PSEUDONORMALITY AND STARCOMPACTNESS
OF $\sigma$-PRODUCTS

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(Communicated by Alan Dow)

Abstract. In this paper we shall prove the following: For every non-trivial $\sigma$-product $\sigma$, of uncountable number of spaces, having at least two points, $\sigma \setminus \sigma_n$ is not pseudonormal. And every non-trivial $\sigma$-product is not strongly starcompact.

1. Introduction

Throughout this paper we assume that each space is a $T_1$-space having at least two points. We recall the definition of $\sigma$-products which were introduced by H. H. Corson [3].

Definition 1. Let $S = \{X_\alpha | \alpha \in \Omega\}$ be a family of spaces. “$\sigma = \sigma(S)$ is called a $\sigma$-product of $S$” if there is a point $x^* = (x^*_\alpha)_{\alpha \in \Omega} \in X = \prod \{X_\alpha | \alpha \in \Omega\}$ (called the base point of $\sigma$) such that $\sigma$ is the subspace of $X$ consisting of $x \in X$ such that $Q(x)$ is finite. Here $Q(x) = \{\alpha | x^*_\alpha \neq x_\alpha\}$. Let $[\Omega]^n = \{\alpha \subset \Omega : |\alpha| = n\}$ for each $n \in \omega$ and put $[\Omega]^<\omega = \bigcup([\Omega]^n : n \in \omega)$. Here $|\alpha|$ denotes the cardinal number of $\alpha$.

$\Sigma = \{x \in X : |Q(x)| \leq \omega\}$ is called a $\Sigma$-product of $S$.

A $\sigma$-product $\sigma$ (resp. $\Sigma$-product $\Sigma$) is called non-trivial if $\sigma \neq X$ (resp. $\Sigma \neq X$).

Let $X$ be a space and $\tau$ be an infinite cardinal number such that $|\Omega| = \tau$. In case $X_\alpha = X$ for each $\alpha \in \Omega$, let us denote $\sigma(S)$ by $\sigma(X^\tau)$. For $a \in X$, we denote by $a^* = (a_\alpha)_{\alpha \in \Omega}$, $a_\alpha = a$ for each $\alpha \in \Omega$.

For a finite subset $F$ of $\Omega$, $\Pi \{X_\alpha | \alpha \in F\}$ is said to be a finite subproduct of $\sigma$.

The following fact concerning $\sigma$-products is known.

Fact. Let $\sigma = \sigma(S)$ and $\sigma_n = \{x \in \sigma : |Q(x)| \leq n\}$ for each $n \in \omega$. Then $\sigma_n$ is closed in $\sigma$.

In this paper we investigate normality-type properties of special subspaces of $\sigma$-products and compactness-type properties of $\sigma$-products.

In 1959, Corson [3] proved that for every non-trivial $\Sigma$-product $\Sigma$, a subspace $\Sigma \setminus \{x\}$ is not normal for every point $x \in \Sigma$. In 1978, A. P. Kombarov [5] proved that if a set $Z$ is closed in the $\tau$-envelope $Y = Y(x^*, \tau) = \{y \in X = \prod \{X_\alpha | \alpha \in \Omega\} : |Q(y)| < \tau\}$ and $|\bigcup \{Q(z) : z \in Z\}| < \tau$, then $Y \setminus Z$ is a non-normal subset.
of $Y$. As corollaries of this theorem we have non-normality of $\Sigma \setminus \{x\}$, where $\Sigma$ is a non-trivial $\Sigma$-product and $x \in \Sigma$, and non-normality of $\sigma \setminus \{x\}$, where $\sigma$ is a non-trivial $\sigma$-product of uncountable number of spaces and $x \in \sigma$.

A space $X$ is called pseudonormal if any two disjoint closed sets, one of which is countable, are separated by open sets in $X$. Obviously, any normal space is pseudonormal.

In 1996, Kombarov \cite{9} proved that if $Y$ is a $\tau$-envelope of spaces $X_{\alpha}, \alpha \in \Omega$, $|\Omega| \geq \max\{\omega_1, \tau\}$, then a subspace $Y \setminus \{x\}$ is not pseudonormal for every $x \in Y$. In particular he obtained the following.

**Theorem A** (Kombarov \cite{9}). Let $\mathcal{S} = \{X_{\alpha}|\alpha \in \Omega\}$ be a family of spaces such that $|\Omega| \geq \omega_1$ and let $\sigma = \sigma(\mathcal{S})$. Then $\sigma \setminus \sigma_0$ is not pseudonormal.

In this paper we shall prove a generalization of Theorem A.

2. Normality and Pseudonormality

**Theorem 1.** Let $\mathcal{S} = \{X_{\alpha}|\alpha \in \Omega\}$ be a family of spaces such that $|\Omega| \geq \omega_1$ and let $\sigma = \sigma(\mathcal{S})$. Then $\sigma \setminus \sigma_n$ is not pseudonormal for each $n \in \omega$.

