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ON NONOSCILLATORY SOLUTIONS OF DIFFERENTIAL INCLUSIONS

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Abstract. This paper introduces a nonoscillatory theory for differential inclusions based on fixed point theory for multivalued maps.

1. Introduction

This paper presents new nonoscillatory results for the differential inclusions

$$(a(t)\,y'(t))' \in e(t) + F(t, y(t)), \quad t \geq t_0 \geq 0, \tag{1.1}$$

and

$$\frac{d}{dt}\left(a(t)\,\frac{d}{dt}(y(t) + p\,y[t-\tau])\right) \in F(t, y(t)), \quad t \geq t_0. \tag{1.2}$$

Recall that a nontrivial solution of (1.1) or (1.2) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. In the single valued case many nonoscillatory results are available in the literature [1], [6], [9], [10], [12], [13], [15], [17]–[21] for (1.1) and (1.2). However, to our knowledge, no nonoscillatory results are available for the multivalued case. Our results rely on two fixed point results, the first known as Ky–Fan's fixed point theorem [2], [8], [11] and the second known as the Fitzpatrick–Petryshyn fixed point theorem [7], and on a compactness criterion [5] in $B[T,\infty)$ (the Banach space of all continuous, bounded real valued functions on $[T,\infty)$ endowed with the usual supremum norm, i.e. $|u|_\infty = \sup_{t\in[T,\infty)} |u(t)|$ for $u \in B[T,\infty)$).

Theorem 1.1. *Let Q be a nonempty, closed, convex subset of a Banach space E and $F : Q \to CK(Q)$ an upper semicontinuous, compact map; here $CK(Q)$ denotes the family of nonempty convex compact subsets of Q. Then there exists $x \in Q$ with $x \in F(x)$.*

Theorem 1.2. *Let Q be a nonempty, closed, convex subset of a Banach space E and $F : Q \to CK(Q)$ an upper semicontinuous, condensing map with $F(Q)$ bounded. Then there exists $x \in Q$ with $x \in F(x)$.*

Theorem 1.3. *Let E be an equicontinuous and uniformly bounded subset of the Banach space $B[T,\infty)$. If E is equiconvergent at ∞, it is also relatively compact.*

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2. Differential inclusions

In this section we discuss the differential inclusion

$$(a(t)\,y'(t))' \in e(t) + F(t, y(t)), \quad t \geq t_0 \geq 0; \tag{2.1}$$

the functions a and e are single valued and F is a multifunction.

For our first result we assume the following conditions hold:

$$a \in C([t_0, \infty), \mathbf{R}^+), \tag{2.2}$$

$$e \in L^1([t_0, \infty), \mathbf{R}^+), \tag{2.3}$$

$$\exists\, \eta \in C([t_0, \infty), \mathbf{R}) \ \text{ with } \ (a(t)\,\eta'(t))' = e(t) \ \text{ for a.e. } \ t \geq t_0 \tag{2.4}$$

and

$$\begin{cases} F : [t_0, \infty) \times \mathbf{R} \to CK(\mathbf{R}) \ \text{ is a } \ L^1\text{-Carathéodory} \\ \text{multifunction. By this we mean:} \\ \text{(a) for each measurable } \ u : [t_0, \infty) \to \mathbf{R}, \ \text{ the map} \\ t \mapsto F(t, u(t)) \ \text{ has measurable single valued selections,} \\ \text{(b) for a.e. } \ t \in [t_0, \infty), \ \text{ the map } \ u \mapsto F(t, u) \\ \text{is upper semicontinuous,} \\ \text{(c) for each } \ r > 0 \ \text{ there exists } \ h_r \in L^1[t_0, \infty) \ \text{ with} \\ |F(t, u)| \leq h_r(t) \ \text{ for a.e. } t \in [t_0, \infty) \ \text{ and all } \ u \in \mathbf{R} \ \text{ with} \\ |u| \leq r; \ \text{ here } \ |F(x, u)| = \sup\{|v| : \ v \in F(x, u)\} \end{cases} \tag{2.5}$$

and

$$\inf_{t \in [t_0, \infty)} \eta(t) > -\infty. \tag{2.6}$$

Remark 2.1. In (2.5), part (a) could be replaced by: the map $t \mapsto F(t, u)$ is measurable for all $u \in \mathbf{R}$.

