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PSEUDONORMALITY AND STARCOMPACTNESS OF σ-PRODUCTS

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Abstract. In this paper we shall prove the following: For every non-trivial σ-product σ, of uncountable number of spaces, having at least two points, $\sigma \setminus \sigma_n$ is not pseudonormal. And every non-trivial σ-product is not strongly starcompact.

1. Introduction

Throughout this paper we assume that each space is a T_1-space having at least two points. We recall the definition of σ-products which were introduced by H. H. Corson [4].

Definition 1. Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be a family of spaces. "$\sigma = \sigma(\mathcal{S})$ is called a σ-product of $\mathcal{S}$" if there is a point $x^* = (x^*_\alpha)_{\alpha\in\Omega} \in X = \Pi\{X_\alpha | \alpha \in \Omega\}$ (called the base point of σ) such that σ is the subspace of X consisting of $x \in X$ such that $Q(x)$ is finite. Here $Q(x) = \{\alpha | \alpha \in \Omega, x_\alpha \neq x^*_\alpha\}$. Let $[\Omega]^n = \{a \subset \Omega : |a| = n\}$ for each $n \in \omega$ and put $[\Omega]^{<\omega} = \bigcup\{[\Omega]^n : n \in \omega\}$. Here $|a|$ denotes the cardinal number of a.

$\Sigma = \{x \in X : |Q(x)| \leq \omega\}$ is called a Σ-product of $\mathcal{S}$.

A σ-product σ (resp. Σ-product Σ) is called non-trivial if $\sigma \neq X$ (resp. $\Sigma \neq X$).

Let X be a space and τ be an infinite cardinal number such that $|\Omega| = \tau$. In case $X_\alpha = X$ for each $\alpha \in \Omega$, let us denote $\sigma(\mathcal{S})$ by $\sigma(X^\tau)$. For $a \in X$, we denote by $a^* = (a_\alpha)_{\alpha\in\Omega}, a_\alpha = a$ for each $\alpha \in \Omega$.

For a finite subset F of $\Omega, \Pi\{X_\alpha | \alpha \in F\}$ is said to be a finite subproduct of σ.

The following fact concerning σ-products is known.

Fact. Let $\sigma = \sigma(\mathcal{S})$ and $\sigma_n = \{x \in \sigma : |Q(x)| \leq n\}$ for each $n \in \omega$. Then σ_n is closed in σ.

In this paper we investigate normality-type properties of special subspaces of σ-products and compactness-type properties of σ-products.

In 1959, Corson [4] proved that for every non-trivial Σ-product Σ, a subspace $\Sigma \setminus \{x\}$ is not normal for every point $x \in \Sigma$. In 1978, A. P. Kombarov [8] proved that if a set Z is closed in the τ-envelope $Y = Y(x^*, \tau) = \{y \in X = \Pi\{X_\alpha | \alpha \in \Omega\} : |Q(y)| < \tau\}$ and $|\bigcup\{Q(z) : z \in Z\}| < \tau$, then $Y \setminus Z$ is a non-normal subset

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of Y. As corollaries of this theorem we have non-normality of $\Sigma \setminus \{x\}$, where Σ is a non-trivial Σ-product and $x \in \Sigma$, and non-normality of $\sigma \setminus \{x\}$, where σ is a non-trivial σ-product of uncountable number of spaces and $x \in \sigma$.

A space X is called pseudonormal if any two disjoint closed sets, one of which is countable, are separated by open sets in X. Obviously, any normal space is pseudonormal.

In 1996, Kombarov [9] proved that if Y is a τ-envelope of spaces $X_\alpha, \alpha \in \Omega$, $|\Omega| \geq max\{\omega_1, \tau\}$, then a subspace $Y \setminus \{x\}$ is not pseudonormal for every $x \in Y$. In particular he obtained the following.

Theorem A (Kombarov [9])**.** *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be a family of spaces such that $|\Omega| \geq \omega_1$ and let $\sigma = \sigma(\mathcal{S})$. Then $\sigma \setminus \sigma_0$ is not pseudonormal.*

In this paper we shall prove a generalization of Theorem A.

2. Normality and pseudonormality

Theorem 1. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be a family of spaces such that $|\Omega| \geq \omega_1$ and let $\sigma = \sigma(\mathcal{S})$. Then $\sigma \setminus \sigma_n$ is not pseudonormal for each $n \in \omega$.*

Lemma 1. *Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ such that $|\Omega| \geq \omega_1$. Let $\sigma = \sigma(2^{\omega_1})$ be the σ-product with the base point 0^*. Here $2 = \{0, 1\}$ is the discrete space of two points. Then there is a homeomorphism f from σ onto $f(\sigma) \subset \sigma'$ such that $f(0^*) = x^*$ and $f(\sigma \setminus \sigma_n)$ is a closed subset of $\sigma' \setminus \sigma'_n$. Here $\sigma' = \sigma(\mathcal{S})$ with the base point x^*.*

Proof. Let us choose a point $a_\alpha \in X_\alpha$ such that $a_\alpha \neq x^*_\alpha$ for each $\alpha \in \Omega$. Let us consider $\omega_1 \subset \Omega$. Let $f : \sigma \to \sigma'$ as follows: for each $x = (x_\alpha)_{\alpha \in \omega_1} \in \sigma$, let $f(x) = (y_\alpha)_{\alpha \in \Omega}$ be

$$y_\alpha = \begin{cases} a_\alpha & \text{if } \alpha \in \omega_1 \text{ and } x_\alpha = 1, \\ x^*_\alpha & \text{otherwise.} \end{cases}$$

