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APPLICATION OF A RIESZ-TYPE FORMULA TO WEIGHTED BERGMAN SPACES

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ABSTRACT. Let $\mathbb{D}$ denote the unit disk in the complex plane. We consider a class of superbiharmonic weight functions $w\colon \mathbb{D} \to \mathbb{R}^+$ whose growth are subject to the condition $0 \le w(z) \le C(1-|z|)$ for some constant C. We first establish a Reisz-type representation formula for w, and then use this formula to prove that the polynomials are dense in the weighted Bergman space with weight w.

1. INTRODUCTION

We denote by $\mathbb{D}$ the unit disk and by $\mathbb{T}$ the unit circle in the complex plane. A *weight function*, or simply a *weight*, in $\mathbb{D}$ is any continuous positive function $w\colon \mathbb{D} \to [0, +\infty[$.

The problem. A real-valued function w defined on the unit disk $\mathbb{D}$ is said to be *superbiharmonic* provided that w is locally integrable and the bi-laplacian $\Delta^2 w$ is a positive distribution on $\mathbb{D}$; here Δ stands for the Laplace operator defined by

$$\Delta = \Delta_z = \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big), \qquad z = x + iy.$$

We consider a class of superbiharmonic weight functions w satisfying condition

$$(\mathbf{A}) \qquad 0 \le w(z) \le C(1-|z|), \quad \text{for every} \quad z \in \mathbb{D},$$

for some constant C. We shall prove that if w is a superbiharmonic function satisfying condition (A), then w enjoys a Riesz-type representation formula in terms of the biharmonic Green function for the operator Δ^2, and the harmonic compensator. In the following we shall briefly introduce these notions.

The biharmonic Green function for the operator Δ^2 in the unit disk is the function $\Gamma(z,\zeta)$ which solves, for fixed ζ, the following boundary value problem:

$$\begin{cases} \Delta_z^2 \Gamma(z,\zeta) = \delta_\zeta(z) & z \in \mathbb{D}, \\ \Gamma(z,\zeta) = 0, & z \in \mathbb{T}, \\ \partial_{n(z)} \Gamma(z,\zeta) = 0, & z \in \mathbb{T}, \end{cases}$$

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where the symbol δ_ζ stands for the unit point mass at $\zeta \in \mathbb{D}$, and $\partial_{n(z)}$ denotes the inward normal derivative in the sense of distributions. It is well-known [4] that the biharmonic Green function has the form

$$\Gamma(z,\zeta) = |z-\zeta|^2 \log\left|\frac{z-\zeta}{1-\bar{\zeta}z}\right|^2 + \big(1-|z|^2\big)\big(1-|\zeta|^2\big), \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D}.$$

We define the *harmonic compensator* $H(\zeta, z)$ by

$$H(\zeta,z) = \big(1-|z|^2\big)\, P(z,\zeta) = \frac{(1-|z|^2)^2}{|1-z\bar{\zeta}|^2}, \qquad (\zeta,z) \in \mathbb{T}\times\mathbb{D},$$

where $P(z,\zeta)$ denotes the Poisson kernel for the unit disk. It turns out that superbiharmonic functions w satisfying condition (A) can be represented by a Riesz-type formula which takes into account the growth of w as z approaches the boundary. Indeed, we shall see that

$$\text{(1–1)} \quad w(z) = \int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2 w(\zeta)\, dA(\zeta) + \int_{\mathbb{T}} H(\zeta,z)\partial_{n(\zeta)} w(\zeta)\, d\sigma(\zeta), \qquad z \in \mathbb{D},$$

where $dA = \pi^{-1} dx\, dy$ is the normalized area measure in $\mathbb{D}$ and $d\sigma = (2\pi)^{-1}\, d\theta$ denotes the normalized arc-length measure on $\mathbb{T}$. Moreover, the normal derivative in the second term should be understood in a generalized sense: $\partial_{n(\zeta)} w(\zeta)\, d\sigma(\zeta)$ is the weak-star limit of a family of positive continuous functions on the unit circle, so that it can be identified with a positive measure on the unit circle $\mathbb{T}$.

