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# PSEUDONORMALITY AND STARCOMPACTNESS OF $\sigma$-PRODUCTS

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ABSTRACT. In this paper we shall prove the following: For every non-trivial $\sigma$-product $\sigma$, of uncountable number of spaces, having at least two points, $\sigma \setminus \sigma_n$ is not pseudonormal. And every non-trivial $\sigma$-product is not strongly starcompact.

## 1. INTRODUCTION

Throughout this paper we assume that each space is a $T_1$-space having at least two points. We recall the definition of $\sigma$-products which were introduced by H. H. Corson [4].

**Definition 1.** Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be a family of spaces. "$\sigma = \sigma(\mathcal{S})$ is called a $\sigma$-product of $\mathcal{S}$" if there is a point $x^* = (x^*_\alpha)_{\alpha\in\Omega} \in X = \Pi\{X_\alpha | \alpha \in \Omega\}$ (called the base point of $\sigma$) such that $\sigma$ is the subspace of $X$ consisting of $x \in X$ such that $Q(x)$ is finite. Here $Q(x) = \{\alpha | \alpha \in \Omega, x_\alpha \neq x^*_\alpha\}$. Let $[\Omega]^n = \{a \subset \Omega : |a| = n\}$ for each $n \in \omega$ and put $[\Omega]^{<\omega} = \bigcup\{[\Omega]^n : n \in \omega\}$. Here $|a|$ denotes the cardinal number of $a$.

$\Sigma = \{x \in X : |Q(x)| \leq \omega\}$ is called a $\Sigma$-product of $\mathcal{S}$.

A $\sigma$-product $\sigma$ (resp. $\Sigma$-product $\Sigma$) is called non-trivial if $\sigma \neq X$ (resp. $\Sigma \neq X$).

Let $X$ be a space and $\tau$ be an infinite cardinal number such that $|\Omega| = \tau$. In case $X_\alpha = X$ for each $\alpha \in \Omega$, let us denote $\sigma(\mathcal{S})$ by $\sigma(X^\tau)$. For $a \in X$, we denote by $a^* = (a_\alpha)_{\alpha\in\Omega}, a_\alpha = a$ for each $\alpha \in \Omega$.

For a finite subset $F$ of $\Omega, \Pi\{X_\alpha | \alpha \in F\}$ is said to be a finite subproduct of $\sigma$.

The following fact concerning $\sigma$-products is known.

**Fact.** Let $\sigma = \sigma(\mathcal{S})$ and $\sigma_n = \{x \in \sigma : |Q(x)| \leq n\}$ for each $n \in \omega$. Then $\sigma_n$ is closed in $\sigma$.

In this paper we investigate normality-type properties of special subspaces of $\sigma$-products and compactness-type properties of $\sigma$-products.

In 1959, Corson [4] proved that for every non-trivial $\Sigma$-product $\Sigma$, a subspace $\Sigma \setminus \{x\}$ is not normal for every point $x \in \Sigma$. In 1978, A. P. Kombarov [8] proved that if a set $Z$ is closed in the $\tau$-envelope $Y = Y(x^*, \tau) = \{y \in X = \Pi\{X_\alpha | \alpha \in \Omega\} : |Q(y)| < \tau\}$ and $|\bigcup\{Q(z) : z \in Z\}| < \tau$, then $Y \setminus Z$ is a non-normal subset

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of $Y$. As corollaries of this theorem we have non-normality of $\Sigma \smallsetminus \{x\}$, where $\Sigma$ is a non-trivial $\Sigma$-product and $x \in \Sigma$, and non-normality of $\sigma \smallsetminus \{x\}$, where $\sigma$ is a non-trivial $\sigma$-product of uncountable number of spaces and $x \in \sigma$.

A space $X$ is called pseudonormal if any two disjoint closed sets, one of which is countable, are separated by open sets in $X$. Obviously, any normal space is pseudonormal.

In 1996, Kombarov [9] proved that if $Y$ is a $\tau$-envelope of spaces $X_\alpha, \alpha \in \Omega$, $|\Omega| \geq max\{\omega_1, \tau\}$, then a subspace $Y \smallsetminus \{x\}$ is not pseudonormal for every $x \in Y$. In particular he obtained the following.

**Theorem A** (Kombarov [9]). *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be a family of spaces such that $|\Omega| \geq \omega_1$ and let $\sigma = \sigma(\mathcal{S})$. Then $\sigma \smallsetminus \sigma_0$ is not pseudonormal.*

In this paper we shall prove a generalization of Theorem A.

## 2. Normality and pseudonormality

**Theorem 1.** *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be a family of spaces such that $|\Omega| \geq \omega_1$ and let $\sigma = \sigma(\mathcal{S})$. Then $\sigma \smallsetminus \sigma_n$ is not pseudonormal for each $n \in \omega$.*

**Lemma 1.** *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ such that $|\Omega| \geq \omega_1$. Let $\sigma = \sigma(2^{\omega_1})$ be the $\sigma$-product with the base point $0^*$. Here $2 = \{0, 1\}$ is the discrete space of two points. Then there is a homeomorphism $f$ from $\sigma$ onto $f(\sigma) \subset \sigma'$ such that $f(0^*) = x^*$ and $f(\sigma \smallsetminus \sigma_n)$ is a closed subset of $\sigma' \smallsetminus \sigma'_n$. Here $\sigma' = \sigma(\mathcal{S})$ with the base point $x^*$.*

