left\|\frac{d}{b}\right\|(\lambda_k)\right) \quad \forall k \geq 1.$$

Substituting (3.21) in (3.19) and taking into account the equality $\left\|\frac{d}{b}\right\|(\lambda_k) = \frac{1}{\rho(\lambda_k)}$ evoked by (3.6), we establish the desired relation (1.6) for any n -canonical measure μ with a positive integer n such that

$$n = \sum_{k \geq 1} \left(1 - \mu_k \left\|\frac{d}{b}\right\|(\lambda_k)\right) = \sum_{k \geq 1} \left(1 - \frac{\mu_k}{\rho(\lambda_k)}\right).$$

With the help of an integral representation of the functions from \mathcal{N} (see [1, III, §1, (3)]) and (1.1), it is possible to approximate any non-canonical but discrete measure from V_μ by canonical measures with their orders n increasing to infinity, and, due to equality (1.6) established for canonical measures, to get for such a measure a convergence to infinity of the series in the left side of (1.6). Finally, statement (B) of the theorem represents a simple reformulation of (A) with the help of Cassier's theorem. For $n = 0$ statements (A) and (B) are evident, and this completes the proof of Theorem 1.

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INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF UKRAINE, TERESCHENKOV-SKAJA 3, KYIV 01601, UKRAINE

E-mail address: andrew@bakan.kiev.ua