Application to weighted Bergman spaces. We say that a function f, analytic in $\mathbb{D}$, belongs to the *weighted Bergman space* $L^p_a(\mathbb{D}, w)$, $0 < p < +\infty$, provided that the following integral is finite:

$$\|f\|^p_{L^p_a(\mathbb{D},w)} = \int_{\mathbb{D}} |f(z)|^p w(z)\, dA(z) < +\infty.$$

If $1 \le p < +\infty$, it follows that $L^p_a(\mathbb{D}, w)$ is a Banach space of analytic functions with norm $\|\cdot\|_{L^p_a(\mathbb{D},w)}$, and for $0 < p < 1$, it is a quasi-Banach space. For $p = 2$, the space $L^2_a(\mathbb{D}, w)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{L^2_a(\mathbb{D},w)} = \int_{\mathbb{D}} f(z)\overline{g(z)} w(z)\, dA(z), \qquad f, g \in L^2_a(\mathbb{D}, w).$$

In the case where the weight function w is identically 1, the corresponding space is called the *Bergman space* and is denoted by $L^p_a(\mathbb{D})$. Note that condition (A) entails

$$\int_{\mathbb{D}} w(z)\, dA(z) < +\infty,$$

which guarantees that the polynomials are contained in the weighted Bergman space $L^p_a(\mathbb{D}, w)$.

We devote the last section to an application of the representation formula (1–1) to the weighted Bergman space $L^p_a(\mathbb{D}, w)$ whose (non-radial) weight w is superbiharmonic and satisfies condition (A). It is shown that the polynomials are dense in such weighted Bergman spaces. We remark that if the weight function w is radial (w depends only on $|z|$), the result is well-known [9]. The question of weighted polynomial approximation for such weights was raised by H. Hedenmalm in [5], p. 114. We answer this question in the positive.

If we weaken condition (A) above, it is possible to find a representation formula for w with one more term. This is done in [1], where the author joint with Hedenmalm succeeded to establish a Riesz representation formula for w.

2. A Riesz-type representation formula

In this section we will find a representation formula for a superbiharmonic function w satisfying the condition $0 \le w(z) \le C(1-|z|)$ for some constant C. The following lemma asserts that $\Delta^2 w(z)$ is integrable against the area measure $(1-|z|^2)^2\, dA(z)$ in the unit disk. The (rather long) proof uses the assumption (A) on w, twice the application of Green's formula together with some sharp estimates of the laplacians and the normal derivatives involved. We omit the details and refer the reader to [1], Lemma 3.1.

Lemma 2.1. *Let w be a superbiharmonic function satisfying $0 \le w(z) \le C(1-|z|)$. Then*

$$\int_{\mathbb{D}} \left(1-|z|^2\right)^2 \Delta^2 w(z)\, dA(z) < +\infty.$$

Proof. The assertion is a consequence of Lemma 3.1 in [1]. □

In the following lemma we collect some facts on the biharmonic Green function for the unit disk. The proof of parts (c) and (d) can be found in [2], Lemma 2.3. Part (a) can be verified by a direct argument using the defining formula for $\Gamma(z,\zeta)$, and finally, (b) is an immediate consequence of (a).

Lemma 2.2. *Let $\Gamma(z,\zeta)$ denote the biharmonic Green function for the unit disk. Then*

(a) *$\Gamma(z,\zeta)$ has the power series representation*

$$\Gamma(z,\zeta) = \sum_{n=0}^{\infty} \frac{\left[\left(1-|z|^2\right)\left(1-|\zeta|^2\right)\right]^{n+2}}{(n+1)(n+2)\,|1-\overline{z}\zeta|^{2(n+1)}}, \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D},$$

(b) *$\Gamma(z,\zeta) > 0$, for every $(z,\zeta) \in \mathbb{D}\times\mathbb{D}$,*