*Proof.* Let us choose a point $a_\alpha \in X_\alpha$ such that $a_\alpha \neq x^*_\alpha$ for each $\alpha \in \Omega$. Let us consider $\omega_1 \subset \Omega$. Let $f : \sigma \to \sigma'$ as follows: for each $x = (x_\alpha)_{\alpha \in \omega_1} \in \sigma$, let $f(x) = (y_\alpha)_{\alpha \in \Omega}$ be

$$y_\alpha = \begin{cases} a_\alpha & \text{if } \alpha \in \omega_1 \text{ and } x_\alpha = 1, \\ x^*_\alpha & \text{otherwise.} \end{cases}$$

Then $f$ has the desired properties. To prove that $f(\sigma \smallsetminus \sigma_n)$ is a closed subset of $\sigma' \smallsetminus \sigma'_n$, let $y \in (\sigma' \smallsetminus \sigma'_n) \smallsetminus f(\sigma \smallsetminus \sigma_n)$. Then $Q(y) \cap (\Omega \smallsetminus \omega_1) \neq \emptyset$. Let us choose an element $\alpha \in Q(y) \cap (\Omega \smallsetminus \omega_1)$ and put $U = \{z \in \sigma' \smallsetminus \sigma'_n | z_\alpha \neq x^*_\alpha\}$. Then $U$ is an open neighborhood of $y$ in $\sigma' \smallsetminus \sigma'_n$ such that $U \cap f(\sigma \smallsetminus \sigma_n) = \emptyset$.

Since pseudonormality is inherited by closed subspaces, Theorem 1 follows from Proposition 1 below by using Lemma 1.

**Proposition 1.** *Let $\sigma = \sigma(2^{\omega_1})$ be the $\sigma$-product with the base point $0^*$. Then $\sigma \smallsetminus \sigma_n$ is not pseudonormal for each $n \in \omega$.*

*Proof.* We denote $\sigma = \{f : \omega_1 \to 2 | Q(f) \text{ is finite}\}$. Here $Q(f) = \{\alpha \in \omega_1 | f(\alpha) = 1\}$.

Put $G = \sigma \smallsetminus \sigma_n$. Let us choose a subset A $\subset \omega_1$ such that $|A| = n$. For each $\alpha \in \omega_1$, let us define $f^\alpha : \omega_1 \to 2$ as follows:

$$f^\alpha(\beta) = \begin{cases} 1 & \text{if } \beta \in A \cup \{\alpha\}, \\ 0 & \text{if } \beta \in \omega_1 \smallsetminus (A \cup \{\alpha\}). \end{cases}$$

Then

(1) $f^\alpha \in \sigma_{n+1} \smallsetminus \sigma_n$ for each $\alpha \in \omega_1 \smallsetminus A$.

Let us choose subsets $\Gamma_1$ and $\Gamma_2$ of $\omega_1$ such that $|\Gamma_1| = \omega_1, |\Gamma_2| = \omega, \Gamma_1 \cap \Gamma_2 = \emptyset, (\Gamma_1 \cup \Gamma_2) \cap A = \emptyset, \omega_1 = \Gamma_1 \cup \Gamma_2 \cup A$. Put $E = \{f^\alpha | \alpha \in \Gamma_1\}$ and $F = \{f^\alpha | \alpha \in \Gamma_2\}$.

Then $E \cap F = \emptyset, |F| = \omega$ and

(i) $E$ and $F$ are closed subsets in $G$;

(ii) $E$ and $F$ are not separated by open sets in $G$.

*Proof of* (i). To prove that $E$ is closed in $G$, let $f \in G \setminus E$. If $|Q(f)| \geq n+2$, then $f \notin \sigma_{n+1}$. Put $U = G \setminus \sigma_{n+1}$. Then $U$ is a neighborhood of $f$ in $G$ such that $U \cap E = \emptyset$. If $|Q(f)| = n+1$, then $Q(f) \setminus A \neq \emptyset$ because $|A| = n$. Let $\alpha \in Q(f) \setminus A$. Then $f(\alpha) = 1$. If $|A \cap Q(f)| = n$, then $\alpha \in \Gamma_2$ because $f \notin E$. Put $U = \{g \in G | g(\alpha) = 1\}$. Then $U$ is a neighborhood of $f$ in $G$ such that $U \cap E = \emptyset$. If $|A \cap Q(f)| < n$, then there exists an element $\beta \in A$ such that $f(\beta) = 0$. Put $U = \{g \in G | g(\beta) = 0\}$. Then $U$ is a neighborhood of $f$ in $G$ such that $U \cap E = \emptyset$. Therefore it is proved that $E$ is closed in $G$. Quite similarly it is proved that $F$ is closed in $G$.

*Proof of* (ii). Let $U$ be an arbitrary open set in $G$ such that $E \subset U$. Then, for each $\alpha \in \Gamma_1$, there is a finite set $r_\alpha$ of $\omega_1$ such that $A \cup \{\alpha\} \subset r_\alpha$ and $f^\alpha \in U_\alpha \equiv \{g \in G | g(\beta) = 1 \text{ for each } \beta \in A \cup \{\alpha\}, g(\beta) = 0 \text{ for each } \beta \in r_\alpha \setminus (A \cup \{\alpha\})\} \subset U$. By Šanin's lemma, there are an uncountable set $\Gamma^* \subset \Gamma_1$ and a finite set $r^* \subset \omega_1$ such that $\{r_\alpha \setminus r^* | \alpha \in \Gamma^*\}$ is disjoint. Since $\Gamma_2$ is infinite and $r^*$ is finite, $\Gamma_2 \setminus r^* \neq \emptyset$. Let $\alpha^* \in \Gamma_2 \setminus r^*$. Then

(2) $f^{\alpha^*} \in F \cap clU$.

