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SEMIPRIME CROSSED PRODUCTS OVER COPOINTED HOPF ALGEBRAS

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Abstract. We prove a result on the transfer of essentiality of extensions of modules over subnormalizing extensions of rings, then apply it to look at the semiprimeness of Hopf-Galois extensions, in particular that of crossed products.

The following is an important open question in the theory of Hopf algebra actions (see [12, Question 7.4.9, p.121]):

If H is a finite-dimensional and semisimple Hopf algebra over the field k, and A is semiprime, is any crossed product $A\#_\sigma H$ semiprime?

This possible extension of the Maschke-type theorem for crossed products [2], was inspired by the fact that it holds in the following cases:

1. when $H = kG$ (G is a finite group, and $\mid G \mid^{-1} \in k$) (this is the Fisher-Montgomery theorem, [9]), or

2. when $H = kG^*$ [6].

The first case was extended in [3] to the case where H is semisimple, pointed and cocommutative. The aim of the present note is to try to extend the second case, exploiting the fact that the dual Hopf algebra H^* is pointed in this case. The result of this effort is Corollary 5. The technique that we use involves the so-called essential form of Maschke's theorem. This approach was first used to provide another proof for the Fisher-Montgomery theorem in [15]. Various partial answers to the above question were also given in [1], [17], [13].

Throughout, H will denote a finite-dimensional Hopf algebra over the field k. For all unexplained notation or definitions, the reader is referred to [12].

Recall that if R is a subring of S, the extension $R \subset S$ is called a *subnormalizing* (or *triangular*) extension if there exist elements $x_1, x_2, x_3, \ldots \in S$ such that $S = \sum_{i=1}^{\infty} Rx_i$ and for any j we have $\sum_{i=1}^{j} Rx_i = \sum_{i=1}^{j} x_iR$. A subnormalizing extension is called *finite* if the sequence $x_1, x_2, x_3, \ldots \in S$ is finite (see [18] or [11]). Our first result is

Theorem 1. *Let $R \subset S$ be a subnormalizing extension of rings, such that the elements $x_1, x_2, x_3, \ldots \in S$ from the definition form a basis for S as a left and a right R-module. If $M \subset N$ is an essential extension of left R-modules, and N is nonsingular, then $S \otimes_R M \subset S \otimes_R N$ is an essential extension of R-modules.*

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Proof. Suppose the extension $S \otimes_R M \subset S \otimes_R N$ is not essential. We say $w \in S \otimes_R N$, $w \neq 0$, is a *nasty* element if any R-multiple of it is either 0 or not in $S \otimes_R M$. It is clear that a nonzero R-multiple of a nasty element is again nasty. Among all nasty elements choose $x_1 \otimes n_1 + \ldots + x_t \otimes n_t \in S \otimes_R N$ with j minimal subject to $n_j \notin M$, $n_{j+1} \in M, \ldots, n_t \in M$.

For any i there exists an automorphism $\sigma_i : R \longrightarrow R$ such that $ax_i = x_i\sigma_i(a) +$ *lower terms* for all $a \in R$. If $a \in R$, then

$$aw = \ldots + x_j \otimes (\sigma_j(a)n_j + n') + x_{j+1} \otimes n'_{j+1} + \ldots + x_t \otimes n'_t$$

where $n', n'_{j+1}, \ldots n'_t \in M$, and by the minimality of j we have that

$$\mathrm{l.ann}_R(w) = \sigma_j^{-1}(\mathrm{l.ann}_R(\widehat{n_j})),$$

where $\widehat{n_j} = n_j + M \in M/N$. But $\mathrm{l.ann}_R(\widehat{n_j})$ is an essential ideal of R, because the extension $M \subset N$ is essential. Thus $\mathrm{l.ann}_R(w)$ is also an essential ideal.

Now, for $a \in \mathrm{l.ann}_R(w)$ we have

$$0 = aw = x_1 \otimes n'_1 + \ldots + x_{t-1} \otimes n'_{t-1} + x_t \otimes \sigma_t(a)n_t,$$

so $\mathrm{l.ann}_R(w) \subseteq \sigma_t^{-1}(\mathrm{l.ann}_R(n_t))$, and hence $\mathrm{l.ann}_R(n_t)$ is also an essential ideal, contradicting the fact that N is nonsingular.

This implies that $S \otimes_R M$ has no nasty elements and the result follows. □

The next lemma is an application of the Taft-Wilson theorem, and was proved in [7] (see also [8, Exercise 7.7.9, p.338]).

Lemma 2. *Let H be a finite-dimensional pointed Hopf algebra acting on the algebra A. Then $A\#H$ is a finite subnormalizing extension of A, and the elements $x_1, \ldots, x_m \in A\#H$ may be chosen to form a basis of $H \subset A\#H$.*

We are now in a position to state and prove our main result.

Theorem 3. *Let H be a finite-dimensional pointed Hopf algebra, and A a left H-module algebra with subalgebra of invariants A^H, such that A/A^H is a faithfully flat right H^*-Galois extension. Then the following hold:*

a) *If $M \subset_A N$ is an essential extension of left A-modules, and N is left nonsingular, then N is an essential extension of M as left A^H-modules.*

b) *If A is H-semiprime and A is a left nonsingular ring, then A is semiprime.*

c) *If A^H is semiprime and A is a left nonsingular ring, then A is semiprime.*

Proof. a) The proof uses the duality approach of [6] as in [16] and [4], in the form for Hopf algebra actions which was also used in [7]: since the induced functor $A\#H \otimes_A - : A - mod \longrightarrow (A\#H)\#H^* - mod$ is an equivalence, it follows that

$$A\#H \otimes_A M \subset A\#H \otimes_A N \tag{1}$$

is an essential extension of left $(A\#H)\#H^*$-modules.

