ON A PROBLEM OF J. P. WILLIAMS

EDWARD KISSIN AND VICTOR S. SHULMAN

Abstract. Let $B(H)$ be the algebra of all bounded operators on a Hilbert space $H$. Let $g$ be a continuous function on the closed disk $D$ and let

$$
\|g(A)X - Xg(A)\| \leq C\|AX - XA\|,
$$

for some $C > 0$, for all $X \in B(H)$ and all $A \in B(H)$ with $\|A\| \leq 1$. Then $g$ is differentiable on $D$. The paper shows that the function $g$ may have a discontinuous derivative.

1. Introduction

Let $B(H)$ be the algebra of all bounded operators on a Hilbert space $H$ and $B_1$ be the unit ball of $B(H)$. For $A, B \in B(H)$, we denote by $[A, B]$ their commutator $AB - BA$. Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk. In his paper [7], Williams raised the following problem. If $g$ is a continuous complex-valued function on $D$, possessing the property

$$
\|[g(A), X]\| \leq C\|[A, X]\|,
$$

for some $C > 0$, for any $X \in B(H)$ and any normal operator $A$ in $B_1$, must $g$ always be continuously differentiable on $D$?

It should be noted that Johnson and Williams proved earlier [2, Theorem 4.1] that $g$ must be differentiable on $D$ and therefore analytic in the interior $D^\circ$ of $D$, and its derivative must be bounded on $D$.

We will show that the answer to Williams’s problem is negative. Moreover, we will show that the function on $D$ may have a discontinuous derivative even if it satisfies (1) for all (not necessarily normal) contractions $A$.

The authors are grateful to the referee for his useful suggestions.

2. Fully Operator Lipschitz functions

We denote by $A(D)$ the disk algebra: the algebra of all continuous complex-valued functions on $D$ which are analytic on $D^\circ$. The algebra $A(D)$ is a closed subalgebra of the algebra $C(D)$ of all continuous complex-valued functions on $D$ with the norm $\|g\| = \sup_{z \in D} |g(z)|$. The subalgebra $P(D)$ of all polynomials on $D$ is dense in $A(D)$ (see, for example, [4, §3.2.13]).

Received by the editors March 19, 2001 and, in revised form, July 6, 2001.

2000 Mathematics Subject Classification. Primary 47A56.

©2002 American Mathematical Society
By von Neumann’s theorem (see [3] Proposition I.8.3), \( \|p(A)\| \leq \|p\| \) for any polynomial \( p \) and any \( A \in B_1 \). Therefore functions from \( A(D) \) act on \( B_1 \) and
\[
\|g(A)\| \leq \|g\|, \quad \text{for any } g \in A(D) \text{ and } A \in B_1.
\]
We call a function \( g \in A(D) \) Fully Operator Lipschitzian if there is \( C > 0 \) such that
\[
\|g(A) - g(B)\| \leq C\|A - B\|, \quad \text{for } A, B \in B_1.
\]
The class of Fully Operator Lipschitz functions is contained in the wider class of Operator Lipschitz functions on \( D \) which consists of all continuous functions on \( D \) satisfying inequality (3) for all normal operators in \( B_1 \) (see [3]). The function \( g(z) = \bar{z} \), for example, is Operator Lipschitzian on \( D \), since \( \|A^* - B^*\| = \|A - B\| \), for all normal \( A, B \in B_1 \). However, it is not Fully Operator Lipschitzian. Both classes of functions are important for applications in mathematical physics and have attracted much attention (see, for example, Bibliography in [1]).

**Proposition 1.** A function \( g \in A(D) \) is Fully Operator Lipschitzian if and only if there is \( C > 0 \) such that
\[
\|g(A), X\| \leq C\|A, X\|, \quad \text{for } A \in B_1 \text{ and } X \in B(H).
\]

**Proof.** If \( A, B \in B_1 \), the operator \( L = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \) on \( H \oplus H \) belongs to the unit ball of \( B(H \oplus H) \). Let \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Clearly, (1) holds for operators on \( H \oplus H \). Hence
\[
\|g(L), X\| \leq C\|L, X\| \quad \text{which implies (3)}.
\]

Conversely, let \( \|A\| < 1 \). For \( X \in B(H) \), the operators \( A(t) = e^{tX}Ae^{-tX} \) belong to \( B_1 \) for sufficiently small \( t \). If (3) holds then, taking into account that \( g(e^{tX}Ae^{-tX}) = e^{tX}g(A)e^{-tX} \), we obtain
\[
\|g(A) - e^{tX}g(A)e^{-tX}\| = \|g(A) - g(e^{tX}Ae^{-tX})\| \leq C\|A - e^{tX}Ae^{-tX}\|.
\]
Dividing through by \( t \) and taking the limit as \( t \to 0 \), we have that (4) holds.

Let \( \|A\| = 1 \), \( X \in B(H) \). For \( r < 1 \), \( \|g(rA), X\| \leq C\|rA, X\| \). Taking the limit as \( r \to 1 \), we obtain that (4) holds.

It follows from Proposition 1 that our aim is to construct a Fully Operator Lipschitz function with discontinuous derivative.