**Lemma 1.** Let $\mathcal{S} = \{X_{\alpha}|\alpha \in \Omega\}$ such that $|\Omega| \geq \omega_1$. Let $\sigma = \sigma(2^{\omega_1})$ be the $\sigma$-product with the base point $0^*$. Here $2 = \{0, 1\}$ is the discrete space of two points. Then there is a homeomorphism $f$ from $\sigma$ onto $f(\sigma) \subset \sigma'$ such that $f(0^*) = x^*$ and $f(\sigma \setminus \sigma_n)$ is a closed subset of $\sigma' \setminus \sigma'_n$. Here $\sigma' = \sigma(\mathcal{S})$ with the base point $x^*$.

**Proof.** Let us choose a point $a_{\alpha} \in X_{\alpha}$ such that $a_{\alpha} \neq x_{\alpha}^*$ for each $\alpha \in \Omega$. Let us consider $\omega_1 \subset \Omega$. Let $f : \sigma \to \sigma'$ as follows: for each $x = (x_{\alpha})_{\alpha \in \omega_1} \in \sigma$, let $f(x) = (y_{\alpha})_{\alpha \in \Omega}$ be

$$y_{\alpha} = \begin{cases} a_{\alpha} & \text{if } \alpha \in \omega_1 \text{ and } x_{\alpha} = 1, \\ x_{\alpha}^* & \text{otherwise.} \end{cases}$$

Then $f$ has the desired properties. To prove that $f(\sigma \setminus \sigma_n)$ is a closed subset of $\sigma' \setminus \sigma'_n$, let $y \in (\sigma' \setminus \sigma'_n) \cap f(\sigma \setminus \sigma_n)$. Then $Q(y) \cap (\Omega \setminus \omega_1) \neq \emptyset$. Let us choose an element $\alpha \in Q(y) \cap (\Omega \setminus \omega_1)$ and put $U = \{z \in \sigma' \setminus \sigma'_n|z_{\alpha} \neq x_{\alpha}^*\}$. Then $U$ is an open neighborhood of $y$ in $\sigma' \setminus \sigma'_n$ such that $U \cap f(\sigma \setminus \sigma_n) = \emptyset$.

Since pseudonormality is inherited by closed subspaces, Theorem 1 follows from Proposition 1 below by using Lemma 1.

**Proposition 1.** Let $\sigma = \sigma(2^{\omega_1})$ be the $\sigma$-product with the base point $0^*$. Then $\sigma \setminus \sigma_n$ is not pseudonormal for each $n \in \omega$.

**Proof.** We denote $\sigma = \{f : \omega_1 \to 2|Q(f) \text{ is finite}\}$. Here $Q(f) = \{\alpha \in \omega_1|f(\alpha) = 1\}$.

Put $G = \sigma \setminus \sigma_n$. Let us choose a subset $A \subset \omega_1$ such that $|A| = n$. For each $\alpha \in \omega_1$, let us define $f^\alpha : \omega_1 \to 2$ as follows:

$$f^\alpha(\beta) = \begin{cases} 1 & \text{if } \beta \in A \cup \{\alpha\}, \\ 0 & \text{if } \beta \in \omega_1 \setminus (A \cup \{\alpha\}). \end{cases}$$

Then

(1) $f^\alpha \in \sigma_{\alpha+1} \setminus \sigma_n$ for each $\alpha \in \omega_1 \setminus A$.

Let us choose subsets $\Gamma_1$ and $\Gamma_2$ of $\omega_1$ such that $|\Gamma_1| = \omega_1, |\Gamma_2| = \omega, \Gamma_1 \cap \Gamma_2 = \emptyset, (\Gamma_1 \cup \Gamma_2) \cap A = \emptyset, \omega_1 = \Gamma_1 \cup \Gamma_2 \cup A$. Put $E = \{f^\alpha|\alpha \in \Gamma_1\}$ and $F = \{f^\alpha|\alpha \in \Gamma_2\}$.
Then \( E \cap F = \emptyset, |F| = \omega \) and

(i) \( E \) and \( F \) are closed subsets in \( G \);

(ii) \( E \) and \( F \) are not separated by open sets in \( G \).

**Proof of (i).** To prove that \( E \) is closed in \( G \), let \( f \in G \setminus E \). If \( |Q(f)| \geq n + 2 \), then \( f \notin \sigma_{n+1} \). Put \( U = G \setminus \sigma_{n+1} \). Then \( U \) is a neighborhood of \( f \) in \( G \) such that \( U \cap E = \emptyset \). If \( |Q(f)| = n + 1 \), then \( Q(f) \setminus A \neq \emptyset \) because \( |A| = n \). Let \( \alpha \in Q(f) \setminus A \). Then \( f(\alpha) = 1 \). If \( |A \cap Q(f)| = n \), then \( \alpha \in \Gamma_2 \) because \( f \notin E \). Put \( U = \{ g \in G | g(\alpha) = 1 \} \). Then \( U \) is a neighborhood of \( f \) in \( G \) such that \( U \cap E = \emptyset \). Therefore it is proved that \( E \) is closed in \( G \). Quite similarly it is proved that \( F \) is closed in \( G \).