(c) *for every $(z,\zeta) \in \mathbb{D}\times\mathbb{D}$ we have*

$$\frac{1}{2}\frac{\left(1-|z|^2\right)^2\left(1-|\zeta|^2\right)^2}{|1-\overline{\zeta}z|^2} \le \Gamma(z,\zeta) \le \frac{\left(1-|z|^2\right)^2\left(1-|\zeta|^2\right)^2}{|1-\overline{\zeta}z|^2},$$

(d) *for every $(z,\zeta) \in \mathbb{D}\times\mathbb{D}$ we have*

$$\left(1-|z|\right)^2\left(1-|\zeta|^2\right)^2 \le 2\Gamma(z,\zeta) \le \left(1+|z|\right)^2\left(1-|\zeta|^2\right)^2.$$

We now combine the preceding lemmas and obtain the following proposition.

Proposition 2.3. *Let w be a superbiharmonic weight function satisfying the condition $0 \le w(z) \le C(1-|z|)$. Let $\Gamma(z,\zeta)$ denote the biharmonic Green function for the unit disk. Then*

$$\int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2 w(\zeta) dA(\zeta) < +\infty, \qquad z \in \mathbb{D}.$$

Proof. According to Lemma 2.2(d), for every $(z,\zeta) \in \mathbb{D}\times\mathbb{D}$,

$$\left(1-|z|\right)^2\left(1-|\zeta|^2\right)^2 \le 2\,\Gamma(z,\zeta) \le \left(1+|z|\right)^2\left(1-|\zeta|^2\right)^2.$$

Hence

$$\Gamma(z,\zeta) \le 2\big(1-|\zeta|^2\big)^2, \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D},$$

from which follows

$$\int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2 w(\zeta)\,dA(\zeta) \le 2\int_{\mathbb{D}} \big(1-|\zeta|^2\big)^2 \Delta^2 w(\zeta)\,dA(\zeta) < +\infty,$$

in accordance with Lemma 2.1. □

Proposition 2.4. *Let w be a superbiharmonic weight function satisfying the condition $0 \le w(z) \le C(1-|z|)$. Define the function v by the formula:*

$$v(z) = \int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2 w(\zeta)\,dA(\zeta), \qquad z \in \mathbb{D}.$$

Then for every integer k we have

$$\lim_{r\to 1^-} \int_{\mathbb{T}} \frac{\overline{z}^k v(rz)}{1-r^2}\,d\sigma(z) = 0.$$

In fact,

$$\sup_k \left| \int_{\mathbb{T}} \frac{\overline{z}^k v(rz)}{1-r^2}\,d\sigma(z)\right| \le \int_{\mathbb{T}} \frac{v(rz)}{1-r^2}\,d\sigma(z) \to 0, \quad \textit{as} \quad r \to 1^-.$$

Proof. For an integer k and $0<r<1$ we define

$$C(k,r) = \int_{\mathbb{T}} \frac{\overline{z}^k v(rz)}{1-r^2}\,d\sigma(z). \tag{2–1}$$

We first make the following simple, but important, observation. Since the function $v(rz)/(1-r^2)$ is nonnegative, it follows that for every integer k we have

$$0 \le |C(k,r)| \le \int_{\mathbb{T}} \left|\frac{\overline{z}^k v(rz)}{1-r^2}\right| d\sigma(z) = C(0,r).$$

Hence it suffices to verify the validity of the statement of the proposition for $k=0$; that is to show that $C(0,r)\to 0$ as $r \to 1^-$. To this end, we note that

$$0 \le C(0,r) = \int_{\mathbb{T}}\int_{\mathbb{D}} \frac{\Gamma(rz,\zeta)}{1-r^2}\Delta^2 w(\zeta)\,dA(\zeta)\,d\sigma(z).$$

