A BLOCK THEORETIC ANALOGUE
OF A THEOREM OF GLAUBERMAN AND THOMPSON

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Abstract. If \( p \) is an odd prime, \( G \) a finite group and \( P \) a Sylow-\( p \)-subgroup of \( G \), a theorem of Glauberman and Thompson states that \( G \) is \( p \)-nilpotent if and only if \( N_G(Z(J(P)))) \) is \( p \)-nilpotent, where \( J(P) \) is the Thompson subgroup of \( P \) generated by all abelian subgroups of \( P \) of maximal order. Following a suggestion of G. R. Robinson, we prove a block-theoretic analogue of this theorem.

\textbf{Theorem.} Let \( p \) be an odd prime and let \( k \) be an algebraically closed field of characteristic \( p \). Let \( G \) be a finite group, \( b \) a block of \( kG \), and \( P \) a defect group of \( b \). Set \( N = N_G(Z(J(P)))) \) and let \( c \) be the unique block of \( kN \) such that \( \text{Br}_P(c) = \text{Br}_P(b) \); that is, \( c \) is the Brauer correspondent of \( b \). Then \( kGb \) is nilpotent if and only if \( kNc \) is nilpotent.

We refer to \cite{5} and \cite{7} for accounts on the terminology from group theory and block theory, respectively, involved in the theorem above and its proof. Nilpotent blocks, introduced by Broué and Puig in \cite{3}, are the block theoretic analogue of the notion of \( p \)-nilpotent groups; the principal block of \( kG \) is nilpotent if and only if \( G \) is \( p \)-nilpotent. Thus, in this case, our theorem is equivalent to the theorem of Glauberman and Thompson. The proof proceeds in two steps. We reduce to the case where \( G \) is the normaliser of a \( b \)-centric Brauer pair (following the lines of the proof of \cite[Ch. 8, Theorem 3.1]{5}), and then we apply results of Kulshammer and Puig in \cite{6} to transport the problem back to the analogous group theoretic statement.

\textbf{Proof.} We fix a block \( e_P \) of \( kC_G(P) \) such that \( \text{Br}_P(b)e_P = e_P \); that is, \( (P, e_P) \) is a maximal \( b \)-Brauer pair. By \cite{1}, for any subgroup \( Q \) of \( P \) there is a unique block \( e_Q \) of \( kC_G(Q) \) such that \( (Q, e_Q) \subseteq (P, e_P) \). Denote by \( \mathcal{F}_{G,b} \), the category whose objects are the subgroups of \( P \) and whose set of morphisms from a subgroup \( Q \) of \( P \) to another subgroup \( R \) of \( P \) is the set of group homomorphisms \( \varphi : Q \rightarrow R \) for which there exists an element \( x \in G \) satisfying \( \varphi(u) = xux^{-1} \) for all \( u \in Q \) and \( x(Q,e_Q) \subseteq (R,e_R) \). Thus the automorphism group of a subgroup \( Q \) of \( P \) as an object of the category \( \mathcal{F}_{G,b} \) is canonically isomorphic to \( N_G(Q,e_Q)/C_G(Q) \). By Alperin’s fusion theorem, the category \( \mathcal{F}_{G,b} \) is completely determined by the structure of \( P \) and the groups \( N_G(Q,e_Q)/C_G(Q) \) where either \( Q = P \) or \( (Q,e_Q) \) is

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an essential $b$-Brauer pair (cf. [4, §48]). Note that $O_p(G) \subseteq Q$ whenever the pair $(Q, e_Q)$ is essential.

By Brauer’s third main theorem (cf. [4, (40.17)]), if $b$ is the principal block of $kG$, then $e_Q$ is the principal block of $kC_G(Q)$, for any subgroup $Q$ of $P$. Thus the above condition $\bar{z}(Q, e_Q) \subseteq (R, e_R)$ is equivalent to $\bar{z}Q \subseteq R$. Therefore, if $b$ is the principal block of $kG$, we write $\mathcal{F}_G$ instead of $\mathcal{F}_{G,b}$.

In general, the definition of $\mathcal{F}_{G,b}$ depends on the choice of a maximal $b$-Brauer pair, but since all maximal $b$-Brauer pairs are $G$-conjugate, it is easy to see that $\mathcal{F}_{G,b}$ is unique up to isomorphism of categories. Note that we always have $\mathcal{F}_P \subseteq \mathcal{F}_{G,b}$.