It is obvious that $f^{\alpha^*} \in F$. Therefore it is sufficient to prove the following.

*Claim.* $f^{\alpha^*} \in clU$.

To prove this, let $V$ be an arbitrary open set in $G$ such that $f^{\alpha^*} \in V$. Then there is a finite set $r$ of $\omega_1$ such that $A \cup \{\alpha^*\} \subset r$ and $f^{\alpha^*} \in V' \equiv \{g \in G | g(\beta) = 1$ for each $\beta \in A \cup \{\alpha^*\}, g(\beta) = 0$ for each $\beta \in r \setminus (A \cup \{\alpha^*\})\} \subset V$.

Since $\Gamma^*$ is uncountable and $r$ and $r^*$ are finite, there is an element $\beta^* \in \Gamma^*$ such that $r \cap (r_{\beta^*} \setminus r^*) = \emptyset$ and $\beta^* \in r_{\beta^*} \setminus r^*$. Then it is easy to see that

(3) $\alpha^* \neq \beta^*, \alpha^*, \beta^* \notin A$.

Define $g^* : \omega_1 \to 2$ by

$$g^*(\alpha) = \begin{cases} 1 & \text{if } \alpha \in A \cup \{\alpha^*, \beta^*\}, \\ 0 & \text{if } \alpha \in \omega_1 \setminus (A \cup \{\alpha^*, \beta^*\}). \end{cases}$$

Then $g^* \in \sigma_{n+2} \setminus \sigma_{n+1}$ and therefore $g^* \in G$. Moreover we have

(4) $g^* \in V' \cap U_{\beta^*}$.

To prove that $g^* \in V'$, let $\beta \in r \setminus (A \cup \{\alpha^*\})$. Then $\beta \notin r_{\beta^*} \setminus r^*$. Thus $\beta \neq \beta^*$. Hence $g(\beta) = 0$. Therefore $g \in V'$. To prove that $g^* \in U_{\beta^*}$, let $\beta \in r_{\beta^*} \setminus (A \cup \{\beta^*\})$. If $\beta \in r^*$, then $\beta \neq \alpha^*$. Thus $g(\beta) = 0$. If $\beta \notin r^*$, then $\beta \in r_{\beta^*} \setminus r^*$. Therefore $\beta \notin r$. Then $\beta \neq \alpha^*$ and so $g(\beta) = 0$. Hence $g \in U_{\beta^*}$.

It is known that there exists a non-normal $\sigma$-product such that each finite subproduct is normal (cf. [3]).

**Theorem 2.** *If each $X_\alpha \in \mathcal{S}$ is normal (resp. pseudonormal), then $\sigma_1$ is normal (resp. pseudonormal).*

*Proof.* We shall write only the proof of normality because the proof of pseudonormality is quite similar. Let $A$ and $B$ be disjoint closed subsets in $\sigma_1$. Let $x^* \in A$. There are a finite set $\{\alpha_i | i = 1, 2, ..., m\}$ of $\Omega$ and open sets $U_{\alpha_i}$ in $X_{\alpha_i}$ such that $x^* \in W \equiv \{x \in \sigma_1 | x_{\alpha_i} \in U_{\alpha_i}$ for $i = 1, 2, ..., m\}$ and $clW \cap B = \emptyset$. Since $X_{\alpha_i}$ is normal, there is an open set $U'_{\alpha_i}$ in $X_{\alpha_i}$ such that $x^*_{\alpha_i} \in U'_{\alpha_i}$ and $clU'_{\alpha_i} \subset U_{\alpha_i}$ for

$i = 1, 2, ..., m$. Put $W' \equiv \{x \in \sigma_1 | x_{\alpha_i} \in U'_{\alpha_i}$ for $i = 1, 2, ..., m\}$. Then $W'$ is open in $\sigma_1$, $x^* \in W'$ and $clW' \cap B = \emptyset$.

Let $Y_\alpha = \{x \in \sigma_1 | x_\beta = x^*_\beta$ for each $\beta \neq \alpha\}$, i.e., $Y_\alpha = X_\alpha \times \{z^*_\alpha\}$. Here $z^*_\alpha = (x^*_\beta)_{\beta \in \Omega \setminus \{\alpha\}}$. Then it is easy to see that

(1) $Y_\alpha \subset W'$ for each $\alpha \in \Omega \setminus \{\alpha_i | i = 1, 2, ..., m\}$;

(2) $Y_\alpha$ is closed in $\sigma_1$.

Put $A_i = A \cap Y_{\alpha_i} \setminus W'$ and $B_i = B \cap Y_{\alpha_i} \setminus W'$. Then $A_i$ and $B_i$ are disjoint closed sets in $Y_{\alpha_i}$ and $x^* \notin A_i, x^* \notin B_i$. Since $Y_{\alpha_i}$ is normal, there are open sets $V_i$ and $V'_i$ in $X_{\alpha_i}$ such that $V_i \cap V'_i = \emptyset, x^*_{\alpha_i} \notin V_i \cup V'_i, A_i \subset V_i \times \{z^*_\alpha\}$ and $B_i \subset V'_i \times \{z^*_\alpha\}$. Put $G_i = \{x \in \sigma_1 | x_{\alpha_i} \in V_i\} \setminus \bigcup \{Y_{\alpha_j} | j = 1, 2, ..., m; j \neq i\}$ and $H_i = \{x \in \sigma_1 | x_{\alpha_i} \in V'_i \setminus clU'_{\alpha_i}\} \setminus \bigcup \{Y_{\alpha_j} | j = 1, 2, ..., m; j \neq i\}$. Then $G_i$ and $H_i$ are open sets in $\sigma_1$ such that

(3) $A_i \subset G_i, B_i \subset H_i$;

(4) $G_i \cap H_i = \emptyset$.