**Proof of (ii).** Let \( U \) be an arbitrary open set in \( G \) such that \( E \subset U \). Then, for each \( \alpha \in \Gamma_1 \), there is a finite set \( r_\alpha \) of \( \omega_1 \) such that \( A \cup \{ \alpha \} \subset r_\alpha \) and \( f^\alpha \in U_{\alpha} \equiv \{ g \in G | g(\beta) = 1 \} \) for each \( \beta \in A \cup \{ \alpha \}, g(\beta) = 0 \) for each \( \beta \in r_\alpha \setminus (A \cup \{ \alpha \}) \} \subset U \). By \( \hat{\alpha} \text{SANIN'S lemma, there are an uncountable set } \Gamma^* \subset \Gamma_1 \text{ and a finite set } r^* \subset \omega_1 \text{ such that } \{ r_\alpha \setminus r^* | \alpha \in \Gamma^* \} \text{ is disjoint. Since } \Gamma_2 \text{ is infinite and } r^* \text{ is finite, } \Gamma_2 \setminus r^* \neq \emptyset \).

Let \( \alpha^* \in \Gamma_2 \setminus r^* \). Then

(2) \( f^{\alpha^*} \in F \cap clU \).

It is obvious that \( f^{\alpha^*} \in F \). Therefore it is sufficient to prove the following.

Claim. \( f^{\alpha^*} \in clU \).

To prove this, let \( V \) be an arbitrary open set in \( G \) such that \( f^{\alpha^*} \in V \). Then there is a finite set \( r \) of \( \omega_1 \) such that \( A \cup \{ \alpha^* \} \subset r \) and \( f^{\alpha^*} \in V \equiv \{ g \in G | g(\beta) = 1 \} \) for each \( \beta \in A \cup \{ \alpha^* \}, g(\beta) = 0 \) for each \( \beta \in r \setminus (A \cup \{ \alpha^* \}) \} \subset V \).

Since \( \Gamma^* \) is uncountable and \( r \) and \( r^* \) are finite, there is an element \( \beta^* \in \Gamma^* \) such that \( r \cap (r^* \setminus \beta^*) = \emptyset \) and \( \beta^* \in r \setminus r^* \). Then it is easy to see that

(3) \( \alpha^* \neq \beta^*, \alpha^*, \beta^* \notin A \).

Define \( g^* : \omega_1 \to 2 \) by

\[
g^*(\alpha) = \begin{cases} 1 & \text{if } \alpha \in A \cup \{ \alpha^*, \beta^* \}, \\ 0 & \text{if } \alpha \in \omega_1 \setminus (A \cup \{ \alpha^*, \beta^* \}). \end{cases}
\]

Then \( g^* \in \sigma_{n+2} \setminus \sigma_{n+1} \) and therefore \( g^* \in G \). Moreover we have

(4) \( g^* \in V' \cap U_{\beta^*} \).

To prove that \( g^* \in V' \), let \( \beta \in r \setminus (A \cup \{ \alpha^* \}) \). Then \( \beta \notin r_{\beta^*} \setminus r^* \). Thus \( \beta \neq \beta^* \).

Hence \( g(\beta) = 0 \). Therefore \( g \in V' \). To prove that \( g^* \in U_{\beta^*} \), let \( \beta \in r_{\beta^*} \setminus (A \cup \{ \beta^* \}) \). If \( \beta \in r^* \), then \( \beta \neq \alpha^* \). Thus \( g(\beta) = 0 \). If \( \beta \notin r^* \), then \( \beta \in r_{\beta^*} \setminus r^* \). Therefore \( \beta \neq \beta^* \).

It is known that there exists a non-normal \( \sigma \)-product such that each finite sub-product is normal (cf. [3]).

**Theorem 2.** If each \( X_\alpha \in S \) is normal (resp. pseudonormal), then \( \sigma_1 \) is normal (resp. pseudonormal).

**Proof.** We shall write only the proof of normality because the proof of pseudonormality is quite similar. Let \( A \) and \( B \) be disjoint closed subsets in \( \sigma_1 \). Let \( x^* \in A \).

There are finite sets \( \{ \alpha_i | i = 1, 2, ..., m \} \) of \( \Omega \) and open sets \( U_{\alpha_i} \) in \( X_{\alpha_i} \) such that \( x^* \in W \equiv \{ x \in \sigma_1 | x_{\alpha_i} \in U_{\alpha_i} \text{ for } i = 1, 2, ..., m \} \) and \( clW \cap B = \emptyset \). Since \( X_{\alpha_i} \) is normal, there is an open set \( U'_{\alpha_i} \) in \( X_{\alpha_i} \) such that \( x_{\alpha_i}^* \in U'_{\alpha_i} \) and \( clU'_{\alpha_i} \subset U_{\alpha_i} \), for
Then (Bing’s Example G ([1]))

Let \( Y_\alpha = \{x \in \sigma_1 | x_\beta = x_\beta^* \text{ for each } \beta \neq \alpha \} \), i.e., \( Y_\alpha = X_n \times \{x_\alpha^* \} \). Here

\[ z_\alpha^* = (x_\beta^*)_{\beta \in \Omega \setminus \{\alpha\}}. \]

Then it is easy to see that

1. \( Y_\alpha \subset W' \) for each \( \alpha \in \Omega \setminus \{\alpha_i | i = 1, 2, \ldots, m\} \);
2. \( Y_\alpha \) is closed in \( \sigma_1 \).