Then f has the desired properties. To prove that $f(\sigma \setminus \sigma_n)$ is a closed subset of $\sigma' \setminus \sigma'_n$, let $y \in (\sigma' \setminus \sigma'_n) \setminus f(\sigma \setminus \sigma_n)$. Then $Q(y) \cap (\Omega \setminus \omega_1) \neq \emptyset$. Let us choose an element $\alpha \in Q(y) \cap (\Omega \setminus \omega_1)$ and put $U = \{z \in \sigma' \setminus \sigma'_n | z_\alpha \neq x^*_\alpha\}$. Then U is an open neighborhood of y in $\sigma' \setminus \sigma'_n$ such that $U \cap f(\sigma \setminus \sigma_n) = \emptyset$.

Since pseudonormality is inherited by closed subspaces, Theorem 1 follows from Proposition 1 below by using Lemma 1.

Proposition 1. *Let $\sigma = \sigma(2^{\omega_1})$ be the σ-product with the base point 0^*. Then $\sigma \setminus \sigma_n$ is not pseudonormal for each $n \in \omega$.*

Proof. We denote $\sigma = \{f : \omega_1 \to 2 | Q(f) \text{ is finite}\}$. Here $Q(f) = \{\alpha \in \omega_1 | f(\alpha) = 1\}$.

Put $G = \sigma \setminus \sigma_n$. Let us choose a subset A $\subset \omega_1$ such that $|A| = n$. For each $\alpha \in \omega_1$, let us define $f^\alpha : \omega_1 \to 2$ as follows:

$$f^\alpha(\beta) = \begin{cases} 1 & \text{if } \beta \in A \cup \{\alpha\}, \\ 0 & \text{if } \beta \in \omega_1 \setminus (A \cup \{\alpha\}). \end{cases}$$

Then

(1) $f^\alpha \in \sigma_{n+1} \setminus \sigma_n$ for each $\alpha \in \omega_1 \setminus A$.

Let us choose subsets Γ_1 and Γ_2 of ω_1 such that $|\Gamma_1| = \omega_1, |\Gamma_2| = \omega, \Gamma_1 \cap \Gamma_2 = \emptyset, (\Gamma_1 \cup \Gamma_2) \cap A = \emptyset, \omega_1 = \Gamma_1 \cup \Gamma_2 \cup A$. Put $E = \{f^\alpha | \alpha \in \Gamma_1\}$ and $F = \{f^\alpha | \alpha \in \Gamma_2\}$.

Then $E \cap F = \emptyset, |F| = \omega$ and

(i) E and F are closed subsets in G;

(ii) E and F are not separated by open sets in G.

Proof of (i). To prove that E is closed in G, let $f \in G \setminus E$. If $|Q(f)| \geq n+2$, then $f \notin \sigma_{n+1}$. Put $U = G \setminus \sigma_{n+1}$. Then U is a neighborhood of f in G such that $U \cap E = \emptyset$. If $|Q(f)| = n+1$, then $Q(f) \setminus A \neq \emptyset$ because $|A| = n$. Let $\alpha \in Q(f) \setminus A$. Then $f(\alpha) = 1$. If $|A \cap Q(f)| = n$, then $\alpha \in \Gamma_2$ because $f \notin E$. Put $U = \{g \in G | g(\alpha) = 1\}$. Then U is a neighborhood of f in G such that $U \cap E = \emptyset$. If $|A \cap Q(f)| < n$, then there exists an element $\beta \in A$ such that $f(\beta) = 0$. Put $U = \{g \in G | g(\beta) = 0\}$. Then U is a neighborhood of f in G such that $U \cap E = \emptyset$. Therefore it is proved that E is closed in G. Quite similarly it is proved that F is closed in G.

Proof of (ii). Let U be an arbitrary open set in G such that $E \subset U$. Then, for each $\alpha \in \Gamma_1$, there is a finite set r_α of ω_1 such that $A \cup \{\alpha\} \subset r_\alpha$ and $f^\alpha \in U_\alpha \equiv \{g \in G | g(\beta) = 1$ for each $\beta \in A \cup \{\alpha\}, g(\beta) = 0$ for each $\beta \in r_\alpha \setminus (A \cup \{\alpha\})\} \subset U$. By Šanin's lemma, there are an uncountable set $\Gamma^* \subset \Gamma_1$ and a finite set $r^* \subset \omega_1$ such that $\{r_\alpha \setminus r^* | \alpha \in \Gamma^*\}$ is disjoint. Since Γ_2 is infinite and r^* is finite, $\Gamma_2 \setminus r^* \neq \emptyset$. Let $\alpha^* \in \Gamma_2 \setminus r^*$. Then

(2) $f^{\alpha^*} \in F \cap clU$.

It is obvious that $f^{\alpha^*} \in F$. Therefore it is sufficient to prove the following.

Claim. $f^{\alpha^*} \in clU$.

To prove this, let V be an arbitrary open set in G such that $f^{\alpha^*} \in V$. Then there is a finite set r of ω_1 such that $A \cup \{\alpha^*\} \subset r$ and $f^{\alpha^*} \in V' \equiv \{g \in G | g(\beta) = 1$ for each $\beta \in A \cup \{\alpha^*\}, g(\beta) = 0$ for each $\beta \in r \setminus (A \cup \{\alpha^*\})\} \subset V$.