*Proof of* (3). Let $x \in A_i \cup B_i$. Then $x_\alpha = x^*_\alpha$ for each $\alpha \neq \alpha_i$. Since $x \notin W', x_{\alpha_i} \neq x^*_{\alpha_i}$. Thus $x \notin Y_{\alpha_j}$ if $j \neq i$. Hence, if $x \in A_i$, then $x \in G_i$. Let $x \in B_i$. Then $x_{\alpha_i} \in V'_i$. If $x_{\alpha_i} \in clU'_{\alpha_i}$, then $x_{\alpha_i} \in U_{\alpha_i}$. Since $x_{\alpha_j} = x^*_{\alpha_j} \in U_{\alpha_j}$ for each $j \neq i, x \in W$. Therefore $x \notin B$. This is a contradiction. Thus $x_{\alpha_i} \notin clU'_{\alpha_i}$. Hence $x \in H_i$.

Put $G = W' \cup \bigcup_{i=1}^m G_i$ and $H = \bigcup_{i=1}^m H_i$. Then $G$ and $H$ are open sets in $\sigma_1$ such that

(5) $A \subset G, B \subset H$

and

(6) $G \cap H = \emptyset$.

(5) is obvious. (6) follows from (4) and (7) and (8) below.

(7) $W' \cap H_i = \emptyset$ for each i.

(8) $i \neq j \Rightarrow G_i \cap H_j = \emptyset$.

(7) is obvious.

*Proof of* (8). If $x \in G_i \cap H_j$, then $x_{\alpha_i} \neq x^*_{\alpha_i}$ and $x_{\alpha_j} \neq x^*_{\alpha_j}$. Thus $|Q(x)| \geq 2$, which contradicts $x \in \sigma_1$.

**Example 1.** There exists a $\sigma$-product such that each finite subproduct is normal and $\sigma_2$ is not normal.

To prove Example 1, we shall use the following lemma.

**Lemma 2** ([2])**.** *Let $X$ be a space and $A$ be a closed set of $X$ which is not a $G_\delta$-subset of $X$. Let $F$ be Bing's Example G or H constructed by $P = X \setminus A$. Then $X \times F$ is not normal.*

**Definition 2** (Bing's Example G ([1]))**.** Let $P$ be an uncountable set and $Q = \{q | q \subset P\}$. Put $F = \{f : Q \to 2\}$. For each $p \in P$, define $f_p$ as follows:

$$f_p(q) = \begin{cases} 1 & \text{if } p \in q, \\ 0 & \text{if } p \notin q. \end{cases}$$

Put $F_P = \{f_p |\ p \in P\}$. Define the topology of $F$ as follows: each $f_p$ has a neighborhood base in Cartesian product topology and for each $f \in F \setminus F_P, \{f\}$ is open. For each $r \in R = Q^{<\omega}$, put $V(f_p; r) = \{f \in F | f(q) = f_p(q)$ for each $q \in r\}$. Then $\mathcal{V}(f_p) = \{V(f_p; r) | r \in R\}$ is a neighborhood base of $f_p$.

*Proof of Lemma* 2. In case $F$ is Bing's Example H, the proof is in [2]. The proof is quite similar for the case of Bing's Example G. But, since [2] is not widely known, we shall sketch the proof of the case of Bing's Example G. Let $C = A \times F$ and $D = \{\langle p, f_p\rangle | p \in P\}$. Then $C$ and $D$ are disjoint closed subsets in $X \times F$ and are not separated by open sets in $X \times F$. To show this, let $O$ be an arbitrary open set in $X \times F$ such that $D \subset O$. For each $p \in P$, there is a member $V(f_p; r_p)$ of $\mathcal{V}(f_p)$ such that $\bigcup_{p\in P}(\{p\} \times V(f_p; r_p)) \subset O$. Let us put $P_i = \{p | p \in P, |r_p| = i\}$ for each $i < \omega$. Since $A$ is not a $G_\delta$-set of $X$, there is an $i$ such that $A \cap cl(P_i) \neq \emptyset$. Let us fix this $i$. Let $x_0 \in A \cap cl(P_i)$. Let us put $P_i(r) = \{p \in P_i | r_p \supset r\}$ for each $r \in R$. Then we can prove that there exists an element $r^* \in R$ satisfying the following conditions: (1) $x_0 \in cl(P_i(r^*)$, (2) $x_0 \notin cl(P_i(r^* \cup \{q\})$ for each $q \in Q \setminus r^*$.
Let us put $R^* = \{s | s \subset r^*\}$. For each $s \in R^*$, we define an element $q_s$ of $Q$ by $q_s = \bigcap\{q | q \in s\} \setminus \bigcup\{q | q \in r^* \setminus s\}$. Then $\{q_s | s \in R^*\}$ is a finite cover of $P$. Therefore, we can choose a member $s_0$ of $R^*$ such that $x_0 \in cl(P_i(r^*) \cap q_{s_0})$. For this $s_0$, we choose an element $p^* \in q_{s_0}$. Then $\langle x_0, f_{p^*}\rangle \in C$. Next we shall prove that $\langle x_0, f_{p^*}\rangle \in cl(O)$. Let $U$ be an arbitrary open neighborhood of $x_0$ in $X$ and $V(f_{p^*}; r)$ be an arbitary member of $\mathcal{V}(f_{p^*})$. Then we can choose an open neighborhood $U'$ of $x_0$ in $X$ such that $U' \cap (\bigcup\{P_i(r^* \cup \{q\}) | q \in r \setminus r^*\}) = \emptyset$. Then $U \cap U' \cap P_i(r^*) \cap q_{s_0} \neq \emptyset$. Let $p \in U \cap U' \cap P_i(r^*) \cap q_{s_0}$. Then $r_p \cap (r \setminus r^*) = \emptyset$. Define $f : Q \to 2$ by