Put \( A_i = A \cap Y_{\alpha_i} \setminus W' \) and \( B_i = B \cap Y_{\alpha_i} \setminus W' \). Then \( A_i \) and \( B_i \) are disjoint closed sets in \( Y_{\alpha_i} \) and \( x_\alpha^* \notin A_i, x_\alpha^* \notin B_i \). Since \( Y_{\alpha_i} \) is normal, there are open sets \( V_i \) and \( V_i' \) in \( X_{\alpha_i} \) such that \( V_i \cap V_i' = \emptyset, x_\alpha^* \notin V_i \cup V_i', A_i \subset V_i \times \{x_\alpha^* \} \) and \( B_i \subset V_i' \times \{x_\alpha^* \} \). Put \( G_i = \{x \in \sigma_1 | x_{\alpha_i} \in V_i \} \setminus \bigcup \{Y_{\alpha_j} | j \neq i \} \) and \( H_i = \{x \in \sigma_1 | x_{\alpha_i} \in V_i' \setminus \text{cl}U_{\alpha_i}' \} \setminus \bigcup \{Y_{\alpha_j} | j \neq i \} \). Then \( G_i \) and \( H_i \) are open sets in \( \sigma_1 \) such that

3. \( A_i \subset G_i, B_i \subset H_i \);
4. \( G_i \cap H_i = \emptyset \).

**Proof of (3).** Let \( x \in A_i \cup B_i \). Then \( x_\alpha = x_\alpha^* \) for each \( \alpha \neq \alpha_i \). Since \( x \notin W', x_{\alpha_i} \neq x_{\alpha_i}^* \). Thus \( x \notin Y_{\alpha_i} \) if \( j \neq i \). Hence, if \( x \in A_i \), then \( x \in G_i \). Let \( x \in B_i \). Then \( x_{\alpha_i} \in V_i' \). If \( x_{\alpha_i} \in \text{cl}U_{\alpha_i}' \), then \( x_{\alpha_i} \in U_{\alpha_i} \). Since \( x_{\alpha_i} = x_{\alpha_i}^* \in U_{\alpha_i} \) for each \( j \neq i, x \in W \). Therefore \( x \notin B_i \). This is a contradiction. Thus \( x_{\alpha_i} \notin \text{cl}U_{\alpha_i}' \). Hence \( x \in H_i \).

Put \( G = W' \cup \bigcup_{i=1}^m G_i \) and \( H = \bigcup_{i=1}^m H_i \). Then \( G \) and \( H \) are open sets in \( \sigma_1 \) such that

5. \( A \subset G, B \subset H \)

and

6. \( G \cap H = \emptyset \).

(5) is obvious, (6) follows from (4) and (7) and (8) below.

7. \( W' \cap H_i = \emptyset \) for each \( i \).

8. \( i \neq j \Rightarrow G_i \cap H_j = \emptyset \).

(7) is obvious.

**Proof of (8).** If \( x \in G_i \cap H_j \), then \( x_{\alpha_i} \neq x_{\alpha_i}^* \) and \( x_{\alpha_i} \neq x_{\alpha_j}^* \). Thus \( |Q(x)| \geq 2 \), which contradicts \( x \in \sigma_1 \).

**Example 1.** There exists a \( \sigma \)-product such that each finite subproduct is normal and \( \sigma_2 \) is not normal.

To prove Example 1, we shall use the following lemma.

**Lemma 2 ([2]).** Let \( X \) be a space and \( A \) be a closed set of \( X \) which is not a \( G_\delta \)-subset of \( X \). Let \( F \) be Bing’s Example \( G \) or \( H \) constructed by \( P = X \setminus A \). Then \( X \times F \) is not normal.

**Definition 2** (Bing’s Example G ([3])). Let \( P \) be an uncountable set and \( Q = \{q | q \subset P \} \). Put \( F = \{f : Q \rightarrow 2 \} \). For each \( p \in P \), define \( f_p \) as follows:

\[ f_p(q) = \begin{cases} 1 & \text{if } p \in q, \\ 0 & \text{if } p \notin q. \end{cases} \]

Put \( F_P = \{f_p | p \in P \} \). Define the topology of \( F \) as follows: each \( f_p \) has a neighborhood base in Cartesian product topology and for each \( f \in F \setminus F_P, \{f \} \) is open. For each \( r \in R = Q^{<\omega} \), put \( V(f_p; r) = \{f \in F | f(q) = f_p(q) \} \) for each \( q \in r \).

