ON ANTI-AUTOMORPHISMS OF THE FIRST KIND IN DIVISION RINGS

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(Communicated by Martin Lorenz)

Abstract. We provide a short proof of the well-known fact that a division ring of finite degree over its center that admits an anti-automorphism of the first kind, i.e., an anti-automorphism that fixes the center elementwise, also admits an involutive anti-automorphism.

Albert has shown that a division ring of finite degree over its center that admits an anti-automorphism of the first kind also admits an involutive anti-automorphism; cf. Theorem X.19 of [1]. A proof of this theorem can now be found in a number of textbooks, e.g. [2], [3], [4], [5], [6], or [7]. The purpose of this note is to give another short proof.

Definition 1. A finite dimensional central simple algebra $A$ is called a crossed product if $A$ has a maximal subfield $K$ which is Galois over the center of $A$.

Theorem 2. Let $A$ be a crossed product and let $F$ be the center of $A$. Suppose that $A$ allows an anti-automorphism that fixes $F$ elementwise. Then $A$ allows an involutive anti-automorphism.

Proof. Let $\sigma : A \to A$ be an anti-automorphism that fixes $F$ elementwise. Since $A$ is a crossed product, there exists a maximal subfield $K$ of $A$ such that $K/F$ is Galois. Denote the corresponding Galois group by $G$. By Theorem 3.1.27 of [5] for every $g \in G$ there exists an element $j_g \in A$ with $g(k) = j_g k j_g^{-1}$ for all $k \in K$, such that $A = \bigoplus_{g \in G} K j_g$. Using the Skolem-Noether theorem, we can assume that $\sigma$ fixes $K$ elementwise and so for every $g \in G$ we have $j_g k j_g^{-1} = g(k) = \sigma(g(k)) = \sigma(j_g k j_g^{-1}) = \sigma(j_g)^{-1} k \sigma(j_g)$. Therefore $\sigma(j_g) j_g \in C_A(K) = K$. Hence $\sigma(j_g) = a_g j_g^{-1}$ for some $a_g \in K^*$. This implies $\sigma^2(j_g) = \sigma(j_g^{-1}) a_g = j_g$, and thus $\sigma^2$ is the identity.

Corollary 3. Let $D$ be a division ring of finite degree over its center $F$. Suppose that $D$ allows an anti-automorphism that fixes $F$ elementwise. Then $D$ allows an involutive anti-automorphism.

Proof. There exists an $n \in \mathbb{N}$ such that $M_n(D)$ is a crossed product by Theorem V.1 of [1]. We can extend $\sigma$ to an anti-automorphism $\overline{\sigma}$ of $M_n(D) = D \otimes F M_n(F)$ by letting $\overline{\sigma}$ act as the transposition map on $1 \otimes M_n(F)$. Theorem [4] implies that
there exists an involutive anti-automorphism on $M_n(D)$. By Theorem 3.1.70 of [5], we conclude that $D$ allows an involutive anti-automorphism.

**Acknowledgment**

The authors would like to express their gratitude to Zinovy Reichstein for pointing out a generalization of their original proof.

**References**


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