By Lemma 2, we can apply Theorem 1 for the subnormalizing extension $A \subset A\#H$ to obtain that (1) is an essential extension of left A-modules, and so it is also an essential extension of left $A\#H$-modules.

Now we have the functorial isomorphisms of left $A\#H$-modules $A\#H \otimes_A M \simeq A \otimes_{A^H} M$, and $A\#H \otimes_A N \simeq A \otimes_{A^H} N$, so $A \otimes_{A^H} M \subset A \otimes_{A^H} N$ is an essential extension of left $A\#H$-modules. But the induced functor $A \otimes_{A^H} - : A^H - mod \longrightarrow A\#H - mod$ is also an equivalence, since A/A^H is a faithfully flat Galois extension, and therefore we have that $M \subset_{A^H} N$ is an essential extension.

b) The proof is similar to the proof of the Fisher-Montgomery theorem, as given in [15], but we sketch it for the convenience of the reader. Let $0 \neq N$ be an ideal of A such that $N^2 = 0$. Then $I = \mathrm{r.ann}_A(N)$ is an essential left ideal of A. By a), we get that I is an essential A^H-submodule of A, so $I^H = I \cap A^H$ is an essential left ideal of A^H. Let

$$J = (I : H) = \{a \in I \mid \quad h \cdot a \in I \ \ \forall h \in H\}.$$

Since $I^H \subseteq J \subseteq I$, it follows that $J^H = I^H$, and since the extension is Galois, we get that $J = I^H A$, by [12, Corollary 8.3.10, p.138]. But now

$$\mathrm{l.ann}_A(J) = \mathrm{l.ann}_A(I^H) \supseteq \mathrm{l.ann}_A(I) \supseteq N \neq 0.$$

Thus $L = \mathrm{l.ann}_A(J)$ is a nonzero H-stable ideal of A, which is again generated (as a left ideal) by its intersection with A^H. Hence we have that $L^H \cap I^H \neq 0$. Finally, $J \cap L$ is a nonzero H-stable ideal of A with $(J \cap L)^2 = 0$, a contradiction to the fact that A is H-semiprime.

c) follows from b), because H-stable ideals of A are generated by their intersection with A^H. ☐

The following is a corollary of the proof of Theorem 3, and is presumably known.

Corollary 4. *If G is a finite group acting as automorphisms on the k-algebra R, R^G denotes the subalgebra of invariants, and R/R^G is a Galois extension having an element of trace* 1*, then the following hold:*

a) *If $M \subset_R N$ is an essential extension of left R-modules, then $M \subset N$ is also an essential extension as left R^G-modules.*

b) *If R^G is semiprime, then R is semiprime.*

Proof. As in the proof of part a) of Theorem 3, we want to prove that

$$R * kG \otimes_R M \subset R * kG \otimes_R N$$

is an essential extension of $R*kG$-modules. But since this is an essential extension of left $(R * kG)\#kG^*$-modules, i.e. a gr-essential extension of graded $R * kG$-modules, the claim follows easily from [14, I.2.8, p. 9]. The rest of the proof is the same. ☐

A direct consequence of Theorem 3 is the following.

Corollary 5. *Let H be a finite-dimensional Hopf algebra such that H^* is pointed, and $A\#_\sigma H$ a crossed product with invertible cocycle σ. If A is H-semiprime and $A\#_\sigma H$ is a left nonsingular ring, then $A\#_\sigma H$ is also semiprime.*

Proof. Denote the left weak action of H on A by $h \cdot a$ for $a \in A$ and $h \in H$.

We remark first that if A is H-semiprime, then $A\#_\sigma H$ is H^*-semiprime. Indeed, if N is a nonzero H^*-stable ideal of $A\#_\sigma H$ with $N^2 = 0$, then $N \cap A$ generates N as a left ideal (again by [12, Corollary 8.3.10, p.138]), so $N \cap A \neq 0$, and clearly $(N \cap A)^2 = 0$. Moreover, $N \cap A$ is an H-stable ideal of A, since for $n \in N \cap A$ and $h \in H$, we have by [12, 7.2.3 and 7.2.7] that

$$h \cdot n = \sum (1\#h_1)(n\#1)(\sigma^{-1}(S(h_3), h_4)\#S(h_2)).$$

This provides the desired contradiction.

Now apply part b) of Theorem 3 to finish the proof. ☐

Remark 6. The relationship between the above Corollary and the question mentioned in the beginning of this note is as follows: the condition "$A\#_\sigma H$ is nonsingular" is needed as a replacement for the condition "H is semisimple" (the latter would imply that $H = kG$ for some finite group G, and this would bring us back to the graded case). This can be seen by taking A to be trivial. Then H nonsingular is equivalent to H semisimple, or H semiprime, because H is Frobenius (see [10, 13.2, p.362]).

A simple example where $A\#H$ is nonsingular (in fact, even simple) without H being semisimple is as follows: let $\mathbf{F}_2$ be the field with two elements, $E = \mathbf{F}_2(X)$ the field of rational fractions in the indeterminate X, and let k be the subfield $\mathbf{F}_2(X^2)$. Let $\delta = \frac{d}{dX} \in Der(E)$, and denote by L the 1-dimensional Lie algebra generated by δ. Consider the action of L on E as derivations. Then the invariants of this action are just k itself, and the Hopf algebra is $H = U(L) \simeq H^*$. The extension is Galois. This follows immediately from a result in [5] which may also be found in [12, 8.3.7, p. 137], or it can be seen directly after a short computation. Now, since the extension is Galois, the smash product $E\#H$ is isomorphic to $End_k(E)$, which is a simple ring.

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