Since Γ^* is uncountable and r and r^* are finite, there is an element $\beta^* \in \Gamma^*$ such that $r \cap (r_{\beta^*} \setminus r^*) = \emptyset$ and $\beta^* \in r_{\beta^*} \setminus r^*$. Then it is easy to see that

(3) $\alpha^* \neq \beta^*, \alpha^*, \beta^* \notin A$.

Define $g^* : \omega_1 \to 2$ by

$$g^*(\alpha) = \begin{cases} 1 & \text{if } \alpha \in A \cup \{\alpha^*, \beta^*\}, \\ 0 & \text{if } \alpha \in \omega_1 \setminus (A \cup \{\alpha^*, \beta^*\}). \end{cases}$$

Then $g^* \in \sigma_{n+2} \setminus \sigma_{n+1}$ and therefore $g^* \in G$. Moreover we have

(4) $g^* \in V' \cap U_{\beta^*}$.

To prove that $g^* \in V'$, let $\beta \in r \setminus (A \cup \{\alpha^*\})$. Then $\beta \notin r_{\beta^*} \setminus r^*$. Thus $\beta \neq \beta^*$. Hence $g(\beta) = 0$. Therefore $g \in V'$. To prove that $g^* \in U_{\beta^*}$, let $\beta \in r_{\beta^*} \setminus (A \cup \{\beta^*\})$. If $\beta \in r^*$, then $\beta \neq \alpha^*$. Thus $g(\beta) = 0$. If $\beta \notin r^*$, then $\beta \in r_{\beta^*} \setminus r^*$. Therefore $\beta \notin r$. Then $\beta \neq \alpha^*$ and so $g(\beta) = 0$. Hence $g \in U_{\beta^*}$.

It is known that there exists a non-normal σ-product such that each finite subproduct is normal (cf. [3]).

Theorem 2. *If each $X_\alpha \in \mathcal{S}$ is normal (resp. pseudonormal), then σ_1 is normal (resp. pseudonormal).*

Proof. We shall write only the proof of normality because the proof of pseudonormality is quite similar. Let A and B be disjoint closed subsets in σ_1. Let $x^* \in A$. There are a finite set $\{\alpha_i | i = 1, 2, ..., m\}$ of Ω and open sets U_{α_i} in X_{α_i} such that $x^* \in W \equiv \{x \in \sigma_1 | x_{\alpha_i} \in U_{\alpha_i}$ for $i = 1, 2, ..., m\}$ and $clW \cap B = \emptyset$. Since X_{α_i} is normal, there is an open set U'_{α_i} in X_{α_i} such that $x^*_{\alpha_i} \in U'_{\alpha_i}$ and $clU'_{\alpha_i} \subset U_{\alpha_i}$ for

$i = 1, 2, ..., m$. Put $W' \equiv \{x \in \sigma_1 | x_{\alpha_i} \in U'_{\alpha_i}$ for $i = 1, 2, ..., m\}$. Then W' is open in σ_1, $x^* \in W'$ and $clW' \cap B = \emptyset$.

Let $Y_\alpha = \{x \in \sigma_1 | x_\beta = x^*_\beta$ for each $\beta \neq \alpha\}$, i.e., $Y_\alpha = X_\alpha \times \{z^*_\alpha\}$. Here $z^*_\alpha = (x^*_\beta)_{\beta \in \Omega \setminus \{\alpha\}}$. Then it is easy to see that

(1) $Y_\alpha \subset W'$ for each $\alpha \in \Omega \setminus \{\alpha_i | i = 1, 2, ..., m\}$;

(2) Y_α is closed in σ_1.

Put $A_i = A \cap Y_{\alpha_i} \setminus W'$ and $B_i = B \cap Y_{\alpha_i} \setminus W'$. Then A_i and B_i are disjoint closed sets in Y_{α_i} and $x^* \notin A_i, x^* \notin B_i$. Since Y_{α_i} is normal, there are open sets V_i and V'_i in X_{α_i} such that $V_i \cap V'_i = \emptyset, x^*_{\alpha_i} \notin V_i \cup V'_i, A_i \subset V_i \times \{z^*_\alpha\}$ and $B_i \subset V'_i \times \{z^*_\alpha\}$. Put $G_i = \{x \in \sigma_1 | x_{\alpha_i} \in V_i\} \setminus \bigcup \{Y_{\alpha_j} | j = 1, 2, ..., m; j \neq i\}$ and $H_i = \{x \in \sigma_1 | x_{\alpha_i} \in V'_i \setminus clU'_{\alpha_i}\} \setminus \bigcup \{Y_{\alpha_j} | j = 1, 2, ..., m; j \neq i\}$. Then G_i and H_i are open sets in σ_1 such that

(3) $A_i \subset G_i, B_i \subset H_i$;

(4) $G_i \cap H_i = \emptyset$.

Proof of (3). Let $x \in A_i \cup B_i$. Then $x_\alpha = x^*_\alpha$ for each $\alpha \neq \alpha_i$. Since $x \notin W', x_{\alpha_i} \neq x^*_{\alpha_i}$. Thus $x \notin Y_{\alpha_j}$ if $j \neq i$. Hence, if $x \in A_i$, then $x \in G_i$. Let $x \in B_i$. Then $x_{\alpha_i} \in V'_i$. If $x_{\alpha_i} \in clU'_{\alpha_i}$, then $x_{\alpha_i} \in U_{\alpha_i}$. Since $x_{\alpha_j} = x^*_{\alpha_j} \in U_{\alpha_j}$ for each $j \neq i, x \in W$. Therefore $x \notin B$. This is a contradiction. Thus $x_{\alpha_i} \notin clU'_{\alpha_i}$. Hence $x \in H_i$.

