DISTINCT GAPS BETWEEN FRACTIONAL PARTS OF SEQUENCES

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Abstract. Let $\alpha$ be a real number, $N$ a positive integer and $\mathcal{N}$ a subset of \{0, 1, 2, \ldots, N\}. We give an upper bound for the number of distinct lengths of gaps between the fractional parts \{na\}, $n \in \mathcal{N}$.

1. Introduction

Questions on the distribution of fractional parts of sequences have a long history, and among the most intensively studied are those related to polynomial sequences. After the classical work of Weyl [11] on uniform distribution mod 1, other aspects of the distribution of fractional parts of polynomials, especially questions concerned with small fractional parts, have been investigated (see Schmidt [8] and Baker [1]). Recently, the distribution of gaps between fractional parts of sequences has attracted attention. Following the work of Rudnick and Sarnak [5] on the pair correlation of fractional parts of polynomials, other related questions have been studied in [2], [6] and [7]. We mention that the distribution of the local spacings between the fractional parts \{n^d\alpha\}, $n \in \mathbb{N}$, in the case $d = 1$ is completely different than in the case $d > 1$. If $d > 1$ one expects that for almost all $\alpha$ the distribution is Poissonian, and one knows for instance that the pair correlation is Poissonian indeed (see [5]). If $d = 1$ one knows for a fact that the distribution is not Poissonian, and this is a consequence of the following Three Gap Theorem of Steinhaus (see [4], [9] and [10]):

Let $\alpha$ be a real number and $N$ a nonnegative integer. Then the fractional parts \{na\}, $0 \leq n \leq N$, partition the unit interval into $N + 1$ intervals which have at most 3 different lengths.

The correlation of fractional parts \{na\}, $n \in \mathbb{N}$, have been recently investigated by Marklof [3]. In this paper we take a real number $\alpha$, a positive integer $N$, a subset $\mathcal{N}$ of \{0, 1, 2, \ldots, N\} and look at the set of fractional parts

$$\mathcal{M} = \mathcal{M}(\alpha, \mathcal{N}) = \{na : n \in \mathcal{N}\},$$

with the intention of proving a result which is independent of $\mathcal{N}$. Clearly, as far as uniform distribution or small fractional parts are concerned, no such result is possible (for instance $\mathcal{N}$ might coincide with the set of those $1 \leq n \leq N$ for which \{na\} $\in \left[\frac{1}{4}, \frac{3}{4}\right]$). The same goes for the spacing distribution: if $\alpha$ is irrational, then the set \{na\}, $n \in \mathbb{N}$, is dense in \([0, 1]\), and one can choose for large $N$ a sparse
set \( \mathcal{N} \) for which the distribution of \( \mathcal{M} \) in \([0,1]\) approaches any given distribution. What we will do is to look at the gaps between the elements of \( \mathcal{M} \) and see whether any kind of Steinhaus phenomenon still holds in this generality. Thus we arrange the elements of \( \mathcal{M} \) in ascending order and consider the number \( l(\alpha, \mathcal{N}) \) of distinct lengths of gaps between consecutive elements of \( \mathcal{M}(\alpha, \mathcal{N}) \). Hence \( l(\alpha, \mathcal{N}) \leq 3 \) when \( \mathcal{N} = \{0, 1, 2, \ldots, N\} \), by the Three Gap Theorem. For a general subset \( \mathcal{N} \) of \( \{0, 1, 2, \ldots, N\} \), \( l(\alpha, \mathcal{N}) \) can be much larger. For example, if \( N \) is a positive integer, \( 0 < \alpha < \frac{1}{2} \), and \( \mathcal{N} \) consists of the squares \( \{0, 1, 4, 9, \ldots, [\sqrt{N}]^2\} \), then the numbers \( na \), \( n \in \mathcal{N} \), coincide with their fractional parts, and all the gaps between consecutive elements of \( \mathcal{M} \) have distinct lengths. Thus \( l(\alpha, \mathcal{N}) \) can be as large as \( \sqrt{N} \). The object of this paper is to prove the following theorem, which shows that \( l(\alpha, \mathcal{N}) \) cannot be much larger than \( \sqrt{N} \).