It now follows from Fubini's theorem and Lemma 2.2(c) that

$$\begin{aligned} 0 \le C(0,r) &\le \int_{\mathbb{D}}\int_{\mathbb{T}} \frac{\big(1-r^2|z|^2\big)^2\big(1-|\zeta|^2\big)^2}{\big(1-r^2\big)|1-rz\overline{\zeta}|^2}\Delta^2 w(\zeta)\,d\sigma(z)\,dA(\zeta) \\ &= \left(\int_{\mathbb{D}} \big(1-|\zeta|^2\big)^2\Delta^2 w(\zeta)\,dA(\zeta)\right)\int_{\mathbb{T}} \frac{1-r^2}{|1-rz\overline{\zeta}|^2}\,d\,\sigma(z). \end{aligned} \tag{2–2}$$

But

$$\int_{\mathbb{T}} \frac{1}{|1-rz\overline{\zeta}|^2}\,d\,\sigma(z) = \frac{1}{1-|r\zeta|^2}\int_{\mathbb{T}} \frac{1-|r\zeta|^2}{|z-r\zeta|^2}\,d\sigma(z) = \frac{1}{1-|r\zeta|^2}.$$

This together with (2–2) yields

$$0 \le C(0,r) \le \int_{\mathbb{D}} \frac{1-r^2}{1-r^2|\zeta|^2}\big(1-|\zeta|^2\big)^2\Delta^2 w(\zeta)\,dA(\zeta). \tag{2–3}$$

To deal with the area integral appearing in (2–3) we write

$$d\lambda(\zeta) = \big(1 - |\zeta|^2\big)^2 \Delta^2 w(\zeta)\, dA(\zeta), \qquad \zeta \in \mathbb{D}.$$

We note that $d\lambda(\zeta)$ is a positive measure on the unit disk, and according to Lemma 2.1 the integral of the constant function 1 against the measure $d\lambda$ is finite. Hence (2–3) can be written as

$$0 \le C(0,r) \le \int_{\mathbb{D}} \frac{1-r^2}{1-r^2|\zeta|^2}\, d\lambda(\zeta). \tag{2–4}$$

Since the integrand in (2–4) is nonnegative and bounded from above by 1, we can apply the dominated convergence theorem to the right-hand side of (2–4) to obtain

$$0 \le \lim_{r\to 1^-} C(0,r) \le \lim_{r\to 1^-} \int_{\mathbb{D}} \frac{1-r^2}{1-r^2|\zeta|^2}\, d\lambda(\zeta) = 0,$$

from which the proposition follows. □

Corollary 2.5. *With the same conditions as in the preceding proposition*

$$\sup_k \left| \int_{\mathbb{T}} \overline{z}^k v(rz)\, d\sigma(z) \right| = o(1-r) \quad \textit{as} \quad r \to 1^-.$$

We have prepared the ground for the main result of this section.

Theorem 2.6. *Let w be a superbiharmonic weight function satisfying the condition $0 \le w(z) \le C(1-|z|)$. Then*

$$w(z) = \int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2 w(\zeta)\, dA(\zeta) + \int_{\mathbb{T}} H(\zeta,z)\partial_{n(\zeta)} w(\zeta)\, d\sigma(\zeta), \qquad z \in \mathbb{D},$$

where $H(\zeta,z)$ is the harmonic compensator and $\partial_n w$ is the generalized normal derivative of w; in the sense that $\partial_{n(\zeta)} w(\zeta) d\sigma(\zeta)$ is a positive measure on the unit circle.

Proof. We define the real-valued function v by

$$v(z) = \int_{\mathbb{D}} \Gamma(z,\zeta)\Delta^2 w(\zeta)\, dA(\zeta), \qquad z \in \mathbb{D}.$$

According to Proposition 2.3 the function v is well-defined. Moreover, v is nonnegative, because $\Gamma(z,\zeta)$ is positive in $\mathbb{D}\times\mathbb{D}$, and $\Delta^2 w \ge 0$ by our assumption.