Now let $\beta \in \mathbf{R}$ be such that

$$\beta \ > \ - \inf_{t \in [t_0, \infty)} \eta(t), \tag{2.7}$$

and let $d > 0$ be such that

$$\beta \ + \inf_{t \in [t_0, \infty)} \eta(t) \geq d. \tag{2.8}$$

Theorem 2.1. *Suppose* (2.2)–(2.6) *hold, and let* β *(respectively* d*) be chosen as in* (2.7) *(respectively* (2.8)*). Also assume the following three conditions are satisfied*:

$$F : [t_0, \infty) \times (0, \infty) \to CK([0, \infty)), \tag{2.9}$$

$$\exists\, M > d \ \ \textit{with} \ \ M > \beta + \sup_{t \geq t_0} \eta(t), \tag{2.10}$$

and

$$\int^{\infty} \frac{1}{a(s)} \int_s^{\infty} \sup_{w \in [d, M]} |F(t, w)|\, dt\, ds \ < \ \infty. \tag{2.11}$$

Then there is a nonoscillatory solution y *of*

$$(a(t)\,y'(t))' \in e(t) + F(t, y(t)), \quad t \geq T,$$

with

$$\lim_{t\to\infty}(y(t)-\eta(t))=\beta \quad \textit{and} \quad \lim_{t\to\infty} a(t)\,(y(t)-\eta(t))'=0;$$

here T *is chosen as in* (2.12).

Proof. From (2.11) there exists $T \geq t_0$ with

$$\int_T^\infty \frac{1}{a(s)} \int_s^\infty \sup_{w\in[d,M]} |F(t,w)|\,dt\,ds \;\leq\; M - \left[\beta + \sup_{t\geq T}\eta(t)\right]. \tag{2.12}$$

We wish to apply Theorem 1.1 with $E=(B[T,\infty),\,|\,.\,|_\infty)$ and

$$Q=\{y\in B[T,\infty):\;\; d\leq y(t)\leq M \;\text{ for }\; t\geq T\}.$$

Clearly Q is closed and convex. Also if $y\in Q$, then for $t\geq T$ we have from (2.9) and the definition of Q that

$$0\leq u(t)\leq \sup_{w\in[d,M]}|F(t,w)| \;\text{ for each }\; u(t)\in F(t,y(t)).$$

Define a mapping $N:Q\to\mathcal{P}(E)$ (the power set of E) by (here $y\in Q$)

$$N\,y(t)=\beta+\eta(t)+\int_t^\infty \frac{1}{a(s)}\int_s^\infty F(x,y(x))\,dx\,ds \;\text{ for }\; t\geq T.$$

In fact [4, p. 777, Proposition 1.1] guarantees that $N:Q\to C(E)$; here $C(E)$ denotes the family of nonempty, convex subsets of E. We first show

$$N:Q\to C(Q). \tag{2.13}$$

For notational purposes for any $y\in Q$ let

$$\mathcal{F}(y)=\left\{u\in L^1[T,\infty):\; u(t)\in F(t,y(t)) \;\text{ for a.e. }\; t\in[T,\infty)\right\}.$$

Let $y\in Q$, and take $w\in N\,y$. Then there exists $\tau\in\mathcal{F}(y)$ with

$$w(t)=\beta+\eta(t)+\int_t^\infty \frac{1}{a(s)}\int_s^\infty \tau(x)\,dx\,ds \;\text{ for }\; t\geq T.$$

Then for $t\geq T$ we have from (2.12) that

$$\begin{aligned} w(t) &\leq \beta+\sup_{t\geq T}\eta(t)+\int_t^\infty \frac{1}{a(s)}\int_s^\infty \sup_{w\in[d,M]}|F(t,w)|\,dt\,ds \\ &\leq \beta+\sup_{t\geq T}\eta(t)+\left[M-\left\{\beta+\sup_{t\geq T}\eta(t)\right\}\right]=M. \end{aligned}$$

As a result $w(t)\leq M$ for $t\geq T$ for each $w\in N\,y$. On the other hand if $t\geq T$ we have

$$w(t)\geq\beta+\eta(t)\geq\beta+\inf_{t\geq t_0}\eta(t)\geq d.$$

As a result $w(t)\geq d$ for $t\geq T$ for each $w\in N\,y$. Thus (2.13) holds.