3. **FULLY OPERATOR LIPSCHITZ FUNCTIONS WITH DISCONTINUOUS DERIVATIVE**

Consider the following function on \( D \):
\[
h(1) = 0 \quad \text{and} \quad h(z) = (z - 1)^2 \exp((z - 1)^{-1}), \quad \text{for } z \in D, z \neq 1.
\]
Since \( \frac{x^{-1}}{(x-1)^2 + y^2} < 0 \), if \( z = x + iy \in D \setminus 1 \), we have that
\[
\sup_{z \in D \setminus 1} |\exp((z - 1)^{-1})| = \sup_{z \in D \setminus 1} \left| \frac{(x - 1) - iy}{(x-1)^2 + y^2} \right| = \sup_{z \in D \setminus 1} \left| \frac{x - 1}{(x-1)^2 + y^2} \right| < 1.
\]
The function \( h \) is analytic on \( D^0 \) and continuous on \( D \), since, by (5),
\[
|h(z)| = |h(x + iy)| = |z - 1|^2 \exp((z - 1)^{-1}) \leq |z - 1|^2 \to 0,
\]
as \( z \to 1 \). Thus \( h \in \mathcal{A}(D) \). We obtain similarly that
\[
\left| \frac{h(z) - h(1)}{z - 1} \right| = |z - 1| |\exp((z - 1)^{-1})| \leq |z - 1| \to 0,
\]
as \( z \to 1 \), so \( h'(1) = 0 \). We also obtain that
\[
h'(z) = 2(z - 1) \exp((z - 1)^{-1}) - \exp((z - 1)^{-1}), \quad \text{for } z \in D, \ z \neq 1.
\]
We have, as above, that \( (z - 1) \exp((z - 1)^{-1}) \to 0 \), as \( z \to 1 \), while \( \exp((z - 1)^{-1}) \) does not have limit as \( z \to 1 \). Therefore \( h' \) is discontinuous at \( z = 1 \).

**Theorem 2.** The function \( h \) is Fully Operator Lipschitzian.

**Proof.** By Proposition 1 we only need to prove that (4) holds for \( h \). For \( 0 < \lambda < 1 \), set \( h_\lambda(z) = h(\lambda z) \). Every \( h_\lambda \) is analytic in a neighbourhood of \( D \), so it belongs to \( \mathcal{A}(D) \), and \( \|h - h_\lambda\| \to 0 \), as \( \lambda \to 1 \). Hence it follows from (2) that
\[
\|[h(A), X] - [h_\lambda(A), X]\| = \|[h(A) - h_\lambda(A)), X]\|
\leq 2\|h(A) - h_\lambda(A)\| \|X\| \leq 2\|h - h_\lambda\| \|A\| \|X\| \to 0.
\]
For any \( A \in \mathbf{B}_1 \) and \( X \in B(H) \),
\[
\|[h_\lambda(A), X]\| = \|[(\lambda A - 1) \exp((\lambda A - 1)^{-1})(\lambda A - 1), X]\|
\leq 2\|\lambda A - 1\| \|A, X\| + \|[(\lambda A - 1)[\exp((\lambda A - 1)^{-1}), X](\lambda A - 1)]\|
\]
We have that \( \|\lambda A - 1\| < 2 \) and that the function \( \exp((\lambda z - 1)^{-1}) \) belongs to \( \mathcal{A}(D) \). We obtain from (2) and (5) that
\[
\|\exp((\lambda A - 1)^{-1})\| \leq \|\exp((\lambda z - 1)^{-1})\| \leq \sup_{z \in D, \lambda} \|\exp((z - 1)^{-1})\| < 1.
\]
Therefore
\[
\|[h_\lambda(A), X]\| \leq 4\lambda \|[A, X]\| + \|[(\lambda A - 1)[\exp((\lambda A - 1)^{-1}), X](\lambda A - 1)]\|.
\]
It follows from Lemma 2 of (5) that, for any \( B \in B(H) \),
\[
[\exp(B), X] = \int_0^1 \exp(tB)[B, X] \exp((1 - t)B) \, dt.
\]
If \( B \) is invertible, then \( B[B^{-1}, X]B = [X, B] \). Hence
\[
\|[\lambda A - 1][\exp((\lambda A - 1)^{-1}), X](\lambda A - 1)]\|
\leq \|[X, \lambda A - 1]\| \int_0^1 \|\exp(t(A - 1)^{-1})\| \|\exp((1 - t)(A - 1)^{-1})\| \, dt.
\]
As in (7), we have that
\[ \| \exp(t(\lambda A - \mathbf{1})^{-1}) \| < 1 \quad \text{and} \quad \| \exp((1 - t)(\lambda A - \mathbf{1})^{-1}) \| < 1. \]
Therefore
\[ \| (\lambda A - \mathbf{1})[\exp((\lambda A - \mathbf{1})^{-1})X(AA - \mathbf{1})] \| \leq \lambda \| [A, X] \|. \]
Hence we obtain from (8) that \( \| h_\lambda(A), X \| \leq 5\lambda \| [A, X] \| \). Combining this with (6), we conclude that \( \| [h(A), X] \| \leq 5\| [A, X] \| \).

References


School of Communications Technology and Mathematical Sciences, University of North London, Holloway, London N7 8DB, Great Britain
E-mail address: e.kissin@unl.ac.uk

School of Communications Technology and Mathematical Sciences, University of North London, Holloway, London N7 8DB, Great Britain – and – Department of Mathematics, Vologda State Technical University, Vologda, Russia
E-mail address: shulman_y@yahoo.com