We shall first verify that the weight function w can be written as

$$w(z) = v(z) + \big(1-|z|^2\big)h(z), \qquad z \in \mathbb{D},$$

where h is a harmonic function in the unit disk. It is clear that $w-v$ is a biharmonic function, hence according to the Almansi representation formula for biharmornic functions (see [6], Lemma 3.1) there exist two real-valued harmonic functions h and u such that

$$w(z) - v(z) = u(z) + \big(1-|z|^2\big)h(z), \qquad z \in \mathbb{D}.$$

We fix a real number r, $0 < r < 1$, and an integer n. Then we have

$$w(rz) - v(rz) = u(rz) + \big(1-r^2\big)\, h(rz), \qquad z \in \mathbb{T},$$

from which it follows that

$$\int_{\mathbb{T}} \overline{z}^n \big(w(rz) - v(rz)\big)\, d\sigma(z)$$
$$= \int_{\mathbb{T}} \overline{z}^n u(rz)\, d\sigma(z) + (1-r^2) \int_{\mathbb{T}} \overline{z}^n h(rz)\, d\sigma(z)$$
$$= \widehat{u}_r(n) + (1-r^2)\widehat{h}_r(n).$$

Here $\hat{u}(n)$ stands for the n-th Fourier coefficient of u; moreover, $u_r(z) = u(rz)$. Since u and h are harmonic, it follows that

$$\int_{\mathbb{T}} \overline{z}^n w(rz)\, d\sigma(z) - \int_{\mathbb{T}} \overline{z}^n v(rz)\, d\sigma(z) = r^{|n|}\, \widehat{u}(n) + (1-r^2) r^{|n|}\, \widehat{h}(n). \tag{2–5}$$

Letting now $r \to 1^-$, we see that the right-hand side of (2–5) tends to $\widehat{u}(n)$. As for the left-hand side of (2-5) we see that the first integral tends to zero, as $r \to 1^-$. Indeed, we know from our assumption on w that

$$0 \le w(rz) \le C(1 - r|z|) = C(1-r), \quad \text{for} \quad z \in \mathbb{T},$$

and hence

$$0 \le \left| \int_{\mathbb{T}} \overline{z}^n w(rz)\, d\sigma(z) \right| \le \int_{\mathbb{T}} w(rz)\, d\sigma(z) \le C(1-r) \to 0, \quad \text{as} \quad r \to 1^-.$$

The fact that the second integral on the left-hand side of (2–5) approaches zero as $r \to 1^-$ follows from Corollary 2.5. Hence for any integer n, we have $\widehat{u}(n) = 0$, from which it follows that u is identically zero. This implies that

$$w(z) - v(z) = (1 - |z|^2) h(z), \qquad z \in \mathbb{D}, \tag{2–6}$$

where h is some harmonic function in the unit disk $\mathbb{D}$.

We fix once again $0 < r < 1$ and use (2–6) to write

$$\frac{w(rz)}{1-r^2} - \frac{v(rz)}{1-r^2} = h(rz), \qquad z \in \mathbb{T}. \tag{2–7}$$

It then follows from Proposition 2.4 that for every integer k we have

$$\lim_{r \to 1^-} \int_{\mathbb{T}} \overline{z}^k \frac{w(rz)}{1-r^2}\, d\sigma(z) = \lim_{r \to 1^-} \int_{\mathbb{T}} \overline{z}^k h(rz)\, d\sigma(z) = \widehat{h}(k). \tag{2–8}$$

Note that the limit on the right-hand side of (2–8) exists, since h is a harmonic function. In fact the equality (2–8) holds for every trigonometric polynomial p on the unit circle, that is,

$$\lim_{r \to 1^-} \int_{\mathbb{T}} p(z) \frac{w(rz)}{1-r^2}\, d\sigma(z) = \lim_{r \to 1^-} \int_{\mathbb{T}} p(z) h(rz)\, d\sigma(z). \tag{2–9}$$

We shall see that

$$\sup_{0<r<1} \|h_r\|_{L^1(\mathbb{T})} < +\infty. \tag{2–10}$$

To see this, we first use (2–7) to write

$$\int_{\mathbb{T}} |h(rz)|\, d\sigma(z) \le \int_{\mathbb{T}} \frac{w(rz)}{1-r^2}\, d\sigma(z) + \int_{\mathbb{T}} \frac{v(rz)}{1-r^2}\, d\sigma(z). \tag{2–11}$$