**Theorem 1.** For any real number \( \alpha \), any positive integer \( N \) and any subset \( \mathcal{N} \) of \( \{0, 1, 2, \ldots, N\} \) one has

\[
l(\alpha, \mathcal{N}) < (2 + 2\sqrt{2})\sqrt{N}.
\]

**2. Proof of Theorem 1**

Fix a positive integer \( N \), then choose a real number \( \alpha \) and a subset \( \mathcal{N} \) of \( \{0, 1, 2, \ldots, N\} \) such that \( l(\alpha, \mathcal{N}) \) is largest. Note first that for fixed \( \mathcal{N} \), the function \( \alpha \mapsto l(\alpha, \mathcal{N}) \) is periodic mod 1; thus we may assume in what follows that \( 0 \leq \alpha < 1 \). In case \( \alpha = 0 \) all the fractional parts \( \{na\} \) are zero, so the maximum value of \( l(\alpha, \mathcal{N}) \) is attained for some \( \alpha \in (0, 1) \). Next, notice that for \( \mathcal{N} \) fixed, the lengths of the gaps between the elements of \( \mathcal{M}(\alpha, \mathcal{N}) \) are continuous functions of \( \alpha \). Thus there is an \( \varepsilon = \varepsilon(\alpha, \mathcal{N}) > 0 \) such that

\[
l(\beta, \mathcal{N}) \geq l(\alpha, \mathcal{N})
\]

for any \( \beta \in (\alpha - \varepsilon, \alpha + \varepsilon) \). If \( \alpha \) and \( \mathcal{N} \) are chosen as above such that \( l(\alpha, \mathcal{N}) \) is largest, then one has equality in (1). Replacing if necessary \( \alpha \) by an irrational number \( \beta \in (\alpha - \varepsilon, \alpha + \varepsilon) \) we may assume in the following that \( 0 < \alpha < 1 \) is irrational. This last assumption is not essential in our proof, but it makes the presentation cleaner. For instance, in this case the fractional parts \( \{na\} \), \( n \in \mathcal{N} \), will be distinct, and we will discuss in detail the order of these fractional parts. To proceed, recall Dirichlet’s theorem which asserts that for any positive integer \( M \) there are coprime integers \( a, q \) with \( 1 \leq q \leq M \) such that

\[
|\alpha - \frac{a}{q}| < \frac{1}{qM}.
\]

We use (2) with \( M = 2N \), so let \( a \in \mathbb{Z} \) and \( 1 \leq q \leq 2N \) such that \((a, q) = 1\) and

\[
|\alpha - \frac{a}{q}| < \frac{1}{2qN}.
\]

Since \( 0 < \alpha < 1 \) we see that \( 0 \leq a \leq q \). From (3) it follows that for any \( n \in \mathcal{N} \) one has

\[
|na - \frac{na}{q}| < \frac{1}{2q}.
\]

Let us consider the open intervals \( J_k = \left( \frac{k}{q} - \frac{1}{2q}, \frac{k}{q} + \frac{1}{2q} \right) \), \( k = 0, 1, \ldots, q - 1 \). For any \( n \in \mathcal{N} \) let \( k(n) \in \{0, 1, \ldots, q - 1\} \) be such that \( an \equiv k(n) \pmod{q} \). Then
the fractional part \( \{ \frac{m}{n} \} \) coincides with the center \( \frac{k(n)}{q} \) of the interval \( J_{k(n)} \), and from (1) it follows that \( \{ na \} \) belongs to \( J_{k(n)} \). Therefore for any \( n, n' \in \mathcal{N} \) for which \( k(n) \neq k(n') \) the order of the elements \( \{ na \}, \{ n' a \} \in \mathcal{M} \) will simply be given by the order of the numbers \( k(n) \) and \( k(n') \). On the other hand, if \( n, n' \in \mathcal{N} \) are such that \( k(n) = k(n') \), then the order of \( \{ na \}, \{ n' a \} \) is determined by the sign of \( \alpha - \frac{a}{q} \) and the order of the numbers \( n \) and \( n' \). To be precise, let \( \alpha - \frac{a}{q} = \eta \) and assume in what follows that \( \eta > 0 \). The case \( \eta < 0 \) is similar and will be left to the reader. Since \( na = n \eta + \frac{ma}{q} \), where as we know \( |n \eta| < \frac{1}{2q} \), the relative “coordinate” of \( \{ na \} \) with respect to the center \( \frac{k(n)}{q} \) of \( J_{k(n)} \) will equal \( n \eta \). With our assumption on \( \eta \), the order of \( \{ na \}, \{ n' a \} \) in case \( k(n) = k(n') \) will be the same as the order of \( n, n' \). Here the condition \( k(n) = k(n') \) is equivalent to the condition \( n \equiv n' \pmod{q} \). We now have a more clear picture of the distribution of the elements of \( \mathcal{M} \). Write \( \mathcal{N} = \bigcup_{r=0}^{q-1} \mathcal{N}_r \), where \( \mathcal{N}_r = \{ n \in \mathcal{N} : n \equiv r \pmod{q} \} \). Each \( \mathcal{N}_r \) corresponds uniquely to a \( J_k \), given by \( k = k(r) \equiv ar \pmod{q} \), respectively \( r = r(k) \equiv ak \pmod{q} \), where \( a \) denotes the inverse of \( a \) mod \( q \). For any \( r \), the map \( n \mapsto \{ na \} \) sends \( \mathcal{N}_r \) monotonically to a subset of \( J_{k(r)} \). We now distinguish two kinds of gaps \( \{(na), \{n'a\}\} \) between consecutive elements \( \{ na \}, \{ n' a \} \) of \( \mathcal{M} \), according as to whether \( k(n) = k(n') \) or \( k(n) \neq k(n') \), and count them separately. Denote by \( l_1 \), respectively \( l_2 \), the number of distinct lengths of gaps of the first kind, respectively of the second kind, between consecutive elements of \( \mathcal{M} \). Some gaps of the first kind might have the same lengths as certain gaps of the second kind. Anyway one has