Next we show

$$N:Q\to C(Q) \;\text{ is a compact map.} \tag{2.14}$$

To see this we will use Theorem 1.3. For $y\in Q$ let $G(y)=N\,y-\eta$. Take any $y\in Q$ and $w\in G\,(y)$. Then there exists $\tau\in\mathcal{F}(y)$ with

$$w(t)=\beta+\int_t^\infty \frac{1}{a(s)}\int_s^\infty \tau(x)\,dx\,ds \;\text{ for }\; t\geq T.$$

Now since $N : Q \to C(Q)$ we have for $t \geq T$ that

$$w(t) \leq M + \max\{|d-\beta|, |M-\beta|\},$$

and so for each $w \in G\,y$ we have that

$$|w|_\infty = \sup_{t\in[T,\infty)} |w(t)| \leq M + \max\{|d-\beta|, |M-\beta|\} \quad \text{for each } \ y \in Q.$$

Thus the set

$$Y = \{N\,y - \theta : \ y \in Q\} \quad \text{is a uniformly bounded subset of } \ B[T,\infty).$$

Also for each $t \geq T$ we have

$$|w(t) - \beta| \leq \int_t^\infty \frac{1}{a(s)} \int_s^\infty \sup_{w\in[d,M]} |F(x,w)|\,dx\,ds \tag{2.15}$$

for $w \in G\,y,\ y \in Q$. Now (2.11) and (2.15) guarantee that the set Y is equiconvergent at ∞. Also for $t_1,\ t_2$ with $T \leq t_1 \leq t_2$ we have

$$|w(t_2) - w(t_1)| \leq \int_{t_1}^{t_2} \frac{1}{a(s)} \int_s^\infty \sup_{w\in[d,M]} |F(x,w)|\,dx\,ds$$

for $w \in G\,y,\ y \in Q$. Now Theorem 1.3 guarantees that Y is relatively compact in $B[T,\infty)$, and as a result (2.14) holds.

It remains to show that

$$N : Q \to CK(Q) \ \text{ is an upper semicontinuous map.} \tag{2.16}$$

From (2.16) and [2, p. 465] it is sufficient to show that the graph of N, $\mathcal{G}(N)$, is closed. Consider $(x_n, y_n) \in \mathcal{G}(N)$ with $(x_n, y_n) \to (x, y)$; here $n \in N_0 = \{1, 2, ...\}$. We must show $x \in N\,y$. Fix $t \in [T,\infty)$. Note $|y|_\infty \leq M,\ |y_n|_\infty \leq M$ for $n \in N_0$, since $y,\ y_n \in Q$ for $n \in N_0$. Also there exists $v_n \in \mathcal{F}(y_n)$ with

$$\begin{aligned} x_n(t) &= \beta + \eta(t) + \textstyle\int_t^\infty \frac{1}{a(s)} \int_s^\infty v_n(x)\,dx\,ds \\ &= \beta + \eta(t) + \textstyle\int_t^\infty v_n(x) \int_t^x \frac{1}{a(s)}\,ds\,dx. \end{aligned} \tag{2.17}$$

We must show that there exists $u \in \mathcal{F}(y)$ with

$$x(t) = \beta + \eta(t) + \int_t^\infty \frac{1}{a(s)} \int_s^\infty u(x)\,dx\,ds.$$

Notice (2.5) guarantees that there exists an $h_M \in L^1[T,\infty)$ with $|v_n(s)| \leq h_M(s)$ for a.e. $s \in [T,\infty)$. Consider $\{v_n\}_{n\in N_0}$. Take $k \in N_0$ and $k > t$. From (2.17) we have

$$\left| x_n(t) - \beta - \eta(t) - \int_t^k v_n(x) \int_t^x \frac{1}{a(s)}\,ds\,dx \right| \leq \int_k^\infty h_M(x) \int_t^x \frac{1}{a(s)}\,ds\,dx. \tag{2.18}$$

A standard result from the literature [16, Proposition 1.4] guarantees that $\mathcal{F}_k : B[T,k] = C[T,k] \to L^1[T,k]$ is upper semicontinuous with respect to the weak topology (w–u.s.c.) and also weakly completely continuous; here $\mathcal{F}_k$ is given by

$$\mathcal{F}_k(w) = \left\{u \in L^1[T,k] : \ u(t) \in F(t, w(t)) \ \text{ for a.e. } \ t \in [T,k]\right\}.$$

Now since $v_n \in \mathcal{F}_k(y_n)$ for $n \in N_0$, there exists a $u_k \in L^1[T,k]$ and a subsequence of S of N_0 with v_n converging weakly in $L^1[T,k]$ to u_k (i.e. $v_n \rightharpoonup u_k$ in $L^1[T,k]$) as $n \to \infty$ in S. Now $y_n \to y$ in $C[T,k]$ and $v_n \rightharpoonup u_k$ in $L^1[T,k]$ as $n \to \infty$

in S, together with $v_n \in \mathcal{F}_k(y_n)$ for $n \in S$ and $\mathcal{F}_k : C[T,k] \to L^1[T,k]$ w–u.s.c., implies that

$$u_k \in \mathcal{F}_k(y). \tag{2.19}$$

Note as well that $|y|_k = \sup_{s\in[T,k]} |y(s)| \le M$, $|y_n|_k \le M$ for $n \in S$, and $|u_k(x)| \le h_M(x)$ for a.e. $x \in [T,k]$. Let $n \to \infty$ through S in (2.18) to obtain

$$\left| x(t) - \beta - \eta(t) - \int_t^k u_k(x) \int_t^x \frac{1}{a(s)}\, ds\, dx \right| \le \int_k^\infty h_M(x) \int_t^x \frac{1}{a(s)}\, ds\, dx. \tag{2.20}$$