Then \( \mathcal{V}(f_p) = \{V(f_p; r) | r \in R \} \) is a neighborhood base of \( f_p \).
Proof of Lemma 2. In case $F$ is Bing’s Example H, the proof is in [2]. The proof is quite similar for the case of Bing’s Example G. But, since [2] is not widely known, we shall sketch the proof of the case of Bing’s Example G. Let $C = A \times F$ and $D = \{ (p, f_p) | p \in P \}$. Then $C$ and $D$ are disjoint closed subsets in $X \times F$ and are not separated by open sets in $X \times F$. To show this, let $O$ be an arbitrary open set in $X \times F$ such that $D \subset O$. For each $p \in P$, there is a member $V(f_p; r_p) \subset V(f_p)$ such that $\bigcup_{p \in P} \{ p \} \times V(f_p; r_p) \subset O$. Let us put $P_i = \{ p | p \in P, r_p = i \}$ for each $i < \omega$. Since $A$ is not a $G_\delta$-set of $X$, there is an $i$ such that $A \cap cl(P_i) \neq \emptyset$. Let us fix this $i$. Let $x_0 \in A \cap cl(P_i)$. Let us put $P_i(r) = \{ p \in P_i | r_p \supset r \}$ for each $r \in R$. Then we can prove that there exists an element $r^* \in R$ satisfying the following conditions: (1) $x_0 \notin cl(P_i(r^*))$, (2) $x_0 \notin cl(P_i(r^* \cup \{ q \})$ for each $q \in Q \setminus r^*$. Let us put $R^* = \{ s | s \subset r^* \}$. For each $s \in R^*$, we define an element $q_s$ of $Q$ by $q_s = \bigcap \{ q | q \in s \} \setminus \bigcup \{ q | q \in r^* \setminus s \}$. Then $\{ q_s | s \in R^* \}$ is a finite cover of $P$. Therefore, we can choose a member $s_0$ of $R^*$ such that $x_0 \in cl(P_i(r^*) \cap q_{s_0})$. For this $s_0$, we choose an element $p^* \in q_{s_0}$. Then $\langle x_0, f_{p^*} \rangle \in C$. Next we shall prove that $\langle x_0, f_{p^*} \rangle \in cl(O)$. Let $U$ be an arbitrary open neighborhood of $x_0$ in $X \times V(f_{p^*}; r)$ be an arbitrary member of $V(f_{p^*})$. Then we can choose an open neighborhood $U'$ of $x_0$ in $X$ such that $U' \cap \bigcup \{ P_i(r^*) \cup \{ q \} | q \in r \setminus r^* \} = \emptyset$. Then $U \cap U' \cap P_i(r^*) \cap q_{s_0} \neq \emptyset$. Let $p \in U \cap U' \cap P_i(r^*) \cap q_{s_0}$. Then $r_{p^*} \cup (r \setminus r^*) = \emptyset$. Define $f : Q \rightarrow 2$ by

$$f(q) = \begin{cases} 1 & \text{if } p^* \in q \in r \text{ or } p \in q \in r_{p^*}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in V(f_{p^*}; r) \cap V(f_{p^*}; r_p)$. Therefore $(U \times V(f_{p^*}; r)) \cap (\{ p \} \times V(f_{p^*}; r_p)) \neq \emptyset$.

Proof of Example 1. Let $\sigma = \sigma(2^{\omega_1})$ be the $\sigma$-product with the base point $0^*$. Then

(i) $0^*$ is not a $G_\delta$-set in $\sigma_1$.

Proof. Assume that there exist countable open sets $\{ W_n | n = 1, 2, \ldots \}$ in $\sigma_1$ such that $\{ 0^* \} = \bigcap_{n < \omega} W_n$. Then there are finite sets $a_n \subset \omega_1$ and open sets $U_n, n$ in $X_n$ for each $n \in a_n$ such that $0^* \in U_n \equiv \{ x \in \sigma_1 | x_{\alpha} = 0 \text{ for each } \alpha \in a_n \} \subset W_n$ for each $n$. Then $\{ 0^* \} = \bigcap_{n < \omega} U_n$. Since $\omega_1 \setminus \bigcup_{n < \omega} a_n \neq \emptyset$, choose an element $\alpha \in \omega_1 \setminus \bigcup_{n < \omega} a_n$. Let us define $x = (x_{\beta})_{\beta < \omega_1}$ by $x_\alpha = 1, x_\beta = 0$ if $\beta \neq \alpha$. Then $x \in \bigcap_{n < \omega} U_n \cap \{ 0^* \}$, which contradicts $\{ 0^* \} = \bigcap_{n < \omega} U_n$.

(ii) Put $P = \sigma_1 \setminus \{ 0^* \}$ and let $F$ be Bing’s Example G constructed by $P$. Then $\sigma_1 \times F$ is not normal.

Let $S = \{ 2_{\alpha} | \alpha < \omega_1 \} \cup \{ F \}$ where $2_{\alpha} = 2$ for each $\alpha$ and let $\sigma' = \sigma(S)$ with the base point $\{ 0^*, f^* \}, f^* \in F \setminus F^p$. Then

(iii) $\sigma'_1$ is not normal.

Since normality is inherited by closed subspaces, (iii) follows from (ii) and (iv) below.

(iv) $\sigma_1 \times F$ is a closed subset of $\sigma'_2$.

Proof. It is obvious that $\sigma_1 \times F \subset \sigma'_2$. Let $y = (x, f), x \in \sigma_2 \setminus \sigma_1 \times F$. Then we can denote $y = (x, f), x \in \sigma_2 \setminus \sigma_1 \times F$. Since $(x, f) \notin \sigma_1 \times F, x \notin \sigma_1$. Thus $x \in \sigma_2 \setminus \sigma_1$. Hence $f = f^*$. Since $(\sigma_2 \setminus \sigma_1) \times \{ f^* \}$ is an open set in $(\sigma_2 \setminus \sigma_1) \times F$, $((\sigma_2 \setminus \sigma_1) \times \{ f^* \}) = ((\sigma_2 \setminus \sigma_1) \times F) \cap \sigma'_2$ and $(\sigma_2 \setminus \sigma_1) \times \{ f^* \}$ is an open neighborhood of $y$ in $\sigma'_2$ such that $((\sigma_2 \setminus \sigma_1) \times \{ f^* \}) \cap (\sigma_1 \times F) = \emptyset$. 