Following [3], the block $b$ is called nilpotent if $\mathcal{F}_P = \mathcal{F}_{G,b}$.

If $H$ is any subgroup of $G$ containing $PC_G(P)$, the block $e_P$ determines a unique block $d$ of $kH$ by $\operatorname{Br}_P(d)e_P = e_P$. Then $(P, e_P)$ is also a maximal $d$-Brauer pair, and this gives rise to the Brauer category $\mathcal{F}_{H,d}$ of $kHd$, defined as above for $H$ and $d$ instead of $G$ and $b$.

We are going to frequently use the following fact:

1. If $Q$ is a normal subgroup of $P$ and $H$ a subgroup of $G$ such that $PC_G(Q) \subseteq H \subseteq N_G(Q)$, then
$$\mathcal{F}_{H,d} \subseteq \mathcal{F}_{G,b},$$
where $d$ is the unique block of $kH$ such that $\operatorname{Br}_P(d)e_P = e_P$. In particular, if $kGb$ is nilpotent, then $kHd$ is nilpotent.

Proof. If $(R, f_R)$ is an essential $d$-Brauer pair contained in $(P, e_P)$, then $R$ contains $Q$ as $Q$ is normal in $H$. But then $C_G(R) = C_H(R)$, and hence $f_R = e_R$. Thus $N_H(R, f_R)/C_H(R)$ is a subgroup of $N_G(R, e_R)/C_G(R)$. \hfill \Box

Statement 1 applies to $N, c$ and $Z(J(P))$ instead of $H, d, Q$, respectively. Thus if $kGb$ is nilpotent, so is $kNc$. In order to show the converse, we consider now a minimal counterexample; that is, we assume that $kGb$ is not nilpotent while $kNc$ is nilpotent and that $|G|$ is minimal with this property. Under this assumption, 1 implies the following statement:

2. If $Q$ is a normal subgroup of $P$ and $H$ a subgroup of $G$ such that $PC_G(Q) \subseteq H \subseteq N_G(Q)$, then either $H = G$ or $kHd$ is nilpotent, where $d$ is the unique block of $kH$ such that $\operatorname{Br}_P(d)e_P = e_P$.

Proof. Let $e$ be the unique block of $N \cap H$ such that $\operatorname{Br}_P(e)e_P = e_P$. We have $PC_N(Q) \subseteq N \cap H \subseteq N_G(Q)$, and thus statement 1 implies that $\mathcal{F}_{N \cap H,e} \subseteq \mathcal{F}_{N,e}$. But then $k(N \cap H)e$ is nilpotent, as $kNc$ is also. Therefore, if $H$ is a proper subgroup of $G$, then the induction hypothesis implies that the block $kHd$ is nilpotent. \hfill \Box

3. We have $O_p(G) \neq \{1\}$.

Proof. Since the block $b$ of $kG$ is not nilpotent, there exists a $b$-Brauer pair $(Q, e_Q)$ with $Q \neq 1$ such that $kN_G(Q, e_Q)c_Q$ is not nilpotent. This is because for some nontrivial Brauer pair $(Q, e_Q)$, $N_G(Q, e_Q)/QCG(Q)$ is not a $p$-group. Amongst all such $b$-Brauer pairs, choose $(Q, e_Q)$ such that a defect group $R$ of $kN_G(Q, e_Q)c_Q$ has maximal order. After replacing, if necessary, $(Q, e_Q)$ by a suitable $G$-conjugate, we may assume that $R = N_P(Q)$. We are going to show that $R = P$, or equivalently that $P \subseteq N_G(Q, e_Q)$. We assume that $R$ is a proper subgroup of $P$, and derive
a contradiction. Set $H = N_G(Q, e_Q)$. Clearly $R \subseteq H$. Since $Q \subseteq R$, we have $C_G(R) \subseteq C_G(Q) \subseteq H$. Now $(Q, e_Q) \subseteq (R, e_R)$, and $Q$ is normal in $R$, hence $e_Q$ is the unique block of $kC_G(Q)$ which is $R$-stable and for which $Br_R(e_Q)e_R = e_R$ (cf. [1]).