Put $G = W' \cup \bigcup_{i=1}^m G_i$ and $H = \bigcup_{i=1}^m H_i$. Then G and H are open sets in σ_1 such that

(5) $A \subset G, B \subset H$

and

(6) $G \cap H = \emptyset$.

(5) is obvious. (6) follows from (4) and (7) and (8) below.

(7) $W' \cap H_i = \emptyset$ for each i.

(8) $i \neq j \Rightarrow G_i \cap H_j = \emptyset$.

(7) is obvious.

Proof of (8). If $x \in G_i \cap H_j$, then $x_{\alpha_i} \neq x^*_{\alpha_i}$ and $x_{\alpha_j} \neq x^*_{\alpha_j}$. Thus $|Q(x)| \geq 2$, which contradicts $x \in \sigma_1$.

Example 1. There exists a σ-product such that each finite subproduct is normal and σ_2 is not normal.

To prove Example 1, we shall use the following lemma.

Lemma 2 ([2])**.** *Let X be a space and A be a closed set of X which is not a G_δ-subset of X. Let F be Bing's Example G or H constructed by $P = X \setminus A$. Then $X \times F$ is not normal.*

Definition 2 (Bing's Example G ([1]))**.** Let P be an uncountable set and $Q = \{q | q \subset P\}$. Put $F = \{f : Q \to 2\}$. For each $p \in P$, define f_p as follows:

$$f_p(q) = \begin{cases} 1 & \text{if } p \in q, \\ 0 & \text{if } p \notin q. \end{cases}$$

Put $F_P = \{f_p | \ p \in P\}$. Define the topology of F as follows: each f_p has a neighborhood base in Cartesian product topology and for each $f \in F \setminus F_P, \{f\}$ is open. For each $r \in R = Q^{<\omega}$, put $V(f_p; r) = \{f \in F | f(q) = f_p(q)$ for each $q \in r\}$. Then $\mathcal{V}(f_p) = \{V(f_p; r) | r \in R\}$ is a neighborhood base of f_p.

Proof of Lemma 2. In case F is Bing's Example H, the proof is in [2]. The proof is quite similar for the case of Bing's Example G. But, since [2] is not widely known, we shall sketch the proof of the case of Bing's Example G. Let $C = A \times F$ and $D = \{\langle p, f_p\rangle | p \in P\}$. Then C and D are disjoint closed subsets in $X \times F$ and are not separated by open sets in $X \times F$. To show this, let O be an arbitrary open set in $X \times F$ such that $D \subset O$. For each $p \in P$, there is a member $V(f_p; r_p)$ of $\mathcal{V}(f_p)$ such that $\bigcup_{p\in P}(\{p\} \times V(f_p; r_p)) \subset O$. Let us put $P_i = \{p | p \in P, |r_p| = i\}$ for each $i < \omega$. Since A is not a G_δ-set of X, there is an i such that $A \cap cl(P_i) \neq \emptyset$. Let us fix this i. Let $x_0 \in A \cap cl(P_i)$. Let us put $P_i(r) = \{p \in P_i | r_p \supset r\}$ for each $r \in R$. Then we can prove that there exists an element $r^* \in R$ satisfying the following conditions: (1) $x_0 \in cl(P_i(r^*)$, (2) $x_0 \notin cl(P_i(r^* \cup \{q\})$ for each $q \in Q \setminus r^*$.
Let us put $R^* = \{s | s \subset r^*\}$. For each $s \in R^*$, we define an element q_s of Q by $q_s = \bigcap\{q | q \in s\} \setminus \bigcup\{q | q \in r^* \setminus s\}$. Then $\{q_s | s \in R^*\}$ is a finite cover of P. Therefore, we can choose a member s_0 of R^* such that $x_0 \in cl(P_i(r^*) \cap q_{s_0})$. For this s_0, we choose an element $p^* \in q_{s_0}$. Then $\langle x_0, f_{p^*}\rangle \in C$. Next we shall prove that $\langle x_0, f_{p^*}\rangle \in cl(O)$. Let U be an arbitrary open neighborhood of x_0 in X and $V(f_{p^*}; r)$ be an arbitary member of $\mathcal{V}(f_{p^*})$. Then we can choose an open neighborhood U' of x_0 in X such that $U' \cap (\bigcup\{P_i(r^* \cup \{q\}) | q \in r \setminus r^*\}) = \emptyset$. Then $U \cap U' \cap P_i(r^*) \cap q_{s_0} \neq \emptyset$. Let $p \in U \cap U' \cap P_i(r^*) \cap q_{s_0}$. Then $r_p \cap (r \setminus r^*) = \emptyset$. Define $f : Q \to 2$ by

$$f(q) = \begin{cases} 1 & \text{if } p^* \in q \in r \text{ or } p \in q \in r_p, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in V(f_{p^*}; r) \cap V(f_p; r_p)$. Therefore $(U \times V(f_{p^*}; r)) \cap (\{p\} \times V(f_p; r_p)) \neq \emptyset$.