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3. Starcompactness

It is well known that every non-trivial $\sigma$-product is not countably compact. A space $X$ is called countably compact if every countable open cover of $X$ has a finite subcover, or, which is equivalent, every infinite subset has a limit point. A space $X$ is called strongly starcompact if for every open cover $U$ there exists a finite set $B$ of $X$ such that $st(B, U) = X$. Here $st(B, U) = \bigcup \{ U \in U | U \cap B \neq \emptyset \}$. A space $X$ is called starcompact if for every open cover $U$ there exists a finite subfamily $U'$ of $U$ such that $st(U', U) = X$.

It is known that countably compact $\Rightarrow$ strongly star compact $\Rightarrow$ starcompact, and for $T_2$-spaces, the converses hold.

**Theorem 3.** Every non-trivial $\sigma$-product is not strongly star compact.

**Proof.** Let $\mathcal{S} = \{ X_\alpha | \alpha \in \Omega \}$ be a family of spaces such that $|\Omega| \geq \omega$. Let $\sigma = \sigma(2^\omega)$ be the $\sigma$-product with the base point $0^*$ and let $\sigma' = \sigma(\mathcal{S})$. Let us choose a point $a_\alpha \in X_\alpha$ such that $a_\alpha \neq x^*_\alpha$ for each $\alpha \in \Omega$. Let us consider $\omega \subset \Omega$. Define $f: \sigma \to \sigma'$ as follows: for each $x = (x_\alpha)_{\alpha \in \omega} \in \sigma$, let $f(x) = (y_\alpha)_{\alpha \in \omega}$ be

$$y_\alpha = \begin{cases} a_\alpha & \text{if } \alpha \in \omega \text{ and } x_\alpha = 1, \\ x^*_\alpha & \text{otherwise.} \end{cases}$$

Then $f$ is a homeomorphism from $\sigma$ onto $f(\sigma)$ such that $f(0^*) = x^*$ and $f(\sigma)$ is a closed subset of $\sigma'$. To prove that $f(\sigma)$ is a closed subset of $\sigma'$, let $y \in \sigma' \setminus f(\sigma)$. Then there exists $\alpha \in \Omega \setminus \omega$ such that $y_\alpha \neq x^*_\alpha$ and put $U = \{ z \in \sigma' | z_\alpha \neq x^*_\alpha \}$. Then $U$ is an open neighborhood of $y$ in $\sigma'$ such that $U \cap f(\sigma) = \emptyset$.

Claim. $\sigma'$ is not strongly starcompact.

**Proof.** Let $U_0 = \{ x \in \sigma' | x_0 \neq a_0 \}$ and let $U_n = \{ x \in \sigma' | x_0 \neq x^*_0, x_1 \neq x^*_1, \ldots, x_{n-1} \neq x^*_n, x_n \neq a_n \}$ for each $n \geq 1$. Put $\mathcal{U} = \{ U_n | n \in \omega \} \cup \{ \sigma' \setminus f(\sigma) \}$. Then

(i) $\mathcal{U}$ is an open cover of $\sigma'$;

(ii) there is no finite $B \subset \sigma'$ such that $st(B, \mathcal{U}) = \sigma'$.

Proof of (ii) is easy and so we omit it.

**Proof of (ii).** Let $B$ be a finite set of $\sigma'$. Then $B \subset \sigma'_n$ for some $n$. Since $U_i \cap \sigma'_i = \emptyset$ for each $i \geq n + 1$, $st(B, \mathcal{U}) \subset \bigcup_{i \leq n} U_i \cup (\sigma' \setminus f(\sigma))$. However $\bigcup_{i \leq n} U_i \cup (\sigma' \setminus f(\sigma)) \neq \sigma'$. To show this, let us define $z = (z_\alpha)_{\alpha \in \Omega}$ as follows:

$$z_\alpha = \begin{cases} a_\alpha & \text{if } \alpha \in \omega \text{ and } \alpha \leq n, \\ x^*_\alpha & \text{otherwise.} \end{cases}$$

Then $z \in \sigma'$ and $z \notin \bigcup_{i \leq n} U_i \cup (\sigma' \setminus f(\sigma))$. Therefore $st(B, \mathcal{U}) \neq \sigma'$.

**Theorem 4.** If each $X_\alpha \in \mathcal{S}$ is strongly starcompact (resp. starcompact), then $\sigma_1$ is strongly starcompact (resp. starcompact).

Proofs are easy and so we omit them.

Since for $T_2$-spaces, starcompactness is equivalent to countable compactness, every non-trivial $\sigma$-product of $T_2$-spaces is not starcompact. However, for $T_1$-spaces, non-trivial $\sigma$-product can be starcompact.

We denote $\sigma(X^*)$ with the base point $x^*$ by $\sigma(X^*; x^*)$.