$$f(q) = \begin{cases} 1 & \text{if } p^* \in q \in r \text{ or } p \in q \in r_p, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in V(f_{p^*}; r) \cap V(f_p; r_p)$. Therefore $(U \times V(f_{p^*}; r)) \cap (\{p\} \times V(f_p; r_p)) \neq \emptyset$.

*Proof of Example* 1. Let $\sigma = \sigma(2^{\omega_1})$ be the $\sigma$-product with the base point $0^*$. Then

(i) $0^*$ is not a $G_\delta$-set in $\sigma_1$.

*Proof.* Assume that there exist countable open sets $\{W_n | n = 1, 2, ...\}$ in $\sigma_1$ such that $\{0^*\} = \bigcap_{n<\omega_1} W_n$. Then there are finite sets $a_n \subset \omega_1$ and open sets $U_{n,\alpha}$ in $X_\alpha$ for each $\alpha \in a_n$ such that $0^* \in U_n \equiv \{x \in \sigma_1 | x_\alpha = 0 \text{ for each } \alpha \in a_n\} \subset W_n$ for each $n$. Then $\{0^*\} = \bigcap_{n<\omega} U_n$. Since $\omega_1 \setminus \bigcup_{n<\omega} a_n \neq \emptyset$, choose an element $\alpha \in \omega_1 \setminus \bigcup_{n<\omega} a_n$. Let us define $x = (x_\beta)_{\beta<\omega_1}$ by $x_\alpha = 1, x_\beta = 0$ if $\beta \neq \alpha$. Then $x \in \bigcap_{n<\omega} U_n \setminus \{0^*\}$, which contradicts $\{0^*\} = \bigcap_{n<\omega} U_n$.

(ii) Put $P = \sigma_1 \setminus \{0^*\}$ and let $F$ be Bing's Example G constructed by $P$. Then $\sigma_1 \times F$ is not normal.

Let $\mathcal{S} = \{2_\alpha | \alpha < \omega_1\} \cup \{F\}$ where $2_\alpha = 2$ for each $\alpha$ and let $\sigma' = \sigma(\mathcal{S})$ with the base point $\langle 0^*, f^*\rangle, f^* \in F \setminus F_P$. Then

(iii) $\sigma'_2$ is not normal.

Since normality is inherited by closed subspaces, (iii) follows from (ii) and (iv) below.

(iv) $\sigma_1 \times F$ is a closed subset of $\sigma'_2$.

*Proof.* It is obvious that $\sigma_1 \times F \subset \sigma'_2$. Let $y \in \sigma'_2 \setminus \sigma_1 \times F$. Then we can denote $y = \langle x, f\rangle, x \in \sigma, f \in F$. Since $\langle x, f\rangle \notin \sigma_1 \times F, x \notin \sigma_1$. Thus $x \in \sigma_2 \setminus \sigma_1$. Hence $f = f^*$. Since $(\sigma_2 \setminus \sigma_1) \times \{f^*\}$ is an open set in $(\sigma_2) \times F$, $((\sigma_2 \setminus \sigma_1) \times \{f^*\} = (\sigma_2 \setminus \sigma_1) \times F) \cap \sigma'_2$ and $(\sigma_2 \setminus \sigma_1) \times \{f^*\}$ is an open neighborhood of $y$ in $\sigma'_2$ such that $((\sigma_2 \setminus \sigma_1) \times \{f^*\}) \cap (\sigma_1 \times F) = \emptyset$.

## 3. Starcompactness

It is well known that every non-trivial $\sigma$-product is not countably compact. A space $X$ is called countably compact if every countable open cover of $X$ has a finite subcover, or, which is equivalent, every infinite subset has a limit point. A space $X$ is called strongly starcompact if for every open cover $\mathcal{U}$ there exists a finite set $B$ of $X$ such that $st(B,\mathcal{U}) = X$. Here $st(B,\mathcal{U}) = \bigcup\{U \in \mathcal{U} | U \cap B \neq \emptyset\}$. A space $X$ is called starcompact if for every open cover $\mathcal{U}$ there exists a finite subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $st(\bigcup\mathcal{U}',\mathcal{U}) = X$.

It is known that countably compact $\Rightarrow$ strongly star compact $\Rightarrow$ starcompact, and for $T_2$-spaces, the converses hold.