Set $M = N_G(Z(J(R)))$. Since $C_G(R)$ centralises $Q$ and centralises $Z(J(R))$, we have $C_G(R) \subseteq M \cap H$. Let $d$ be the unique block of $k(M \cap H)$ (having $R$ as defect group) such that $Br_R(d)e_R = e_R$. Let $f$ be the unique block of $kM$ (having $R$ as defect group) such that $Br_R(f)e_R = e_R$. Since $Z(J(R))$ is a normal $p$-subgroup of $M$, $f$ is a central idempotent of $kC_G(Z(J(R)))$ (cf. [1]). Thus there exists a block $f_0$ of $C_G(Z(J(R)))$ such that $f_0 = f$ and $(Z(J(R)), f_0) \subseteq (R, e_R)$ in $M$, and hence in $G$. Since $(R, e_R) \subseteq (P, e_P)$, by the uniqueness of inclusion of Brauer pairs, we must have $f_0 = e_{Z(J(R))}$. Let $M'$ be the stabiliser of $e_{Z(J(R))}$ in $M$. Then $N_P(Z(J(R)))$, and hence $N_P(R)$ is contained in a defect group of $kM'e_{Z(J(R))}$. In particular, the defect groups of $kM'e_{Z(J(R))}$ have order strictly greater than $|R|$. By the maximality of $|R|$, we have that $kM'e_{Z(J(R))}$ is nilpotent. Since $kMf$ is the induced algebra $Ind^M_{M'}(kM'e_{Z(J(R))})$, it follows that $kMf$ is nilpotent. Now $RC_G(Q) \subseteq M \cap H \subseteq N_M(Q)$, and by statement 1 again, it follows that $k(M \cap H)d$ is nilpotent. By the minimality of $|G|$, and the fact that $kHe_Q$ is not nilpotent, it follows that $H = G$ and hence $R = P$, contradicting the assumption $R \neq P$. If $R = P$, then $H$ satisfies the hypothesis of 2 with $d = e_Q$, and $kHe_Q$ is not nilpotent, thus $G = H$. In particular, $Q \subseteq O_p(G) \neq 1$. 

From now on set $Q = O_p(G)$.

4. We have $G = N_G(Q, e_Q)$ and $b = e_Q$.

Proof. Since $G = N_G(Q)$, the block $b$ is contained in $kC_G(Q)$ (cf. [1]) and hence $b = Tr^{G}_{C_G(Q, e_Q)}(e_Q)$. Thus $kb \cong Ind^{G}_{C_G(Q, e_Q)}(kN_G(Q, e_Q)e_Q)$, so that in particular, $kN_G(Q, e_Q)e_Q$ is not nilpotent. Since $P$ is contained in $N_G(Q, e_Q)$, it follows from 2 that $G = N_G(Q, e_Q)$ and hence $b = e_Q$. 

Note that $b$ is a block of any subgroup of $G$ containing $C_G(Q)$. We want to show that actually the pair $(Q, b)$ is $b$-centric (or self-centralising in the terminology of Puig, cf. [7] §41); that is, the block $kC_G(Q)b$ is nilpotent with $Z(Q)$ as defect group. This notion goes back to Brauer [2]. We need the following technical statement.

5. Let $H$ be a subgroup of $G$ containing $P$ and let $d$ be a block of $kH$ having $P$ as defect group. Put $H = H/Q$ and for any element $a$ of $kH$ let $a$ denote the image of $a$ under the canonical surjection of $kH$ onto $kH$. Then $Br_H(d) = Br_H(d')$.

Proof. Since $Q$ is normal in $H$, the block idempotent $d$ is a $k$-linear combination over the set $C_H(Q)_{p'}$ of $p'$-elements in $C_H(Q)$. Write $d = \sum_{g \in C_H(Q)_{p'}} \alpha_g g$ with coefficients $\alpha_g \in k$. So $d = \sum_{g \in C_H(Q)_{p'}} \alpha_g \bar{g}$ and $Br_H(d) = \sum_{g \in C_H(Q)_{p'}} \alpha_g \bar{g}$, where $C_H(P)$ denotes the inverse image in $H$ of $C_H(\bar{P})$.