**Example 2.** There exists a starcompact space $X$ such that $X$ is not a $T_2$-space and not countably compact and (1) $\sigma = \sigma(X^*; a^*)$ is starcompact for some $a \in X$. (2) $\sigma' = \sigma(X^*; b^*)$ is not starcompact for some $b \in X$. 

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Proof. Let $X = \mathbb{R}$ with the topology as follows: let $U(0) = \{U|0 \in U, |X \setminus U| \leq \omega\}$ be the neighborhoods of 0 in $X$ and for each $x \neq 0$, $U(x) = \{U|U$ is a neighborhood of $x$ in usual topology of $\mathbb{R}\}$ be the base of $x$ in $X$. Then $X$ is a $T_1$-space and not a $T_2$-space and $X$ is starcompact and not countably compact. By Theorem 5 below, (1) $\sigma = \sigma(X^\omega;0^*)$ is starcompact. By Theorem 6, (2) $\sigma' = \sigma(X^\omega;1^*)$ is not starcompact.

**Theorem 5.** Let $X$ satisfy the condition: “there exists $a \in X$ such that if $U$ and $V$ are open sets in $X$ and $a \in U$, then $U \cap V \neq \emptyset$”. Let $\tau$ be an infinite cardinal number, and $\sigma = \sigma(X^\tau;a^*)$. Then (i) $\sigma$ is starcompact, (ii) $\sigma_n$ is starcompact ($\forall n$), (iii) $X^* \sigma$ is starcompact. Moreover let $\sigma' = \sigma(X^\omega;b^*)$, $b \in X$, $b \neq a$. Then (iv) $\sigma'_n$ is starcompact ($\forall n$).

Proof. Proof of (i). Let $\mathcal{G}$ be an arbitrary open cover of $\sigma$. Let us choose $G_0 \in \mathcal{G}$ such that $a^* \in G_0$. There are a finite set \{\alpha_i|i = 1,2,...,m\} $\in \mathcal{T}$ and open sets $U_{\alpha_i}$ in $X_{\alpha_i}$ such that $a^* \in W_0 \equiv \{x \in \sigma| x_{\alpha_i} \in U_{\alpha_i} \text{ for } i = 1,2,...,m\} \subset G_0$. For each $x \in \sigma \setminus W_0$, let us choose $G_x \in \mathcal{G}$ such that $x \in G_x$. Then there are a finite set \{\beta_j|j = 1,2,...,k\} and open sets $V_{\alpha_i}$ in $X_{\alpha_i}$ and $V_j$ in $X_{\beta_j}$ such that $x \in W_x \equiv \{y \in \sigma| y_{\alpha_i} \in V_{\alpha_i} \text{ for } i = 1,2,...,m; y_{\beta_j} \in V_j \text{ for } j = 1,2,...,k\} \subset G_2$. Since $a \in U_{\alpha_i}, U_n \cap V_{\alpha_i} \neq \emptyset$ for each $i = 1,2,...,m$. Thus $W_0 \cap W_x \neq \emptyset$ and so $G_0 \cap G_x \neq \emptyset$. Therefore $s\tau(G_0,\mathcal{G}) = \sigma$.

Proofs of (ii) and (iii) are similar.

Proof of (iv). First we define $\mathcal{B}$, $y^*$ as follows: put $\mathcal{B} = \{W|W$ is a basic open set in $\sigma^\prime\}$. Here $W \subset \sigma'$ is called a basic open set in $\sigma'$ if $W = \{x \in \sigma'| x_i \in U_i \text{ for each } i \leq n\}, n \in \omega, U_i$ is an open set in $X_i$ for each $i \leq n$. Define $l(W) = n$. For each $s \in [\omega]^\omega\setminus$, define $y^* = (y^*_i)_{i \in \omega}$ as follows:

\[
y^*_i = \begin{cases} 
    a & \text{if } i \in s, \\
    b & \text{if } i \notin s.
\end{cases}
\]

To prove (iv), let $\mathcal{G}$ be an arbitrary open cover of $\sigma'$. Let us prove that there exists a finite subfamily $\mathcal{G}_n$ of $\mathcal{G}$ such that $s\tau(\bigcup \mathcal{G}_n, \mathcal{G}) \supset \sigma'_n$ for each $n \in \omega$.

(I) Let us choose an element $G_0 \in \mathcal{G}$ such that $b^* \in G_0$. Then there is a set $W_0 \in \mathcal{B}$ such that

(0-1) $b^* \in W_0 \subset G_0$.

Put $l(W_0) = k_0$. Then

(0-2) For each $x \in \sigma'$, if $l > k_0$ for each $l \in Q(x)$, then $x \in W_0$.

(II) For each $n = 1,2,...$, inductively we can choose $k_n, S_n$ and $W_n$ satisfying the conditions:

(1) $k_n \in \omega, k_n < k_{n+1} (\forall n \geq 1), k_0 = k_1$.

(2) $S_n \subset S_{n+1}(\forall n \geq 1)$.

(3) \((n - 1) S_n = \{s|s \subset \omega, 1 \leq |s| \leq n, l \leq k_n (\forall l \in s)\}.

(4) \((n - 2) W_n = \{W_s|s \subset S_n\} \cup \{W_0\} \subset \mathcal{B}. W_n$ is a partial refinement of $\mathcal{G}.