**Theorem 3.** *Every non-trivial $\sigma$-product is not strongly starcompact.*

*Proof.* Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be a family of spaces such that $|\Omega| \geq \omega$. Let $\sigma = \sigma(2^\omega)$ be the $\sigma$-product with the base point $0^*$ and let $\sigma' = \sigma(\mathcal{S})$. Let us choose a point $a_\alpha \in X_\alpha$ such that $a_\alpha \neq x^*_\alpha$ for each $\alpha \in \Omega$. Let us consider $\omega \subset \Omega$. Define $f : \sigma \to \sigma'$ as follows: for each $x = (x_\alpha)_{\alpha\in\omega} \in \sigma$, let $f(x) = (y_\alpha)_{\alpha\in\Omega}$ be

$$y_\alpha = \begin{cases} a_\alpha & \text{if } \alpha \in \omega \text{ and } x_\alpha = 1, \\ x^*_\alpha & \text{otherwise.} \end{cases}$$

Then $f$ is a homeomorhism from $\sigma$ onto $f(\sigma)$ such that $f(0^*) = x^*$ and $f(\sigma)$ is a closed subset of $\sigma'$. To prove that $f(\sigma)$ is a closed subset of $\sigma'$, let $y \in \sigma' \smallsetminus f(\sigma)$. Then there exists $\alpha \in \Omega \smallsetminus \omega$ such that $y_\alpha \neq x^*_\alpha$ and put $U = \{z \in \sigma' | z_\alpha \neq x^*_\alpha\}$. Then $U$ is an open neighborhood of $y$ in $\sigma'$ such that $U \cap f(\sigma) = \emptyset$.

*Claim.* $\sigma'$ is not strongly starcompact.

*Proof.* Let $U_0 = \{x \in \sigma' | x_0 \neq a_0\}$ and let $U_n = \{x \in \sigma' | x_0 \neq x^*_0, x_1 \neq x^*_1, ..., x_{n-1} \neq x^*_{n-1}, x_n \neq a_n\}$ for each $n \geq 1$. Put $\mathcal{U} = \{U_n | n \in \omega\} \cup \{\sigma' \smallsetminus f(\sigma)\}$. Then

(i) $\mathcal{U}$ is an open cover of $\sigma'$;

(ii) there is no finite $B \subset \sigma'$ such that $st(B,\mathcal{U}) = \sigma'$.

Proof of (i) is easy and so we omit it.

*Proof of* (ii). Let $B$ be a finite set of $\sigma'$. Then $B \subset \sigma'_n$ for some $n$. Since $U_i \cap \sigma'_n = \emptyset$ for each $i \geq n+1$, $st(B,\mathcal{U}) \subset \bigcup_{i\leq n} U_i \cup (\sigma' \smallsetminus f(\sigma))$. However $\bigcup_{i\leq n} U_i \cup (\sigma' \smallsetminus f(\sigma)) \neq \sigma'$. To show this, let us define $z = (z_\alpha)_{\alpha\in\Omega}$ as follows:

$$z_\alpha = \begin{cases} a_\alpha & \text{if } \alpha \in \omega \text{ and } \alpha \leq n, \\ x^*_\alpha & \text{otherwise.} \end{cases}$$

Then $z \in \sigma'$ and $z \notin \bigcup_{i\leq n} U_i \cup (\sigma' \smallsetminus f(\sigma))$. Therefore $st(B,\mathcal{U}) \neq \sigma'$.

**Theorem 4.** *If each $X_\alpha \in \mathcal{S}$ is strongly starcompact (resp. starcompact), then $\sigma_1$ is strongly starcompact (resp. starcompact).*

Proofs are easy and so we omit them.

Since for $T_2$-spaces, starcompactness is equivalent to countable compactness, every non-trivial $\sigma$-product of $T_2$-spaces is not starcompact. However, for $T_1$-spaces, non-trivial $\sigma$-product can be starcompact.

We denote $\sigma(X^\tau)$ with the base point $x^*$ by $\sigma(X^\tau; x^*)$.

**Example 2.** There exists a starcompact space $X$ such that $X$ is not a $T_2$-space and not countably compact and (1) $\sigma = \sigma(X^\omega; a^*)$ is starcompact for some $a \in X$. (2) $\sigma' = \sigma(X^\omega; b^*)$ is not starcompact for some $b \in X$.

*Proof.* Let $X = \mathbf{R}$ with the topology as follows: let $\mathcal{U}(0) = \{U | 0 \in U, |X \setminus U| \le \omega\}$ be the neighborhoods of 0 in $X$ and for each $x \neq 0, \mathcal{U}(x) = \{U | U$ is a neighborhood of $x$ in usual topology of $\mathbf{R}\}$ be the base of $x$ in $X$. Then $X$ is a $T_1$-space and not a $T_2$-space and $X$ is starcompact and not countably compact. By Theorem 5 below, (1) $\sigma = \sigma(X^\omega; 0^*)$ is starcompact. By Theorem 6, (2) $\sigma' = \sigma(X^\omega; 1^*)$ is not starcompact.

**Theorem 5.** *Let $X$ satisfy the condition: "there exists $a \in X$ such that if $U$ and $V$ are open sets in $X$ and $a \in U$, then $U \cap V \neq \emptyset$". Let $\tau$ be an infinite cardinal number, and $\sigma = \sigma(X^\tau; a^*)$. Then* (i) *$\sigma$ is starcompact,* (ii) *$\sigma_n$ is starcompact* $(\forall n)$, (iii) *$X^\tau$ is starcompact. Moreover let $\sigma' = \sigma(X^\omega; b^*), b \in X$, $b \neq a$. Then* (iv) *$\sigma'_n$ is starcompact* $(\forall n)$.