Similarly we can show that there exists $u_{k+1} \in L^1[T,k+1]$ and a subsequence of S, say S_1, with $v_n \rightharpoonup u_{k+1}$ in $L^1[T,k+1]$ as $n \to \infty$ in S_1 and with $u_{k+1} \in \mathcal{F}_{k+1}(y)$. Of course this implies $v_n \rightharpoonup u_{k+1}$ in $L^1[T,k]$ as $n \to \infty$ in S_1 so $u_{k+1}(x) = u_k(x)$ for a.e. $x \in [T,k]$. In addition note $|u_{k+1}(x)| \le h_M(x)$ for a.e. $x \in [T,k+1]$. Continue and construct u_{k+2}, u_{k+3},... . For $l \in \{k, k+1, ...\} \equiv P$ let $u_l^\star(x)$ be any extension to $[T,\infty)$ of u_l with $|u_l^\star(x)| \le h_M(x)$ for a.e. $x \in (l,\infty)$. Also let

$$\begin{aligned} \mathcal{F}_l^\star(w) = \{v \in L^1[T,\infty) : \quad & v(x) \in F(x, w(x)) \text{ for a.e. } x \in [T,l], \\ & |v(x)| \le h_M(x) \text{ for a.e. } x \in [T,\infty)\}. \end{aligned}$$

Now $\{u_l^\star\}_{l\in P}$ is a weakly compact sequence in $L^1[T,\infty)$ (see [3] or [5]) so there exists a subsequence which converges weakly to a function $u \in L^1[T,\infty)$. Note $u(x) = u_k(x)$ for a.e. $x \in [T,k]$ since $u_{k+m}(x) = u_k(x)$ for a.e. $x \in [T,k]$, here $m \in N_0$. This together with (2.20) yields

$$\left| x(t) - \beta - \eta(t) - \int_t^k u(x) \int_t^x \frac{1}{a(s)}\, ds\, dx \right| \le \int_k^\infty h_M(x) \int_t^x \frac{1}{a(s)}\, ds\, dx. \tag{2.21}$$

We next claim that

$$\mathcal{F}(y) = \bigcap_{l\in K} \mathcal{F}_l^\star(y) \quad (\text{here } K = \{[T]+1, [T]+2, ...\}) \tag{2.22}$$

(and $\mathcal{F}(w)$ is nonempty, closed and convex). Note first that $|y|_\infty \le M$ so $|F(x, y(x))| \le h_M(x)$ for a.e. $x \in [T,\infty)$. Let y_k be the restriction to the interval $[T,k]$, $k \in N_0$, of y. Note [8, 11] or [14] that

$$\mathcal{F}_k(y_k) = \{v \in L^1[T,k] : \ v(x) \in F(x, y_k(x)) \text{ for a.e. } x \in [T,k]\}$$

is closed in $L^1[T,k]$ for all $k \in K$. Let

$$\begin{aligned} \mathcal{F}_k^\star(y_k) = \{v \in L^1[T,\infty) : \quad & v \in \mathcal{F}_k(y_k) \text{ for } x \in [T,k] \\ & \text{and } v(x) = 0 \text{ for } x > k\}. \end{aligned}$$

It is immediate that $\mathcal{F}_k^\star(y_k)$ is a closed set in $L^1[T,\infty)$ for each $k \in K$. Let

$$\begin{aligned} R_k = \{v \in L^1[T,\infty) : \quad & v(x) = 0 \text{ for } x \in [T,k], \\ & |v(x)| \le h_M(x) \text{ for a.e. } x \in (k,\infty)\} \end{aligned}$$

and notice it is clear that

$$\mathcal{F}_k^\star(y) = \mathcal{F}_k^\star(y_k) \oplus R_k.$$

It is clear that $\mathcal{F}_k^\star(y)$ is a closed set in $L^1[T,\infty)$. Also for each $k \in K$ we have $\mathcal{F}(y) \subseteq \mathcal{F}_k^\star(y)$ and so

$$\mathcal{F}(y) \subseteq \bigcap_{l\in K} \mathcal{F}_l^\star(y).$$

On the other hand if $v \in \mathcal{F}_l^\star(y)$ for each $l \in K$, then $v(x) \in F(x, y(x))$ for a.e. $x \in [T,\infty)$ and so