The first integral in (2–11) is bounded by the constant 1, as was observed earlier. The uniform boundedness of the second integral in (2–11) follows from Proposition 2.4 for $k = 0$. Hence (2–10) holds. This L^1-boundedness of the functions h_r implies that there exists a unique real-valued Borel measure μ on the unit circle such that h is the Poisson integral of this measure (see [10], Theorem 11.30); or

$$h(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{\zeta}z|^2}\, d\mu(\zeta), \qquad z \in \mathbb{D}. \tag{2–12}$$

Moreover, the measure μ is the weak-star limit of the measures $d\mu_r = h_r d\sigma$ (see [8], p. 33). This means that for every trigonometric polynomial p on the unit circle we have

$$\begin{aligned}\int_{\mathbb{T}} p(z)\, d\mu(z) &= \lim_{r\to 1^-} \int_{\mathbb{T}} p(z) h_r(z)\, d\sigma(z)\\ &= \lim_{r\to 1^-} \int_{\mathbb{T}} p(z) \frac{w(rz)}{1-r^2}\, d\sigma(z) - \lim_{r\to 1^-} \int_{\mathbb{T}} p(z) \frac{v(rz)}{1-r^2}\, d\sigma(z).\end{aligned}$$

It is implicit in (2–9) that the last limit in the above displayed formula is zero from which it follows that

$$\int_{\mathbb{T}} p(z)\, d\mu(z) = \lim_{r\to 1^-} \int_{\mathbb{T}} p(z) \frac{w(rz)}{1-r^2}\, d\sigma(z). \tag{2–13}$$

Now, (2–13) implies that for every nonnegative trigonometric polynomial p on the unit circle

$$\int_{\mathbb{T}} p(z)\, d\mu(z) \ge 0,$$

proving that μ is a positive measure on $\mathbb{T}$. This together with (2–12) implies that h is a positive harmonic function in the unit disk. Therefore we obtain from (2–6)

$$0 \le w(z) - v(z) = \big(1-|z|^2\big) h(z) \le w(z) \le C(1-|z|), \qquad z \in \mathbb{D},$$

so that $0 \le h(z) \le C$, that is, h is a nonnegative bounded harmonic function in the unit disk. It follows from (2–6) and (2–12) that for every $z \in \mathbb{D}$,

$$\begin{aligned} w(z) - v(z) = \big(1-|z|^2\big) h(z) &= \int_{\mathbb{T}} \frac{\big(1-|z|^2\big)^2}{|1-\bar{\zeta}z|^2}\, d\mu(\zeta)\\ &= \int_{\mathbb{T}} H(\zeta, z)\, d\mu(\zeta), \end{aligned} \tag{2–14}$$

where $H(\zeta, z)$ is the harmonic compensator defined by

$$H(\zeta, z) = \big(1-|z|^2\big) P(z,\zeta) = \frac{\big(1-|z|^2\big)^2}{|1-\bar{\zeta}z|^2}, \qquad (\zeta, z) \in \mathbb{T}\times\mathbb{D},$$

in which $P(z,\zeta)$ denotes the Poisson kernel for the unit disk.

The measure μ in the representation formula (2–14) was obtained as a weak-star limit of the functions $w_r(z)/(1-r^2)$, for $z \in \mathbb{T}$. This weak-star limit can be regarded as the generalized normal derivative of the function w on the boundary. The proof of the theorem is now complete. □

3. Application to weighted Bergman spaces

We start with a lemma which says that the harmonic compensator has some kind of monotonicity property with respect to its second argument. The proof uses the method of [2], p. 285 (see also [7], p. 64, where the result is attributed to the current author).