*Proof. Proof of* (i). Let $\mathcal{G}$ be an arbitrary open cover of $\sigma$. Let us choose $G_0 \in \mathcal{G}$ such that $a^* \in G_0$. There are a finite set $\{\alpha_i | i = 1, 2, ..., m\} \subset \tau$ and open sets $U_{\alpha_i}$ in $X_{\alpha_i}$ such that $a^* \in W_0 \equiv \{x \in \sigma | x_{\alpha_i} \in U_{\alpha_i}$ for $i = 1, 2, ..., m\} \subset G_0$. For each $x \in \sigma \setminus W_0$, let us choose $G_x \in \mathcal{G}$ such that $x \in G_x$. Then there are a finite set $\{\beta_j | j = 1, 2, ..., k\}$ and open sets $V_{\alpha_i}$ in $X_{\alpha_i}$ and $V_{\beta_j}$ in $X_{\beta_j}$ such that $x \in W_x \equiv \{y \in \sigma | y_{\alpha_i} \in V_{\alpha_i}$ for $i = 1, 2, ..., m; y_{\beta_j} \in V_{\beta_j}$ for $j = 1, 2, ..., k\} \subset G_x$. Since $a \in U_{\alpha_i}, U_{\alpha_i} \cap V_{\alpha_i} \neq \emptyset$ for each $i = 1, 2, ..., m$. Thus $W_0 \cap W_x \neq \emptyset$ and so $G_0 \cap G_x \neq \emptyset$. Therefore $st(G_0, \mathcal{G}) = \sigma$.
Proofs of (ii) and (iii) are similar.

*Proof of* (iv). First we define $\mathcal{B}, y^s$ as follows: put $\mathcal{B} = \{W | W$ is a basic open set in $\sigma'\}$. Here $W \subset \sigma'$ is called a basic open set in $\sigma'$ if $W = \{x \in \sigma' | x_i \in U_i$ for each $i \le n\}, n \in \omega, U_i$ is an open set in $X_i$ for each $i \le n$. Define $l(W) = n$. For each $s \in [\omega]^{<\omega}$, define $y^s = (y^s_i)_{i \in \omega}$ as follows:

$$y^s_i = \begin{cases} a & \text{if } i \in s, \\ b & \text{if } i \notin s. \end{cases}$$

To prove (iv), let $\mathcal{G}$ be an arbitrary open cover of $\sigma'$. Let us prove that there exists a finite subfamily $\mathcal{G}_n$ of $\mathcal{G}$ such that $st(\bigcup \mathcal{G}_n, \mathcal{G}) \supset \sigma'_n$ for each $n \in \omega$.

(I) Let us choose an element $G_0 \in \mathcal{G}$ such that $b^* \in G_0$. Then there is a set $W_0 \in \mathcal{B}$ such that

(0-1) $b^* \in W_0 \subset G_0$.

Put $l(W_0) = k_0$. Then

(0-2) For $x \in \sigma'$, if $l > k_0$ for each $l \in Q(x)$, then $x \in W_0$.

(II) For each $n = 1, 2, ...$, inductively we can choose $k_n, S_n$ and $\mathcal{W}_n$ satisfying the conditions:

(1) $k_n \in \omega, k_n < k_{n+1} (\forall n \ge 1), k_0 = k_1$.

(2) $S_n \subset S_{n+1} (\forall n \ge 1)$.

(3) $(n-1)$ $S_n = \{s : s \subset \omega, 1 \le |s| \le n, l \le k_n (\forall l \in s)\}$.

$(n-2)$ $\mathcal{W}_n = \{W_s | s \in S_n\} \cup \{W_0\} \subset \mathcal{B}, \mathcal{W}_n$ is a partial refinement of $\mathcal{G}$.

$(n-3)$ $y^s \in W_s (\forall s \in S_n), k_n < l(W_s) \le k_{n+1} (\forall s \in S_n)$.

$(n-4)$ $st(\cup \mathcal{W}_n, \mathcal{G}) \supset \sigma'_n$.

Assume that $k_n, S_n$ and $\mathcal{W}_n$ have been chosen for each $n \le m$. Define $k_{m+1} = max\{l(W_s) | s \in S_m\}$ and $S_{m+1} = \{s : s \subset \omega, 1 \le |s| \le m+1, l \le k_{m+1} (\forall l \in s)\}$.

For each $s \in S_{m+1} \setminus S_m$, choose $G_s \in \mathcal{G}$ and $W_s \in \mathcal{B}$ such that $y^s \in W_s \subset G_s$. Put $\mathcal{W}_{m+1} = \{W_s | s \in S_{m+1}\} \cup \{W_0\}$. Then $k_{m+1}, S_{m+1}$ and $\mathcal{W}_{m+1}$ satisfy the conditions. We only prove $(m+1-4)$ because others are obvious.

*Proof of* $(m+1-4)$. Let $x \in \sigma'_{m+1} \setminus \sigma'_m$. Put $s = Q(x)$. Then $|s| = m+1$. Choose $G_x \in \mathcal{G}$ and $W_x \in \mathcal{B}$ such that $x \in W_x \subset G_x$. Put $s = \{l_i | i = 1, 2, ..., m+1\}$ such that $l_i < l_{i+1} (\forall i)$.

(i) If $l_{m+1} \leq k_{m+1}$, then $s \in S_{m+1}$ and $W_x \cap W_s \neq \emptyset$. To show this, let $W_s = \{y \in \sigma' | y_i \in U_i^s$ for $\forall i \leq l(W_s)\}$ and $W_x = \{y \in \sigma' | y_i \in U_i^x$ for $\forall i \leq l(W_x)\}$. Here we may assume that $l(W_s), l(W_x) \geq k_{m+1}$. Since $a \in U_i^s$ for each $i \in s, U_i^s \cap U_i^x \neq \emptyset$ for each $i \in s$. For each $i \notin s, i \leq min\{l(W_s), l(W_x)\}, b \in U_i^s \cap U_i^x$. Hence $U_i^s \cap U_i^x \neq \emptyset$. Therefore $W_x \cap W_s \neq \emptyset$. Thus $x \in st(\bigcup \mathcal{W}_{m+1}, \mathcal{G})$.