$$\bigcap_{l\in K} \mathcal{F}_l^\star(y) \subseteq \mathcal{F}(y).$$

Thus (2.22) holds and also $\mathcal{F}(y)$ is a closed subset of $L^1[T,\infty)$. Now since u belongs to $\bigcap_{l\in K} \mathcal{F}_l^\star(y)$ (note for each $l \in K$ that $u \in \mathcal{F}_l^\star(y)$) we have $u \in \mathcal{F}(y)$. Let $k \to \infty$ in (2.21) to obtain

$$x(t) - \beta - \eta(t) - \int_t^\infty u(x) \int_t^x \frac{1}{a(s)}\, ds\, dx = 0,$$

and so

$$x(t) = \beta + \eta(t) + \int_t^\infty \frac{1}{a(s)} \int_s^\infty u(x)\, dx\, ds.$$

Consequently $\mathcal{G}(N)$ is closed, so $N : Q \to C(Q)$ is an upper semicontinuous map (see [2, p. 465]); in fact (2.14) guarantees that $N : Q \to CK(Q)$.

Theorem 1.1 guarantees that there exists $y \in Q$ with $y \in N\,y$. That is for every $t \geq T$ we have

$$y(t) \in \beta + \eta(t) + \int_t^\infty \frac{1}{a(s)} \int_s^\infty F(x, y(x))\, dx\, ds,$$

and so

$$(a(t)\, y'(t))' \in e(t) + F(t, y(t)) \quad \text{for a.e.} \quad t \geq T.$$

In addition

$$\begin{aligned} |y(t) - \eta(t) - \beta| \quad &\leq \quad \int_t^\infty \frac{1}{a(s)} \int_s^\infty \sup_{w\in[d,M]} |F(x,w)|\, dx\, ds \\ &\to \quad 0 \quad \text{as} \quad t \to \infty \end{aligned}$$

and

$$|a(t)\,[y(t) - \eta(t)]'| \leq \int_t^\infty \sup_{w\in[d,M]} |F(x,w)|\, dx \to 0 \quad \text{as} \quad t \to \infty.$$

□

We next show that we can obtain an existence result for (2.1) if (2.9) is replaced by

$$F : [t_0,\infty) \times (0,\infty) \to CK((-\infty, 0]). \tag{2.23}$$

Theorem 2.2. *Suppose* (2.2)–(2.6) *and* (2.23) *hold, and let* β *(respectively* d*) be chosen as in* (2.7) *(respectively* (2.8)*). Also assume the following two conditions are satisfied*:

$$\exists\, K > 1 \quad \textit{and} \quad \exists\, M > \frac{d}{K} \quad \textit{with} \quad M \geq \beta + \sup_{t\geq t_0} \eta(t) \tag{2.24}$$

and

$$\int^{\infty} \frac{1}{a(s)} \int_s^{\infty} \sup_{w\in\left[\frac{d}{K},M\right]} |F(t,w)|\,dt\,ds \;<\; \infty. \tag{2.25}$$

Then there is a nonoscillatory solution y *of*

$$(a(t)\,y'(t))' \;\in\; e(t) + F(t,y(t)), \quad t \geq T,$$

with

$$\lim_{t\to\infty} (y(t) - \eta(t)) = \beta \quad \textit{and} \quad \lim_{t\to\infty} a(t)\,(y(t) - \eta(t))' = 0;$$

here T *is chosen as in* (2.26).

Proof. From (2.25) there exists $T \geq t_0$ with

$$\int_T^{\infty} \frac{1}{a(s)} \int_s^{\infty} \sup_{w\in\left[\frac{d}{K},M\right]} |F(t,w)|\,dt\,ds \;\leq\; \left(\frac{K-1}{K}\right) d. \tag{2.26}$$

We will apply Theorem 1.1 with $E = (B[T,\infty),\, |\,.\,|_\infty)$ and

$$Q = \left\{ y \in B[T,\infty) : \;\; \frac{d}{K} \leq y(t) \leq M \;\; \text{for} \;\; t \geq T \right\}.$$

Let N and $\mathcal{F}$ be as in Theorem 2.1. First we show

$$N : Q \to C(Q). \tag{2.27}$$

Let $y \in Q$, and take $w \in N\,y$. Then there exists $\tau \in \mathcal{F}(y)$ with

$$w(t) = \beta + \eta(t) + \int_t^{\infty} \frac{1}{a(s)} \int_s^{\infty} \tau(x)\,dx\,ds \;\; \text{for} \;\; t \geq T.$$