Proof of Example 1. Let $\sigma = \sigma(2^{\omega_1})$ be the σ-product with the base point 0^*. Then

(i) 0^* is not a G_δ-set in σ_1.

Proof. Assume that there exist countable open sets $\{W_n | n = 1, 2, ...\}$ in σ_1 such that $\{0^*\} = \bigcap_{n<\omega_1} W_n$. Then there are finite sets $a_n \subset \omega_1$ and open sets $U_{n,\alpha}$ in X_α for each $\alpha \in a_n$ such that $0^* \in U_n \equiv \{x \in \sigma_1 | x_\alpha = 0 \text{ for each } \alpha \in a_n\} \subset W_n$ for each n. Then $\{0^*\} = \bigcap_{n<\omega} U_n$. Since $\omega_1 \setminus \bigcup_{n<\omega} a_n \neq \emptyset$, choose an element $\alpha \in \omega_1 \setminus \bigcup_{n<\omega} a_n$. Let us define $x = (x_\beta)_{\beta<\omega_1}$ by $x_\alpha = 1, x_\beta = 0$ if $\beta \neq \alpha$. Then $x \in \bigcap_{n<\omega} U_n \setminus \{0^*\}$, which contradicts $\{0^*\} = \bigcap_{n<\omega} U_n$.

(ii) Put $P = \sigma_1 \setminus \{0^*\}$ and let F be Bing's Example G constructed by P. Then $\sigma_1 \times F$ is not normal.

Let $\mathcal{S} = \{2_\alpha | \alpha < \omega_1\} \cup \{F\}$ where $2_\alpha = 2$ for each α and let $\sigma' = \sigma(\mathcal{S})$ with the base point $\langle 0^*, f^*\rangle, f^* \in F \setminus F_P$. Then

(iii) σ'_2 is not normal.

Since normality is inherited by closed subspaces, (iii) follows from (ii) and (iv) below.

(iv) $\sigma_1 \times F$ is a closed subset of σ'_2.

Proof. It is obvious that $\sigma_1 \times F \subset \sigma'_2$. Let $y \in \sigma'_2 \setminus \sigma_1 \times F$. Then we can denote $y = \langle x, f\rangle, x \in \sigma, f \in F$. Since $\langle x, f\rangle \notin \sigma_1 \times F, x \notin \sigma_1$. Thus $x \in \sigma_2 \setminus \sigma_1$. Hence $f = f^*$. Since $(\sigma_2 \setminus \sigma_1) \times \{f^*\}$ is an open set in $(\sigma_2) \times F$, $((\sigma_2 \setminus \sigma_1) \times \{f^*\} = (\sigma_2 \setminus \sigma_1) \times F) \cap \sigma'_2$ and $(\sigma_2 \setminus \sigma_1) \times \{f^*\}$ is an open neighborhood of y in σ'_2 such that $((\sigma_2 \setminus \sigma_1) \times \{f^*\}) \cap (\sigma_1 \times F) = \emptyset$.

3. Starcompactness

It is well known that every non-trivial σ-product is not countably compact. A space X is called countably compact if every countable open cover of X has a finite subcover, or, which is equivalent, every infinite subset has a limit point. A space X is called strongly starcompact if for every open cover $\mathcal{U}$ there exists a finite set B of X such that $st(B,\mathcal{U}) = X$. Here $st(B,\mathcal{U}) = \bigcup\{U \in \mathcal{U} | U \cap B \neq \emptyset\}$. A space X is called starcompact if for every open cover $\mathcal{U}$ there exists a finite subfamily $\mathcal{U}'$ of $\mathcal{U}$ such that $st(\bigcup \mathcal{U}', \mathcal{U}) = X$.

It is known that countably compact $\Rightarrow$ strongly star compact $\Rightarrow$ starcompact, and for T_2-spaces, the converses hold.

Theorem 3. *Every non-trivial σ-product is not strongly starcompact.*

Proof. Let $\mathcal{S} = \{X_\alpha | \alpha \in \Omega\}$ be a family of spaces such that $|\Omega| \geq \omega$. Let $\sigma = \sigma(2^\omega)$ be the σ-product with the base point 0^* and let $\sigma' = \sigma(\mathcal{S})$. Let us choose a point $a_\alpha \in X_\alpha$ such that $a_\alpha \neq x^*_\alpha$ for each $\alpha \in \Omega$. Let us consider $\omega \subset \Omega$. Define $f : \sigma \to \sigma'$ as follows: for each $x = (x_\alpha)_{\alpha \in \omega} \in \sigma$, let $f(x) = (y_\alpha)_{\alpha \in \Omega}$ be

$$y_\alpha = \begin{cases} a_\alpha & \text{if } \alpha \in \omega \text{ and } x_\alpha = 1, \\ x^*_\alpha & \text{otherwise.} \end{cases}$$

Then f is a homeomorhism from σ onto $f(\sigma)$ such that $f(0^*) = x^*$ and $f(\sigma)$ is a closed subset of σ'. To prove that $f(\sigma)$ is a closed subset of σ', let $y \in \sigma' \setminus f(\sigma)$. Then there exists $\alpha \in \Omega \setminus \omega$ such that $y_\alpha \neq x^*_\alpha$ and put $U = \{z \in \sigma' | z_\alpha \neq x^*_\alpha\}$. Then U is an open neighborhood of y in σ' such that $U \cap f(\sigma) = \emptyset$.

Claim. σ' is not strongly starcompact.