\[
(5) \quad l(\alpha, \mathcal{N}) \leq l_1 + l_2.
\]

In order to get an upper bound for \( l_1 \), we allow \( r \) to run over the set \( \{0, 1, \ldots, q - 1\} \) and for each such value of \( r \) we look at the gaps formed by the image of \( \mathcal{N}_r \) in \( J_{k(r)} \). We already know that consecutive elements of \( \mathcal{N}_r \) correspond to consecutive elements of \( \mathcal{M} \). Moreover, if \( n < n' \) are consecutive elements of \( \mathcal{N}_r \), then the length of the gap between \( \{ na \} \) and \( \{ n' a \} \) equals the difference between their coordinates in \( J_{k(r)} \), which is \( (n' - n)\eta \). Thus the lengths of these gaps in \( \mathcal{M} \) are proportional to the lengths of the gaps \( (n' - n) \) in \( \mathcal{N}_r \), by a factor \( \eta \) which is independent of \( r \). It follows that \( l_1 \) equals the cardinality of the set

\[
A = \bigcup_{r=0}^{q-1} \{ n' - n : n, n' \text{ consecutive in } \mathcal{N}_r \}.
\]

Now the point is that since each element of \( A \) is a positive multiple of \( q \), the sum of its \( l_1 \) (distinct) elements will be at least

\[
q + 2q + \cdots + l_1 q = \frac{q l_1 (l_1 + 1)}{2}.
\]

On the other hand, if we add all the elements of \( A \) counted with multiplicities, the sum will equal

\[
\sum_{r=0}^{q-1} \sum_{n, n' \text{ consecutive in } \mathcal{N}_r} (n' - n) = \sum_{r=0}^{q-1} (\max \mathcal{N}_r - \min \mathcal{N}_r) < N q.
\]