(5) \((n - 3) y^* = W_s (\forall s \in S_n), k_n < l(W_s) \leq k_{n+1} (\forall s \in S_n).

(6) \((n - 4) s\tau(\bigcup \mathcal{W}_n, \mathcal{G}) \supset \sigma'_n$.\)

Assume that $k_n, S_n$ and $W_n$ have been chosen for each $n \leq m$. Define $k_{m+1} = \max\{l(W_s)|s \subset S_m\}$ and $S_{m+1} = \{s|s \subset \omega, 1 \leq |s| \leq m + 1, l \leq k_{m+1} (\forall l \in s)\}.

For each $s \in S_{m+1} \setminus S_m$, choose $G_s \in \mathcal{G}$ and $W_s \in \mathcal{B}$ such that $y^* \in W_s \subset G_s$. Put $W_{m+1} = \{W_s|s \in S_{m+1}\} \cup \{W_0\}$. Then $k_{m+1}, S_{m+1}$ and $W_{m+1}$ satisfy the conditions. We only prove $(m + 1 - 4)$ because others are obvious.
Theorem 6. There are a countable closed subset of $A$ of $X$ and a pairwise disjoint open family $\mathcal{U} = \{U(a)|a \in A\}$ such that $a \in U(a)$ for each $a \in A$ and $X \setminus A \neq \emptyset$. Let $\sigma = \sigma(X^\omega; a^*)$, $a \in A$. Then $\sigma$ is not starcompact.

Proof. Let $A = \{a_n|n = 1, 2, ...,\}, U(a_n) = U_n$ for each $n$ and put $U_0 = X \setminus A$. Then $\bigcup_{n \in \omega} U_n = X$. Without loss of generality we may assume that $a = a_1$. For each $k = 1, 2, ..., \Lambda_k \equiv \{(l_0,l_1,...,l_{k-1},1) \in [\omega]^{k+1}|l_0 \neq 1, l_{k-1} \neq 1\}$ and put $\Lambda = \bigcup_{1 \leq k} \Lambda_k$.

Define $G_1 \equiv \{x \in \sigma|x_0 \in U_1\}$ and $G_\lambda \equiv \{x \in \sigma|x_i \in U_i\}$ for $i = 0, 1, ..., k-1; x_k \in U_1\}$ for each $\lambda = (l_0,l_1,...,l_{k-1},1) \in \Lambda$ and put $\mathcal{G} = \{G_\lambda|\lambda \in \Lambda\} \cup \{G_1\}$. Then

(1) $\mathcal{G}$ is an open cover of $\sigma$.

(2) For any finite subfamily $\mathcal{G}'$ of $\mathcal{G}$, $st(\bigcup \mathcal{G}', \mathcal{G}) \neq \sigma$.

Proof of (1). Let $x \in \sigma$. If $x_0 \notin U_1$, then $x_0 \in U_i$ for some $i \neq 1$. Since $|Q(x) = \{|i|x_i \neq a_1\}| < \omega$, there is a $k$ such that $x_i = a_1$ for each $i \geq k$ and $x_{k-1} \neq a_1$. Then $x \in G_\lambda$ for some $\lambda \in \Lambda$.

Proof of (2). Let $\mathcal{G}'$ be an arbitrary finite subfamily of $\mathcal{G}$. Then there exists $k > 1$ such that

(2-1) $G_\lambda \notin \mathcal{G}'$ for each $\lambda \in \bigcup_{m > k} \Lambda_m$.

Define $x = (x_i)_{i \in \omega}$ as follows:

$$x_i = \begin{cases} a_k & \text{if } i \leq k, \\ a_1 & \text{if } i > k. \end{cases}$$

Then

(2-2) $x \notin st(\bigcup \mathcal{G}', \mathcal{G})$.

Proof of (2-2). Let $x \in G \in \mathcal{G}$. Then $G \neq G_1$. Therefore $G = G_\lambda$ with $\lambda = (l_0,l_1,...,l_{m-1},1) \in \Lambda_m$. Then $m > k$. To show this, assume that $m \leq k$. Then $x_m = a_k$ by the definition of $x$. Since $a_k \notin U_1$, $x \notin G_\lambda$, which is a contradiction. Therefore $m > k$. It is easy to see that $l_i = k$ for each $i \leq k$ and $l_i = 1$ for each $i \geq k + 1$. Thus $l_0 \geq 2$ and so $U_{l_0} \cap U_1 = \emptyset$. Hence $G_\lambda \cap G_1 = \emptyset$. If $m \geq k + 2$, then $l_{m-1} = 1$. This contradicts the definition of $\lambda$. Thus $m = k + 1$. Let $G_n \in \mathcal{G}'$ with $n = (s_0,s_1,...,s_{k-1},1) \in \Lambda_k$. Then $t \leq k$. Therefore $l_t = k$. Since $U_k \cap U_1 = \emptyset$, $G_\lambda \cap G_n = \emptyset$.

Remark. For $\sigma$ and $\sigma'$ in Example 2, $\sigma_n$ and $\sigma'_n$ are starcompact for each $n$ by Theorems 5 and 6.
ACKNOWLEDGEMENT

The author is grateful to the referee for his helpful comments.

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