If $t \geq T$ we also have

$$\begin{aligned} w(t) \;\; &\geq \;\; \beta + \eta(t) - \int_T^{\infty} \frac{1}{a(s)} \int_s^{\infty} \sup_{w\in\left[\frac{d}{K},M\right]} |F(t,w)|\,dt\,ds \\ &\geq \;\; \beta + \inf_{t\geq T} \eta(t) - \left(\frac{K-1}{K}\right) d \\ &\geq \;\; d - \left(\frac{K-1}{K}\right) d = \frac{d}{K}. \end{aligned}$$

As a result $w(t) \geq \frac{d}{K}$ for $t \geq T$ for each $w \in N\,y$. On the other hand for $t \geq T$ we have

$$w(t) \leq \beta + \eta(t) \leq \beta + \sup_{t\geq T} \eta(t) \leq M.$$

As a result $w(t) \leq M$ for $t \geq T$ for each $w \in N\,y$. Thus (2.27) holds. Essentially the same reasoning as in Theorem 2.1 guarantees that

$$N : Q \to CK(Q) \;\; \text{is an upper semicontinuous, compact map.}$$

Apply Theorem 1.1 to deduce the result. □

Notice that it is easy to remove assumption (2.23) (respectively (2.9)) in Theorem 2.2 (respectively Theorem 2.1) if we combine the analysis of both theorems. In this case (2.24) has to be adjusted slightly.

Theorem 2.3. *Suppose* (2.2)–(2.6) *hold, and let* β *(respectively* d*) be chosen as in* (2.7) *(respectively* (2.8)*). Also assume that the following two conditions are satisfied*:

$$\exists\, K > 1 \ \text{ and } \ \exists\, M > \frac{d}{K} \ \text{ with } \ M > \beta + \sup_{t \geq t_0} \eta(t) \tag{2.28}$$

and

$$\int^{\infty} \frac{1}{a(s)} \int_s^{\infty} \sup_{w \in \left[\frac{d}{K}, M\right]} |F(t,w)|\, dt\, ds \ < \ \infty. \tag{2.29}$$

Then there is a nonoscillatory solution y *of*

$$(a(t)\, y'(t))' \ \in \ e(t) + F(t, y(t)), \quad t \geq T,$$

with

$$\lim_{t \to \infty} (y(t) - \eta(t)) = \beta \quad \text{and} \quad \lim_{t \to \infty} a(t)\, (y(t) - \eta(t))' = 0;$$

here T *is chosen as in* (2.30).

Proof. From (2.29) there exists $T \geq t_0$ with

(2.30)

$$\int_T^{\infty} \tfrac{1}{a(s)} \int_s^{\infty} \sup_{w \in \left[\frac{d}{K}, M\right]} |F(t,w)|\, dt\, ds \leq \min\left\{ \left(\tfrac{K-1}{K}\right) d, M - \left[\beta + \sup_{t \geq T} \eta(t)\right] \right\}.$$

Let Q and N be as in Theorem 2.2. Let $y \in Q$ and take $w \in N\, y$. As in Theorem 2.2 we have $w(t) \geq \frac{d}{K}$ for $t \geq T$ and as in Theorem 2.1 we have $w(t) \leq M$ for $t \geq T$. Thus $N : Q \to C(Q)$. □

Remark 2.2. Minor adjustments in the analysis of this section would enable us to discuss the more general differential inclusion

$$(a(t)\, (b(t)\, y(t))')' \ \in \ e(t) + F(t, y(t)), \quad t \geq t_0 \geq 0.$$

3. Neutral inclusions

In this section we discuss the neutral inclusion

$$\frac{d}{dt} \left(a(t)\, \frac{d}{dt} (y(t) + p\, y[t - \tau]) \right) \ \in \ F(t, y(t)), \quad t \geq t_0 \geq 0; \tag{3.1}$$

the function a is single valued, p and τ are constants, and F is a multifunction.

For the result in this section we assume the following conditions hold:

$$\tau \in \mathbf{R}^+, \tag{3.2}$$

$$a \in C([t_0, \infty), \mathbf{R}^+), \tag{3.3}$$

and

$$F : [t_0, \infty) \times \mathbf{R} \to CK(\mathbf{R}) \ \text{ is an } \ L^1\text{-Carathéodory multifunction.} \tag{3.4}$$

Theorem 3.1. *Suppose* (3.2)–(3.4) *hold. Also assume the following three conditions are satisfied*:

$$F : [t_0, \infty) \times (0, \infty) \to CK((-\infty, 0]), \tag{3.5}$$

$$|p| \ \neq \ 1, \tag{3.6}$$

and

$$\exists\, K > 0 \quad \textit{with} \quad \int^{\infty} \frac{1}{a(s)} \int_s^{\infty} \sup_{w \in \left[\frac{K}{2}, K\right]} |F(t,w)|\, dt\, ds \;<\; \infty. \tag{3.7}$$