Lemma 3.1. *Let $z = e^{i\theta} \in \mathbb{T}$ be fixed, and consider*

$$H(e^{i\theta}, \zeta) = \frac{\big(1-|\zeta|^2\big)^2}{\big|1-\overline{\zeta}e^{i\theta}\big|^2}, \qquad \zeta \in \mathbb{D}.$$

Then for $0 < r < 1$ and $|\zeta| < r$, we have

$$\frac{d}{dr}\left(rH\Big(e^{i\theta}, \frac{\zeta}{r}\Big)\right) > 0.$$

Proof. For $0 < r < 1$ and $|\zeta| < r$ we have

$$H\Big(e^{i\theta}, \frac{\zeta}{r}\Big) = \frac{\big(r^2-|\zeta|^2\big)^2}{r^2\Big|r-\overline{\zeta}e^{i\theta}\Big|^2}.$$

A computation reveals that

$$\frac{d}{dr}H\Big(e^{i\theta}, \frac{\zeta}{r}\Big) = 4\frac{r^2-|\zeta|^2}{r\Big|r-\overline{\zeta}e^{i\theta}\Big|^2} - \frac{\big(r^2-|\zeta|^2\big)^2}{\Big|r-\overline{\zeta}e^{i\theta}\Big|^2}\left(\frac{2}{r^3} + \frac{1}{r^2\big(r-\overline{\zeta}e^{i\theta}\big)} + \frac{1}{r^2\big(r-\zeta e^{-i\theta}\big)}\right),$$

and therefore

$$r\frac{d}{dr}H\Big(e^{i\theta}, \frac{\zeta}{r}\Big) = \frac{4\big(r^2-|\zeta|^2\big)}{\Big|r-\overline{\zeta}e^{i\theta}\Big|^2} - \frac{\big(r^2-|\zeta|^2\big)^2}{\Big|r-\overline{\zeta}e^{i\theta}\Big|^2}\left(\frac{2}{r^2} + \frac{1}{r\big(r-\overline{\zeta}e^{i\theta}\big)} + \frac{1}{r\big(r-\zeta e^{-i\theta}\big)}\right).$$

It then follows that

$$\begin{aligned}\frac{d}{dr}\left(rH\Big(e^{i\theta}, \frac{\zeta}{r}\Big)\right) &= H\Big(e^{i\theta}, \frac{\zeta}{r}\Big) + r\frac{d}{dr}H\Big(e^{i\theta}, \frac{\zeta}{r}\Big)\\ &= \frac{\big(r^2-|\zeta|^2\big)^2}{r\Big|r-\overline{\zeta}e^{i\theta}\Big|^2}\left(\frac{r-|\zeta|}{r\big(r+|\zeta|\big)} + \frac{2}{r-|\zeta|} - \frac{1}{r-\overline{\zeta}e^{i\theta}} - \frac{1}{r-\zeta e^{-i\theta}}\right)\\ &\geq \frac{\big(r^2-|\zeta|^2\big)^2(r-|\zeta|)}{r^2\Big|r-\overline{\zeta}e^{i\theta}\Big|^2(r+|\zeta|)} > 0,\end{aligned}$$

because for $|\zeta| < r$ we have

$$\frac{1}{r-\zeta e^{-i\theta}} + \frac{1}{r-\overline{\zeta}e^{i\theta}} = 2\,Re\left(\frac{1}{r-\zeta e^{-i\theta}}\right) \leq \frac{2}{|r-\zeta e^{-i\theta}|} \leq \frac{2}{r-|\zeta|}.$$

The proof is now complete. □

In the next lemma we prove that $r\Gamma\big(z, \frac{\zeta}{r}\big)$ has the same monotonicity property as well.

Lemma 3.2. *For $0 < r < 1$, $z \in \mathbb{D}$, and $|\zeta| < r$, the function $r\Gamma\big(z, \frac{\zeta}{r}\big)$ is an increasing function of r.*

Proof. We shall use the following formula, due to Hadamard, which represents the biharmonic Green function $\Gamma(z,\zeta)$ in terms of the harmonic compensator $H(z,\zeta)$ (see [6], p. 74):

$$\Gamma(z,\zeta)=\frac{1}{\pi}\int_{\max\{|z|,|\zeta|\}}^{1}\int_{-\pi}^{\pi}H\Big(e^{i\theta},\frac{z}{s}\Big)\,H\Big(e^{i\theta},\frac{\zeta}{s}\Big)s\,d\theta\,ds,\qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D}.$$