It follows that \( \frac{l_1 (l_1 + 1)}{2} < N q \), which implies

\[
(6) \quad l_1 < \sqrt{2N}.
\]
We now turn to $l_2$. Some of the above sets $N_r$ might be empty, resulting in some intervals $J_k$ having no points from $M$. Let $0 \leq k_1 < k_2 < \cdots < k_s \leq q - 1$ be those values of $k$ for which $J_k \cap M$ is nonempty. Then for each pair $(k_j, k_{j+1})$ we have exactly one gap of the second kind. Its left and right endpoints are the largest and, respectively, the smallest element of $M \cap J_{k_{j+1}}$. Thus the length of this gap, which we denote by $\delta_j$, is given by
\[
\delta_j = \{\underline{n}_{j+1} \alpha\} - \{\overline{n}_j \alpha\},
\]
where for any $j$, $\underline{n}_j$ and $\overline{n}_j$ stand for the smallest, respectively the largest, element of $N_{r(k_j)}$. The distance between the centers of $J_{k_j}$ and $J_{k_{j+1}}$ equals $\frac{k_{j+1} - k_j}{q}$, and the coordinates of $\{\overline{n}_j \alpha\}$ and $\{\underline{n}_{j+1} \alpha\}$ with respect to these centers are $\overline{n}_j \eta$ and respectively $\underline{n}_{j+1} \eta$. Hence
\[
\delta_j = \frac{k_{j+1} - k_j}{q} + \underline{n}_{j+1} \eta - \overline{n}_j \eta.
\]
A trivial upper bound for $l_2$ is
\[
l_2 \leq s \leq q.
\]
For each positive integer $b$, let $n(b)$ be the number of distinct lengths $\delta_j$ of gaps of the second kind for which $k_{j+1} - k_j = b$. Thus $l_2$ can be written as
\[
l_2 = \sum_{b \geq 1} n(b).
\]
Here we used the fact that if $k_{j+1} - k_j = b \neq b' = k_{j'+1} - k_{j'}$, then $\delta_j \neq \delta_{j'}$. This is a consequence of the inequalities $k_{j+1} - k_j - \frac{1}{2} < q \delta_j < k_{j+1} - k_j + \frac{1}{2}$, which in turn follow from (7) and the inequality $0 \leq n \eta < \frac{1}{2q}$, valid for any $n \in N$. Note that
\[
\sum_{b \geq 1} n(b) b \leq \sum_j (k_{j+1} - k_j) \leq q.
\]
We claim that for any $b$ one has
\[
n(b) \leq \left\lfloor \frac{2N}{q} \right\rfloor + 1,
\]
where $\lfloor \cdot \rfloor$ denotes the greatest integer part function. In order to prove the claim, let $j_1, \ldots, j_{n(b)}$ be such that $\delta_{j_1}, \ldots, \delta_{j_{n(b)}}$ are distinct and
\[
k_{j_1} + k_{j_1} = \cdots = k_{j_{n(b)} + 1} - k_{j_{n(b)}} = b.
\]
By (7) we know that
\[
\delta_j = \frac{b}{q} + \eta(n_{j+1} - \overline{n}_j)
\]
for any $j \in \{j_1, \ldots, j_{n(b)}\}$. The numbers $\delta_{j_1}, \ldots, \delta_{j_{n(b)}}$ being distinct, it follows that as $j$ runs over the set $\{j_1, \ldots, j_{n(b)}\}$, the numbers $n_{j+1} - \overline{n}_j$ are distinct. Recall that $\overline{n}_j \in N_{r(k_j)}$ and $\underline{n}_{j+1} \in N_{r(k_{j+1})}$, so they satisfy the congruences
\[
\overline{n}_j \equiv r(k_j) \equiv \overline{n}_j k_j \pmod{q}
\]
and
\[
\underline{n}_{j+1} \equiv r(k_{j+1}) \equiv \overline{n}_j k_{j+1} \pmod{q}.
\]
Hence
\[ \mu_{j+1} - \pi_j \equiv \pi (k_{j+1} - k_j) \equiv \pi b \pmod{q} \]
for any \( j \in \{j_1, \ldots, j_n(b)\} \). Note also that for any \( j \) one has
\[-N \leq \mu_{j+1} - \pi_j \leq N .\]
There are at most \( 1 + \lfloor 2N/q \rfloor \) integers in the interval \([-N, N]\) which are congruent to \( \pi b \pmod{q} \), and this proves (11). Next, from (9) we know that the left-hand side of (10) is a sum of exactly \( l_2 \) terms, counting with multiplicities. By using (11) one sees that the left-hand side of (10) is at least as large as the sum
\[ \left( 1 + \left\lfloor \frac{2N}{q} \right\rfloor \right) \cdot 1 + \left( 1 + \left\lfloor \frac{2N}{q} \right\rfloor \right) \cdot 2 + \cdots + \left( 1 + \left\lfloor \frac{2N}{q} \right\rfloor \right) u + v (u + 1), \]
where \( u \) and \( v \) are given by
\[ u = \left\lfloor \frac{l_2}{1 + \lfloor 2N/q \rfloor} \right\rfloor \]
and
\[ v = l_2 - \left( 1 + \left\lfloor \frac{2N}{q} \right\rfloor \right) u .\]
We combine this with (10) to derive
\[ \left( 1 + \left\lfloor \frac{2N}{q} \right\rfloor \right) \frac{u (u + 1)}{2} \leq q , \]
which implies
\[ u < \left( \frac{2q}{1 + \lfloor 2N/q \rfloor} \right)^{\frac{1}{2}} . \]
Relations (12) and (13) give
\[ \frac{l_2}{1 + \lfloor 2N/q \rfloor} - 1 < \left( \frac{2q}{1 + \lfloor 2N/q \rfloor} \right)^{\frac{1}{2}} , \]
from which we get the following upper bound for \( l_2 \) :
\[ l_2 < 1 + \left\lfloor \frac{2N}{q} \right\rfloor + \left( 2q \left( 1 + \left\lfloor \frac{2N}{q} \right\rfloor \right) \right)^{\frac{1}{2}} < 1 + \frac{2N}{q} + (2q + 4N)^{\frac{1}{2}} . \]
Since \( q \leq 2N \), from (14) we obtain
\[ l_2 < 1 + \frac{2N}{q} + 2 \sqrt{2N} . \]
This inequality is sharp when \( q \) is at least of the size of \( \sqrt{N} \). If \( q \) is smaller, then we use (5). From (8) and (15) we find that
\[ l_2 < (2 + \sqrt{2}) \sqrt{N} . \]
regardless of the size of \( q \). On combining (5), (6) and (11) we get
\[ l(\alpha, N) < (2 + 2\sqrt{2}) \sqrt{N} , \]
which completes the proof of Theorem 1.
References


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