Then

$$\frac{d}{dt}\left(a(t)\,\frac{d}{dt}(y(t) + p\,y[t-\tau])\right) \;\in\; F(t, y(t)), \quad t \geq T,$$

has a bounded nonoscillatory solution; here $T \geq t_0$ *is suitably chosen (see the proof of the theorem).*

Remark 3.1. One cannot expect an analogue of Theorem 3.1 for the case $|p| = 1$ even in the neutral equation case [6, Chapter 5] (see also [1, Chapter 3] for some partial results when $|p| = 1$).

Proof. The proof will be broken into two cases, namely $|p| < 1$ and $|p| > 1$.

Case (I). $|p| < 1$.

Choose $T \geq t_0$ so that $t - \tau > t_0$ for $t \geq T$ and

$$\int_T^{\infty} \frac{1}{a(s)} \int_s^{\infty} \sup_{w \in \left[\frac{K}{2}, K\right]} |F(t,w)|\, dt\, ds \;\leq\; \frac{1}{4}\,(1 - |p|)\, K. \tag{3.8}$$

Let $T_1 = T - \tau$. We wish to apply Theorem 1.2 with $E = (B[T_1, \infty), |\,.\,|_\infty)$ and

$$Q = \left\{ y \in B[T_1, \infty) : \;\; \frac{K}{2} \leq y(t) \leq K \;\; \text{for} \;\; t \geq T_1 \right\}.$$

Define the single valued map $N_1 : Q \to E$ and the multivalued map $N_2 : Q \to \mathcal{P}(E)$ as follows (here $y \in Q$):

$$N_1\, y(t) = \begin{cases} \frac{3}{4}\,(1+p)\,K - p\,y[T-\tau], \; T_1 \leq t \leq T, \\ \frac{3}{4}\,(1+p)\,K - p\,y[t-\tau], \; t \geq T, \end{cases}$$

and

$$N_2\, y(t) = \begin{cases} \{0\}, \quad T_1 \leq t \leq T, \\ -\int_T^t \frac{1}{a(s)} \int_s^{\infty} F(x, y(x))\, dx\, ds, \; t \geq T. \end{cases}$$

First we show

$$N = N_1 + N_2 : Q \to C(Q). \tag{3.9}$$

For notational purposes for any $y \in Q$ let

$$\mathcal{F}\,(y) = \left\{ u \in L^1[T_1, \infty) : \; u(t) \in F(t, y(t)) \;\; \text{for a.e.} \;\; t \in [T_1, \infty) \right\}.$$

Let $y \in Q$, so $\frac{K}{2} \leq y(t) \leq K$ for $t \in [T_1, \infty)$, and take $w \in N_2\, y$. Then there exists $\tau_0 \in \mathcal{F}(y)$ with

$$w(t) = \begin{cases} \{0\}, \quad T_1 \leq t \leq T, \\ -\int_T^t \frac{1}{a(s)} \int_s^{\infty} \tau_0(x)\, dx\, ds, \; t \geq T. \end{cases}$$

Our discussion is broken into two subcases, namely $0 \leq p < 1$ and $-1 < p < 0$.

Subcase (i). $0 \le p < 1$.

If $T_1 \le t \le T$, then clearly

$$\begin{aligned} N_1\, y(t) + w(t) &= \frac{3}{4}\,(1+p)\,K - p\,y[T-\tau] \ge \frac{3}{4}\,(1+p)\,K - p\,K \\ &= \left(\frac{3}{4} - \frac{1}{4}\,p\right)\,K \ge \frac{K}{2} \end{aligned}$$

and

$$N_1\, y(t) + w(t) \le \frac{3}{4}\,(1+p)\,K - p\,\frac{K}{2} = \left(\frac{3}{4} + \frac{1}{4}\,p\right)\,K \le K.$$

If $t \ge T$, then (3.5) implies

$$N_1\, y(t) + w(t) \ge \frac{3}{4}\,(1+p)\,K - p\,y[t-\tau] \ge \frac{3}{4}\,(1+p)\,K - p\,K \ge \frac{K}{2},$$

and (3.8) implies

$$\begin{aligned} N_1\, y(t) + w(t) &\le \frac{3}{4}\,(1+p)\,K - p\,y[t-\tau] + \frac{1}{4}\,(1-p)\,K \\ &\le \frac{3}{4}\,(1+p)\,K - p\,\frac{K}{2} + \frac{1}{4}\,(1-p)\,K \;=\; K. \end{aligned}$$

As a result $\frac{K}{2} \le N_1\, y(t) + w(t) \le K$ for $t \ge T_1$ for each $w \in N_2\, y$. Thus (3.9) holds in this case.