Proof. Let $U_0 = \{x \in \sigma' | x_0 \neq a_0\}$ and let $U_n = \{x \in \sigma' | x_0 \neq x^*_0, x_1 \neq x^*_1, ..., x_{n-1} \neq x^*_{n-1}, x_n \neq a_n\}$ for each $n \geq 1$. Put $\mathcal{U} = \{U_n | n \in \omega\} \cup \{\sigma' \setminus f(\sigma)\}$. Then

(i) $\mathcal{U}$ is an open cover of σ';

(ii) there is no finite $B \subset \sigma'$ such that $st(B,\mathcal{U}) = \sigma'$.

Proof of (i) is easy and so we omit it.

Proof of (ii). Let B be a finite set of σ'. Then $B \subset \sigma'_n$ for some n. Since $U_i \cap \sigma'_n = \emptyset$ for each $i \geq n+1$, $st(B,\mathcal{U}) \subset \bigcup_{i \leq n} U_i \cup (\sigma' \setminus f(\sigma))$. However $\bigcup_{i \leq n} U_i \cup (\sigma' \setminus f(\sigma)) \neq \sigma'$. To show this, let us define $z = (z_\alpha)_{\alpha \in \Omega}$ as follows:

$$z_\alpha = \begin{cases} a_\alpha & \text{if } \alpha \in \omega \text{ and } \alpha \leq n, \\ x^*_\alpha & \text{otherwise.} \end{cases}$$

Then $z \in \sigma'$ and $z \notin \bigcup_{i \leq n} U_i \cup (\sigma' \setminus f(\sigma))$. Therefore $st(B,\mathcal{U}) \neq \sigma'$.

Theorem 4. *If each $X_\alpha \in \mathcal{S}$ is strongly starcompact (resp. starcompact), then σ_1 is strongly starcompact (resp. starcompact).*

Proofs are easy and so we omit them.

Since for T_2-spaces, starcompactness is equivalent to countable compactness, every non-trivial σ-product of T_2-spaces is not starcompact. However, for T_1-spaces, non-trivial σ-product can be starcompact.

We denote $\sigma(X^\tau)$ with the base point x^* by $\sigma(X^\tau; x^*)$.

Example 2. There exists a starcompact space X such that X is not a T_2-space and not countably compact and (1) $\sigma = \sigma(X^\omega; a^*)$ is starcompact for some $a \in X$. (2) $\sigma' = \sigma(X^\omega; b^*)$ is not starcompact for some $b \in X$.

Proof. Let $X = \mathbf{R}$ with the topology as follows: let $\mathcal{U}(0) = \{U | 0 \in U, |X \setminus U| \leq \omega\}$ be the neighborhoods of 0 in X and for each $x \neq 0, \mathcal{U}(x) = \{U | U$ is a neighborhood of x in usual topology of $\mathbf{R}\}$ be the base of x in X. Then X is a T_1-space and not a T_2-space and X is starcompact and not countably compact. By Theorem 5 below, (1) $\sigma = \sigma(X^\omega; 0^*)$ is starcompact. By Theorem 6, (2) $\sigma' = \sigma(X^\omega; 1^*)$ is not starcompact.

Theorem 5. *Let X satisfy the condition: "there exists $a \in X$ such that if U and V are open sets in X and $a \in U$, then $U \cap V \neq \emptyset$". Let τ be an infinite cardinal number, and $\sigma = \sigma(X^\tau; a^*)$. Then* (i) *$\sigma$ is starcompact,* (ii) *σ_n is starcompact* $(\forall n)$, (iii) *X^τ is starcompact. Moreover let $\sigma' = \sigma(X^\omega; b^*), b \in X$, $b \neq a$. Then* (iv) *σ'_n is starcompact* $(\forall n)$.

Proof. Proof of (i). Let $\mathcal{G}$ be an arbitrary open cover of σ. Let us choose $G_0 \in \mathcal{G}$ such that $a^* \in G_0$. There are a finite set $\{\alpha_i | i = 1, 2, ..., m\} \subset \tau$ and open sets U_{α_i} in X_{α_i} such that $a^* \in W_0 \equiv \{x \in \sigma | x_{\alpha_i} \in U_{\alpha_i}$ for $i = 1, 2, ..., m\} \subset G_0$. For each $x \in \sigma \setminus W_0$, let us choose $G_x \in \mathcal{G}$ such that $x \in G_x$. Then there are a finite set $\{\beta_j | j = 1, 2, ..., k\}$ and open sets V_{α_i} in X_{α_i} and V_{β_j} in X_{β_j} such that $x \in W_x \equiv \{y \in \sigma | y_{\alpha_i} \in V_{\alpha_i}$ for $i = 1, 2, ..., m; y_{\beta_j} \in V_{\beta_j}$ for $j = 1, 2, ..., k\} \subset G_x$. Since $a \in U_{\alpha_i}, U_{\alpha_i} \cap V_{\alpha_i} \neq \emptyset$ for each $i = 1, 2, ..., m$. Thus $W_0 \cap W_x \neq \emptyset$ and so $G_0 \cap G_x \neq \emptyset$. Therefore $st(G_0, \mathcal{G}) = \sigma$.
Proofs of (ii) and (iii) are similar.