Let us denote by

$$K(z,\zeta)=\chi_{\big[\max\{|z|,|\zeta|\},\,1\big]},\qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D},$$

where χ_I stands for the characteristic function of the interval I. Assuming that $|\zeta|<r$, we obtain

$$r\,\Gamma\Big(z,\frac{\zeta}{r}\Big)=\frac{1}{\pi}\int_0^1\int_{-\pi}^{\pi}K\Big(z,\frac{\zeta}{r}\Big)\,H\Big(e^{i\theta},\frac{z}{s}\Big)\,rs\,H\Big(e^{i\theta},\frac{\zeta}{rs}\Big)d\theta\,ds.$$

It is easy to see that the function $K\big(z,\frac{\zeta}{r}\big)$ increases with r. On the other hand, since $\frac{|\zeta|}{r}<s$, we have $|\zeta|<rs$, and hence the function

$$rs\,H\Big(e^{i\theta},\frac{|\zeta|}{rs}\Big)$$

increases with r, according to Lemma 3.1. This completes the proof. □

Proposition 3.3. *Let w be a superbiharmonic weight function satisfying the condition $0\le w(z)\le C(1-|z|)$. Then the function $rw(\frac{z}{r})$, for $0<r<1$ and $|z|<r$, is increasing in r.*

Proof. This is an immediate consequence of Theorem 2.6, Lemma 3.1, and Lemma 3.2. □

Lemma 3.4. *Suppose that μ is a finite positive measure on a measure space X, $0<p<+\infty$, and f_n, f are measurable functions such that*

$$\limsup_{n\to+\infty}\int_X|f_n|^p d\mu\le\int_X|f|^p d\mu<+\infty,$$

and $f_n\to f$ almost everywhere with respect to the measure μ. Then

$$\lim_{n\to+\infty}\int_X|f-f_n|^p d\mu=0.$$

Proof. This is a well-known statement; for a proof the reader is referred to [3], p. 21, or [7], p. 66. □

We can now state the main result of this section.

Theorem 3.5. $(0<p<+\infty)$ *Let w be a superbiharmonic weight function satisfying the condition $0\le w(z)\le C(1-|z|)$. Then the polynomials are dense in the weighted Bergman space $L^p_a(\mathbb{D},w)$.*

Proof. Let f be a function in the weighted Bergman space $L^p_a(\mathbb{D},w)$, $0<p<+\infty$. We intend to prove that the function f can be approximated by the polynomials in norm. Since the dilation f_r for every $0<r<1$ can be approximated in norm by the polynomials, it suffices to show that $f_r\to f$ in norm as $r\to1^-$. It is easy to see that $f_r\to f$ pointwise, hence what we need to verify is

$$\|f_r\|_{L^p_a(\mathbb{D},w)}\to\|f\|_{L^p_a(\mathbb{D},w)},\qquad \text{as}\quad r\to1^-.$$

This together with the standard Lemma 3.4 implies that

$$\|f_r - f\|_{L^p_a(\mathbb{D},w)} \to 0, \qquad \text{as} \quad r \to 1^-,$$

from which the result follows. We start by writing

$$\|f_r\|^p_{L^p_a(\mathbb{D},w)} = \int_{\mathbb{D}} |f_r(z)|^p \, w(z) dA(z).$$

We now replace z with z/r on the right side of the above formula to obtain

$$\|f_r\|^p_{L^p_a(\mathbb{D},w)} = \frac{1}{r^3} \int_{r\mathbb{D}} |f(z)|^p \, rw(\frac{z}{r}) dA(z).$$

According to Proposition 3.3 the integrand is an increasing function of r, hence we can apply the monotone convergence theorem to conclude that

$$\|f_r\|^p_{L^p_a(\mathbb{D},w)} \to \|f\|^p_{L^p_a(\mathbb{D},w)}, \quad \text{as} \quad r \to 1^-,$$

from which the theorem follows. □

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