We claim that $C_H(Q)_{p'} \cap C_H(P) = C_H(P)_{p'}$. To see this, consider the action of an element $g \in C_H(Q)_{p'} \cap C_H(P)$ on an element $u$ of $P$. Since $g$ normalises $P$ and centralises $P/Q$, $g(u) = uv$ for some $v$ in $Q$. Let $n$ be the order of $g$. Since $g$ centralises $Q$, it follows that $u = g^n(u) = uv^n$. But $p$ and $n$ are relatively prime, hence $v = 1$, thereby proving the claim.

The statement is immediate from the above expression for $\bar{d}$. 


6. The blocks $kPC_G(Q)b$ and $kC_G(Q)b$ are nilpotent.

Proof. By a result of Cabanes [4], normal $p$-extensions of nilpotent blocks are nilpotent; thus $kPC_G(Q)b$ is nilpotent if and only if $kC_G(Q)b$ is nilpotent. If $PC_G(Q)$ is a proper subgroup of $G$, then, by 2, $b$ is nilpotent as a block of $PC_G(Q)$, and hence of $C_G(Q)$. Thus we may assume that $G = PC_G(Q)$. We have to show that $kGb$ is a nilpotent block. Set $\bar{G} = G/Q$ and let $\bar{b}$ denote the image of $b$ under the canonical surjection of $kG$ onto $k\bar{G}$. Identify $C_G(Q)/Z(Q)$ with its canonical image in $G$; this is a normal subgroup of $\bar{G}$ of index a $p$-power. Since $b$ is a $k$-linear combination of $p'$-elements in $C_G(Q)$ and $Z(Q) = Q \cap C_G(Q)$ is central in $C_G(Q)$, it is clear that $\bar{b}$ is a block of $kC_G(Q)/Z(Q)$ and hence of $k\bar{G}$. Furthermore, $\bar{P}$ is a defect group of $k\bar{G}$. Let $Z$ be the inverse image in $G$ of $Z(J(\bar{P}))$ and set $H = N_G(Z)$. Then $H$ is the inverse image in $G$ of the group $\bar{H} = kN_G(Z(J(\bar{P})))$. Let $f$ be the block of $kH$ which corresponds to the block $\bar{b}$ of $k\bar{G}$; that is, $Br_P(\bar{b}) = Br_P(f)$. Clearly, $P$ and $C_G(Z)$ are both subgroups of $H$. Since $Z$ properly contains $Q$ and $Q = O_p(G)$, $H$ is a proper subgroup of $G$. Thus by 2, the block $kHd$ is nilpotent where $d$ is the block of $kH$ satisfying $Br_P(d)e_P = e_P$. Since $N_G(P)$ is contained in $H$, we have in fact that $Br_P(d) = Br_P(b)$.

Now, it follows from 5 that

$$Br_P(\bar{d}) = Br_P(d) = Br_P(b) = Br_P(\bar{b}) = Br_P(f).$$

In particular, $df \neq 0$. Since $kHd$ is nilpotent, this means that $f = d$ and hence that $kHf$ is nilpotent. As $G$ is a minimal counterexample to the Theorem, it follows that $k\bar{G}b$ is nilpotent, which implies that $kGb$ is nilpotent. □

7. The group $Q$ is a defect group of $kQG_C(Q)b$.

Proof. Let $R$ be a defect group of $kQG_C(Q)b$. We may assume that $R = QC_P(Q)$. The pair $(R, e_R)$ is a maximal Brauer pair for the block $kQG_C(Q)b$, and hence, by the Frattini argument,


Suppose, if possible, that $Q$ is a proper subgroup of $R$. Then, $N_G(R, e_R)$ is a proper subgroup of $G$ because $Q = O_p(G)$. On the other hand, $N_G(R, e_R)$ satisfies the hypothesis of 2 with $R$ instead of $Q$, since $P$ normalises $R$, and consequently $(R, e_R)$. So $kN_G(R, e_R)e_R$ is nilpotent. In particular, $N_G(R, e_R)/C_G(R)$ is a $p$-group, and hence so is $G/C_G(Q)$. In other words, $G = PC_G(Q)$, and hence $kGb$ is nilpotent by 6, a contradiction. □

We are now in the situation where $kGb$ is an extension of the nilpotent block $kQG_C(Q)b$, and this is where the results of Kulshammer and Puig in [4] come in.