Proof of (iv). First we define $\mathcal{B}, y^s$ as follows: put $\mathcal{B} = \{W | W$ is a basic open set in $\sigma'\}$. Here $W \subset \sigma'$ is called a basic open set in σ' if $W = \{x \in \sigma' | x_i \in U_i$ for each $i \leq n\}, n \in \omega, U_i$ is an open set in X_i for each $i \leq n$. Define $l(W) = n$. For each $s \in [\omega]^{<\omega}$, define $y^s = (y^s_i)_{i \in \omega}$ as follows:

$$y^s_i = \begin{cases} a & \text{if } i \in s, \\ b & \text{if } i \notin s. \end{cases}$$

To prove (iv), let $\mathcal{G}$ be an arbitrary open cover of σ'. Let us prove that there exists a finite subfamily $\mathcal{G}_n$ of $\mathcal{G}$ such that $st(\bigcup \mathcal{G}_n, \mathcal{G}) \supset \sigma'_n$ for each $n \in \omega$.

(I) Let us choose an element $G_0 \in \mathcal{G}$ such that $b^* \in G_0$. Then there is a set $W_0 \in \mathcal{B}$ such that

(0-1) $b^* \in W_0 \subset G_0$.

Put $l(W_0) = k_0$. Then

(0-2) For $x \in \sigma'$, if $l > k_0$ for each $l \in Q(x)$, then $x \in W_0$.

(II) For each $n = 1, 2, ...$, inductively we can choose k_n, S_n and $\mathcal{W}_n$ satisfying the conditions:

(1) $k_n \in \omega, k_n < k_{n+1} (\forall n \geq 1), k_0 = k_1$.

(2) $S_n \subset S_{n+1} (\forall n \geq 1)$.

(3) $(n-1)$ $S_n = \{s : s \subset \omega, 1 \leq |s| \leq n, l \leq k_n (\forall l \in s)\}$.

$(n-2)$ $\mathcal{W}_n = \{W_s | s \in S_n\} \cup \{W_0\} \subset \mathcal{B}, \mathcal{W}_n$ is a partial refinement of $\mathcal{G}$.

$(n-3)$ $y^s \in W_s (\forall s \in S_n), k_n < l(W_s) \leq k_{n+1} (\forall s \in S_n)$.

$(n-4)$ $st(\cup \mathcal{W}_n, \mathcal{G}) \supset \sigma'_n$.

Assume that k_n, S_n and $\mathcal{W}_n$ have been chosen for each $n \leq m$. Define $k_{m+1} = max\{l(W_s) | s \in S_m\}$ and $S_{m+1} = \{s : s \subset \omega, 1 \leq |s| \leq m+1, l \leq k_{m+1} (\forall l \in s)\}$.

For each $s \in S_{m+1} \setminus S_m$, choose $G_s \in \mathcal{G}$ and $W_s \in \mathcal{B}$ such that $y^s \in W_s \subset G_s$. Put $\mathcal{W}_{m+1} = \{W_s | s \in S_{m+1}\} \cup \{W_0\}$. Then k_{m+1}, S_{m+1} and $\mathcal{W}_{m+1}$ satisfy the conditions. We only prove $(m+1-4)$ because others are obvious.

Proof of $(m+1-4)$. Let $x \in \sigma'_{m+1} \setminus \sigma'_m$. Put $s = Q(x)$. Then $|s| = m+1$. Choose $G_x \in \mathcal{G}$ and $W_x \in \mathcal{B}$ such that $x \in W_x \subset G_x$. Put $s = \{l_i | i = 1, 2, ..., m+1\}$ such that $l_i < l_{i+1} (\forall i)$.

(i) If $l_{m+1} \leq k_{m+1}$, then $s \in S_{m+1}$ and $W_x \cap W_s \neq \emptyset$. To show this, let $W_s = \{y \in \sigma' | y_i \in U_i^s$ for $\forall i \leq l(W_s)\}$ and $W_x = \{y \in \sigma' | y_i \in U_i^x$ for $\forall i \leq l(W_x)\}$. Here we may assume that $l(W_s), l(W_x) \geq k_{m+1}$. Since $a \in U_i^s$ for each $i \in s, U_i^s \cap U_i^x \neq \emptyset$ for each $i \in s$. For each $i \notin s, i \leq min\{l(W_s), l(W_x)\}, b \in U_i^s \cap U_i^x$. Hence $U_i^s \cap U_i^x \neq \emptyset$. Therefore $W_x \cap W_s \neq \emptyset$. Thus $x \in st(\bigcup \mathcal{W}_{m+1}, \mathcal{G})$.

(ii) Case $l_{m+1} > k_{m+1}$. If $l_1 > k_0$, then $x \in W_0$ by (0-2).

Suppose $l_1 \leq k_0$. Then there is a j such that $1 \leq j \leq m+1, Q(x) \cap \{l | k_j < l \leq k_{j+1}\} = \emptyset$. Let j be the greatest such number. Then there exists $t \in \{i | i = 1, 2, ..., m+1\}$ such that $l_t \leq k_j$ and $l_{t+1} > k_{j+1}$. Then $t \leq j$. Therefore $s = \{l_i | i \leq t\} \in S_j$. Since $l(W_s) \leq k_{j+1}$, it is easy to see that $W_x \cap W_s \neq \emptyset$.

Since $\mathcal{W}_n$ is a partial refinement of $\mathcal{G}$, there exists $\mathcal{G}_n$ such that $st(\bigcup \mathcal{G}_n, \mathcal{G}) \supset \sigma'_n$.