(ii) Case $l_{m+1} > k_{m+1}$. If $l_1 > k_0$, then $x \in W_0$ by (0-2).

Suppose $l_1 \leq k_0$. Then there is a $j$ such that $1 \leq j \leq m+1, Q(x) \cap \{l | k_j < l \leq k_{j+1}\} = \emptyset$. Let $j$ be the greatest such number. Then there exists $t \in \{i | i = 1, 2, ..., m+1\}$ such that $l_t \leq k_j$ and $l_{t+1} > k_{j+1}$. Then $t \leq j$. Therefore $s = \{l_i | i \leq t\} \in S_j$. Since $l(W_s) \leq k_{j+1}$, it is easy to see that $W_x \cap W_s \neq \emptyset$.

Since $\mathcal{W}_n$ is a partial refinement of $\mathcal{G}$, there exists $\mathcal{G}_n$ such that $st(\bigcup \mathcal{G}_n, \mathcal{G}) \supset \sigma'_n$.

**Theorem 6.** *There are a countable closed subset $A$ of $X$ and a pairwise disjoint open family $\mathcal{U} = \{U(a) | a \in A\}$ such that $a \in U(a)$ for each $a \in A$ and $X \setminus A \neq \emptyset$. Let $\sigma = \sigma(X^\omega; a^*), a \in A$. Then $\sigma$ is not starcompact.*

*Proof.* Put $A = \{a_n | n = 1, 2, ...\}, U(a_n) = U_n$ for each $n$ and put $U_0 = X \setminus A$. Then $\bigcup_{n<\omega} U_n = X$. Without loss of generality we may assume that $a = a_1$. For each $k = 1, 2, ...$, let $\Lambda_k \equiv \{(l_0, l_1, ..., l_{k-1}, 1) \in [\omega]^{k+1} | l_0 \neq 1, l_{k-1} \neq 1\}$ and put $\Lambda = \bigcup_{1 \leq k} \Lambda_k$.

Define $G_1 \equiv \{x \in \sigma | x_0 \in U_1\}$ and $G_\lambda \equiv \{x \in \sigma | x_i \in U_{l_i}$ for $i = 0, 1, ..., k-1; x_k \in U_1\}$ for each $\lambda = (l_0, l_1, ..., l_{k-1}, 1) \in \Lambda$ and put $\mathcal{G} = \{G_\lambda | \lambda \in \Lambda\} \cup \{G_1\}$. Then

(1) $\mathcal{G}$ is an open cover of $\sigma$.

(2) For any finite subfamily $\mathcal{G}'$ of $\mathcal{G}, st(\bigcup \mathcal{G}', \mathcal{G}) \neq \sigma$.

*Proof of* (1). Let $x \in \sigma$. If $x_0 \notin U_1$, then $x_0 \in U_i$ for some $i \neq 1$. Since $|Q(x) = \{i | x_i \neq a_1\}| < \omega$, there is a $k$ such that $x_i = a_1$ for each $i \geq k$ and $x_{k-1} \neq a_1$. Then $x \in G_\lambda$ for some $\lambda \in \Lambda$.

*Proof of* (2). Let $\mathcal{G}'$ be an arbitrary finite subfamily of $\mathcal{G}$. Then there exists $k > 1$ such that

(2-1) $G_\lambda \notin \mathcal{G}'$ for each $\lambda \in \bigcup_{m>k} \Lambda_m$.

Define $x = (x_i)_{i \in \omega}$ as follows:

$$x_i = \begin{cases} a_k & \text{if } i \leq k, \\ a_1 & \text{if } i > k. \end{cases}$$

Then

(2-2) $x \notin st(\bigcup \mathcal{G}', \mathcal{G})$.

*Proof of* (2-2). Let $x \in G \in \mathcal{G}$. Then $G \neq G_1$. Therefore $G = G_\lambda$ with $\lambda = (l_0, l_1, ..., l_{m-1}, 1) \in \Lambda_m$. Then $m > k$. To show this, assume that $m \leq k$. Then $x_m = a_k$ by the definition of $x$. Since $a_k \notin U_1, x \notin G_\lambda$, which is a contradiction. Therefore $m > k$. It is easy to see that $l_i = k$ for each $i \leq k$ and $l_i = 1$ for each $i \geq k+1$. Thus $l_0 \geq 2$ and so $U_{l_0} \cap U_1 = \emptyset$. Hence $G_\lambda \cap G_1 = \emptyset$. If $m \geq k+2$, then $l_{m-1} = 1$. This contradicts the definition of $\lambda$. Thus $m = k+1$. Let $G_\mu \in \mathcal{G}'$ with $\mu = (s_0, s_1, ..., s_{t-1}, 1) \in \Lambda_t$. Then $t \leq k$. Therefore $l_t = k$. Since $U_k \cap U_1 = \emptyset, G_\lambda \cap G_\mu = \emptyset$.

*Remark.* For $\sigma$ and $\sigma'$ in Example 2, $\sigma_n$ and $\sigma'_n$ are starcompact for each $n$ by Theorems 5 and 6.

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