8. There exists a short exact sequence of groups

$$1 \longrightarrow Q \longrightarrow L \longrightarrow G/QC_G(Q) \longrightarrow 1$$

such that $P$ is a Sylow $p$-subgroup of $L$ and such that we have $F_{G,b} = F_L$.

Proof. Note first that $P$ is also a defect group of $\{b\}$ viewed as a point of $G$ on $OQC_G(Q)$ because $P$ is maximal with the property $Br_P(b) \neq 0$. The existence of a canonical short exact sequence of finite groups as stated such that $P$ is a Sylow-$p$-
subgroup of $L$ is a particular case of [5] 1.8]. The equality $\mathcal{F}_{G,b} = \mathcal{F}_L$ is a translation of the statement [6 1.8.2], which requires a brief explanation. Since $Q$ is normal in $L$ and in $G$, it suffices to show that the images in $\text{Aut}(R)$ of $N_G(R,e_R)/C_G(R)$ and $N_L(R)/C_L(R)$ are equal, where $R$ is a subgroup of $P$ containing $Q$. As $(Q,e_Q)$ is $b$-centric and $Q$ is $p$-centric in $L$, it follows from a result of Puig (cf. [7 (41.1), (41.4)]) that $(R,e_R)$ is $b$-centric and $R$ is $p$-centric in $L$ (that is, $Z(R)$ is a Sylow-$p$-subgroup of $C_L(R)$). In particular, $kC_G(R)e_R$ has a unique conjugacy class of primitive idempotents. Setting $H = QC_G(Q)$, we have $C_G(R) = C_H(R)$, hence there is a unique point $\gamma_R$ of $R$ on $kH$ such that $B_{iR}(i)e_R = i$ for some (and hence any) element $i$ of $\gamma_R$. In this way, we get an inclusion preserving bijection, $R_{\gamma_R} \to (R,e_R)$ between local pointed groups $R_{\gamma_R}$ on $kHb$ for which $Q_{\gamma_R} \subseteq R_{\gamma_R} \subseteq P_{\gamma_R}$ and $kGb$-Brauer pairs, $(R,e_R)$ with $(Q,e_Q) \subseteq (R,e_R) \subseteq (P,e_P)$. Further, it is clear that $N_G(R,e_R) = N_G(R_{\gamma_R})$. Thus, setting $\bar{G} = G/QC_G(Q)$, with the notation in [6 1.8] (which is defined in [6] 2.8), we have $E_{G,\bar{G}}(R,e_R) = E_{L,\bar{G}}(R)$ for any subgroup $R$ such that $Q \subseteq R \subseteq P$. By [6] (2.8.1), the canonical maps $E_{G,\bar{G}}(R,e_R) \to E_G(R,e_R)$ and $E_{L,\bar{G}}(R) \to E_{L}(R)$ are surjective. Thus $E_G(R,e_R) = E_{L}(R)$, which implies the equality $\mathcal{F}_{G,b} = \mathcal{F}_L$.  \\ 

9. We have $\mathcal{F}_{N,c} = \mathcal{F}_{N_L(Z(J(P)))}$.  

Proof. Since $Z(J(P))$ is normal in both $N$ and $N_L(Z(J(P)))$, it suffices to show that the images of $N_G(S,f) \cap N$ and $N_L(S) \cap N_L(Z(J(P)))$ in $\text{Aut}(S)$ are equal, where $(S,f)$ is a $c$-Brauer pair contained in $(P,e_P)$ such that $Z(J(P)) \subseteq S$. Note that then $C_G(S) \subseteq N$ and hence $f = e_S$. Also, by 8 we have $\mathcal{F}_{G,b} = \mathcal{F}_L$. Thus for any $x \in N_G(S,e_S)$ there is $y \in N_L(S)$ such that $\gamma u = \gamma y u$ for all $u \in S$. Since $Z(J(P)) \subseteq S$ we have $x \in N_G(S,e_S) \cap N$ if and only if $y \in N_L(S) \cap N_L(Z(J(P)))$, from which the equality 9 follows.  

We conclude the proof of the Theorem as follows. Since $kgb$ is not nilpotent, $L$ is not a $p$-nilpotent group by 8. However, $kNc$ is nilpotent and hence $N_L(Z(J(P)))$ is $p$-nilpotent by 9. This contradicts the normal $p$-complement theorem [5 Ch. 8, Theorem 3.1] of Glauberman and Thompson.  

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