Theorem 6. *There are a countable closed subset A of X and a pairwise disjoint open family $\mathcal{U} = \{U(a) | a \in A\}$ such that $a \in U(a)$ for each $a \in A$ and $X \setminus A \neq \emptyset$. Let $\sigma = \sigma(X^\omega; a^*), a \in A$. Then σ is not starcompact.*

Proof. Put $A = \{a_n | n = 1, 2, ...\}, U(a_n) = U_n$ for each n and put $U_0 = X \setminus A$. Then $\bigcup_{n<\omega} U_n = X$. Without loss of generality we may assume that $a = a_1$. For each $k = 1, 2, ...,$ let $\Lambda_k \equiv \{(l_0, l_1, ..., l_{k-1}, 1) \in [\omega]^{k+1} | l_0 \neq 1, l_{k-1} \neq 1\}$ and put $\Lambda = \bigcup_{1 \leq k} \Lambda_k$.

Define $G_1 \equiv \{x \in \sigma | x_0 \in U_1\}$ and $G_\lambda \equiv \{x \in \sigma | x_i \in U_{l_i}$ for $i = 0, 1, ..., k-1; x_k \in U_1\}$ for each $\lambda = (l_0, l_1, ..., l_{k-1}, 1) \in \Lambda$ and put $\mathcal{G} = \{G_\lambda | \lambda \in \Lambda\} \cup \{G_1\}$. Then

(1) $\mathcal{G}$ is an open cover of σ.

(2) For any finite subfamily $\mathcal{G}'$ of $\mathcal{G}, st(\bigcup \mathcal{G}', \mathcal{G}) \neq \sigma$.

Proof of (1). Let $x \in \sigma$. If $x_0 \notin U_1$, then $x_0 \in U_i$ for some $i \neq 1$. Since $|Q(x) = \{i | x_i \neq a_1\}| < \omega$, there is a k such that $x_i = a_1$ for each $i \geq k$ and $x_{k-1} \neq a_1$. Then $x \in G_\lambda$ for some $\lambda \in \Lambda$.

Proof of (2). Let $\mathcal{G}'$ be an arbitrary finite subfamily of $\mathcal{G}$. Then there exists $k > 1$ such that

(2-1) $G_\lambda \notin \mathcal{G}'$ for each $\lambda \in \bigcup_{m>k} \Lambda_m$.

Define $x = (x_i)_{i \in \omega}$ as follows:

$$x_i = \begin{cases} a_k & \text{if } i \leq k, \\ a_1 & \text{if } i > k. \end{cases}$$

Then

(2-2) $x \notin st(\bigcup \mathcal{G}', \mathcal{G})$.

Proof of (2-2). Let $x \in G \in \mathcal{G}$. Then $G \neq G_1$. Therefore $G = G_\lambda$ with $\lambda = (l_0, l_1, ..., l_{m-1}, 1) \in \Lambda_m$. Then $m > k$. To show this, assume that $m \leq k$. Then $x_m = a_k$ by the definition of x. Since $a_k \notin U_1, x \notin G_\lambda$, which is a contradiction. Therefore $m > k$. It is easy to see that $l_i = k$ for each $i \leq k$ and $l_i = 1$ for each $i \geq k+1$. Thus $l_0 \geq 2$ and so $U_{l_0} \cap U_1 = \emptyset$. Hence $G_\lambda \cap G_1 = \emptyset$. If $m \geq k+2$, then $l_{m-1} = 1$. This contradicts the definition of λ. Thus $m = k+1$. Let $G_\mu \in \mathcal{G}'$ with $\mu = (s_0, s_1, ..., s_{t-1}, 1) \in \Lambda_t$. Then $t \leq k$. Therefore $l_t = k$. Since $U_k \cap U_1 = \emptyset, G_\lambda \cap G_\mu = \emptyset$.

Remark. For σ and σ' in Example 2, σ_n and σ'_n are starcompact for each n by Theorems 5 and 6.

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References

1. R. H. Bing, Metrization of topological spaces, Canad. J. Math., 3 (1951), 175 -186. MR **13:**264f
2. K. Chiba, Two remarks on the normality of product spaces, Reports of Faculty of Science, Shizuoka University, 11 (1976), 17 - 22. MR **56:**1257
3. K. Chiba, The strong paracompactness of σ-products, Scientiae Mathematicae, vol. 2, No. 3 (1999), 285-292. MR **2000g:**54042
4. H. H. Corson, Normality in subsets of product spaces, Amer. J. Math., 81 (1959), 785 - 796. MR **21:**5947
5. R. Engelking, General Topology, Polish Scientific Publishers, Warszawa (1988).
6. W. M. Fleischman, A new extension of countable compactness, Fund. Math., 67 (1970), 1-9. MR **41:**9200
7. S. Ikenaga, A class which contains Lindelöf spaces, separable spaces and countably compact spaces, Memories of Numazu College of Techonology 18 (1983), 105-108.
8. A. P. Kombarov, On a theorem of Corson, Vestnik Moskov. Univ., Ser. Mat., No. 6 (1978), 33-34 (Russian), (Engl. transl.: Moscow Univ. Math. Bull., vol. 33, No. 6 (1978), 28-29. MR **80c:**54016
9. A. P. Kombarov, On expandable discrete collections, Topology and its Applications, 69 (1996), 283-292. MR **97d:**54011
10. M. V. Matveev, A survey on star covering properties, Topology Atlas, preprint No. 330 (1998).

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