Subcase (ii). $-1 < p < 0$.

If $T_1 \le t \le T$, then clearly

$$\begin{aligned} N_1\, y(t) + w(t) &= \frac{3}{4}\,(1+p)\,K - p\,y[T-\tau] \ge \frac{3}{4}\,(1+p)\,K - p\,\frac{K}{2} \\ &= \left(\frac{3}{4} + \frac{1}{4}\,p\right)\,K \ge \frac{K}{2} \end{aligned}$$

and

$$N_1\, y(t) + w(t) \le \frac{3}{4}\,(1+p)\,K - p\,K = \left(\frac{3}{4} - \frac{1}{4}\,p\right)\,K \le K.$$

If $t \ge T$, then (3.5) implies

$$N_1\, y(t) + w(t) \ge \frac{3}{4}\,(1+p)\,K - p\,y[t-\tau] \ge \left(\frac{3}{4} + \frac{1}{4}\,p\right)\,K \ge \frac{K}{2},$$

and (3.8) implies

$$N_1\, y(t) + w(t) \le \frac{3}{4}\,(1+p)\,K - p\,K + \frac{1}{4}\,(1+p)\,K \;=\; K.$$

As a result $\frac{K}{2} \le N_1\, y(t) + w(t) \le K$ for $t \ge T_1$ for each $w \in N_2\, y$. Thus (3.9) holds in this case also.

Essentially the same reasoning as in Theorem 2.1 guarantees that

(3.10) $\qquad N_2 : Q \to C(E)$ is an upper semicontinuous, compact map.

Next we claim that

(3.11) $\qquad N_1 : Q \to E$ is a contractive map.

To see this notice for $y_1,\, y_2 \in Q$ and $T_1 \le t \le T$ we have

$$|N_1\, y_1(t) - N_1\, y_2(t)| = |p\,\{y_1[T-\tau] - y_2[T-\tau]\}| \le |p|\,|y_1 - y_2|_\infty,$$

whereas if $t \geq T$ we have

$$|N_1\, y_1(t) - N_1\, y_2(t)| = |p\,\{y_1[t-\tau] - y_2[t-\tau]\}| \leq |p|\,|y_1 - y_2|_\infty.$$

Combining gives

$$|N_1\, y_1 - N_1\, y_2|_\infty \leq |p|\,|y_1 - y_2|_\infty,$$

so (3.11) is true since $|p| < 1$.

Now (3.9), (3.10) and (3.11) imply that

(3.12) $\quad N : Q \to CK(Q)$ is an upper semicontinuous, condensing map.

Theorem 1.2 implies that there exists $y \in Q$ with $y \in N_1\, y + N_2\, y$. Hence for $t \geq T$ we have

$$y(t) \in \frac{3}{4}(1+p)\,K - p\,y[t-\tau] - \int_T^t \frac{1}{a(s)} \int_s^\infty F(x, y(x))\,dx\,ds,$$

and we are finished.

Case (II). $|p| > 1$.

Choose $T \geq t_0$ so that $t - \tau > t_0$ for $t \geq T$ and

$$\int_T^\infty \frac{1}{a(s)} \int_s^\infty \sup_{w \in \left[\frac{K}{2}, K\right]} |F(t,w)|\,dt\,ds \leq \frac{1}{4}\,(|p| - 1)\,K.$$

Let $T_1 = T - \tau$. Let E and Q be as in Case (I). Define $N_1 : Q \to E$ and $N_2 : Q \to \mathcal{P}(E)$ as follows (here $y \in Q$):

$$N_1\, y(t) = \begin{cases} \frac{3}{4}\left(\frac{1+p}{p}\right) K - \frac{1}{p}\, y[T+\tau], & T_1 \leq t \leq T, \\ \frac{3}{4}\left(\frac{1+p}{p}\right) K - \frac{1}{p}\, y[t+\tau], & t \geq T, \end{cases}$$

and

$$N_2\, y(t) = \begin{cases} \{0\}, & T_1 \leq t \leq T, \\ -\frac{1}{p} \int_{T+\tau}^{t+\tau} \frac{1}{a(s)} \int_s^\infty F(x, y(x))\,dx\,ds, & t \geq T. \end{cases}$$

A slight modification of the argument in Case (I) guarantees that $N : Q \to CK(Q)$ is an upper semicontinuous, condensing map. Now apply Theorem 1.2. □

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