SEMI PRIME CROSSED PRODUCTS
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Abstract. We prove a result on the transfer of essentiality of extensions of modules over subnormalizing extensions of rings, then apply it to look at the semiprimeness of Hopf-Galois extensions, in particular that of crossed products.

The following is an important open question in the theory of Hopf algebra actions (see [12, Question 7.4.9, p.121]):

If $H$ is a finite-dimensional and semisimple Hopf algebra over the field $k$, and $A$ is semiprime, is any crossed product $A\#_\sigma H$ semiprime?

This possible extension of the Maschke-type theorem for crossed products [2], was inspired by the fact that it holds in the following cases:

1. when $H = kG$ ($G$ is a finite group, and $|G|^{-1} \in k$) (this is the Fisher-Montgomery theorem, [9]), or

2. when $H = kG^*$ [6].

The first case was extended in [3] to the case where $H$ is semisimple, pointed and cocommutative. The aim of the present note is to try to extend the second case, exploiting the fact that the dual Hopf algebra $H^*$ is pointed in this case. The result of this effort is Corollary 5. The technique that we use involves the so-called essential form of Maschke’s theorem. This approach was first used to provide another proof for the Fisher-Montgomery theorem in [14]. Various partial answers to the above question were also given in [1], [17], [13].

Throughout, $H$ will denote a finite-dimensional Hopf algebra over the field $k$. For all unexplained notation or definitions, the reader is referred to [12].

Recall that if $R$ is a subring of $S$, the extension $R \subset S$ is called a subnormalizing (or triangular) extension if there exist elements $x_1, x_2, x_3, \ldots \in S$ such that $S = \sum_{i=1}^{\infty} Rx_i$ and for any $j$ we have $\sum_{i=1}^{j} Rx_i = \sum_{i=1}^{j} x_i R$. A subnormalizing extension is called finite if the sequence $x_1, x_2, x_3, \ldots \in S$ is finite (see [18] or [11]). Our first result is

Theorem 1. Let $R \subset S$ be a subnormalizing extension of rings, such that the elements $x_1, x_2, x_3, \ldots \in S$ from the definition form a basis for $S$ as a left and a right $R$-module. If $M \subset N$ is an essential extension of left $R$-modules, and $N$ is nonsingular, then $S \otimes_R M \subset S \otimes_R N$ is an essential extension of $R$-modules.
Proof. Suppose the extension $S \otimes_R M \subset S \otimes_R N$ is not essential. We say $w \in S \otimes_R N$, $w \neq 0$, is a nasty element if any $R$-multiple of it is either 0 or not in $S \otimes_R M$. It is clear that a nonzero $R$-multiple of a nasty element is again nasty. Among all nasty elements choose $x_1 \otimes n_1 + \ldots + x_t \otimes n_t \in S \otimes_R N$ with $j$ minimal subject to $n_j \notin M$, $n_{j+1} \in M$, ..., $n_t \in M$.

For any $i$ there exists an automorphism $\sigma_i : R \to R$ such that $ax_i = x_i \sigma_i(a) + \text{lower terms}$ for all $a \in R$. If $a \in R$, then
\[
aw = \ldots + x_j \otimes (\sigma_j(a) n_j + n') + x_{j+1} \otimes n'_{j+1} + \ldots + x_t \otimes n'_t
\]
where $n', n'_{j+1}, \ldots, n'_{t} \in M$, and by the minimality of $j$ we have that
\[
\lambda_{\text{ann}}(w) = \sigma_j^{-1}(\lambda_{\text{ann}}(n_j)),
\]
where $\widehat{n}_j = n_j + M \in M/N$. But $\lambda_{\text{ann}}(\widehat{n}_j)$ is an essential ideal of $R$, because the extension $M \subset N$ is essential. Thus $\lambda_{\text{ann}}(w)$ is also an essential ideal.

Now, for $a \in \lambda_{\text{ann}}(w)$ we have
\[
0 = aw = x_1 \otimes n'_1 + \ldots + x_t \otimes n'_t + x_t \otimes \sigma_t(a) n_t,
\]
so $\lambda_{\text{ann}}(w) \subseteq \sigma_t^{-1}(\lambda_{\text{ann}}(n_t))$, and hence $\lambda_{\text{ann}}(n_t)$ is also an essential ideal, contradicting the fact that $N$ is nonsingular.

This implies that $S \otimes_R M$ has no nasty elements and the result follows. \qed

The next lemma is an application of the Taft-Wilson theorem, and was proved in [7] (see also [5, Exercise 7.7.9, p.338]).

Lemma 2. Let $H$ be a finite-dimensional pointed Hopf algebra acting on the algebra $A$. Then $A\#H$ is a finite subnormalizing extension of $A$, and the elements $x_1, \ldots, x_m \in A\#H$ may be chosen to form a basis of $H \subset A\#H$.

We are now in a position to state and prove our main result.

Theorem 3. Let $H$ be a finite-dimensional pointed Hopf algebra, and $A$ a left $H$-module algebra with subalgebra of invariants $A^H$, such that $A/A^H$ is a faithfully flat right $H^*$-Galois extension. Then the following hold:

a) If $M \subset A$ $N$ is an essential extension of left $A$-modules, and $N$ is left nonsingular, then $N$ is an essential extension of $M$ as left $A^H$-modules.

b) If $A$ is $H$-semiprime and $A$ is a left nonsingular ring, then $A$ is semiprime.

c) If $A^H$ is semiprime and $A$ is a left nonsingular ring, then $A$ is semiprime.

Proof. a) The proof uses the duality approach of [6], as in [10] and [4], in the form for Hopf algebra actions which was also used in [7]: since the induced functor $A\#H \otimes_A - : A - \mod \to (A\#H)^\#H^* - \mod$ is an equivalence, it follows that
\[
(1) \quad A\#H \otimes_A M \subset A\#H \otimes_A N
\]
is an essential extension of left $(A\#H)^\#H^*$-modules.

By Lemma 2 we can apply Theorem 1 for the subnormalizing extension $A \subset A\#H$ to obtain that (1) is an essential extension of left $A$-modules, and so it is also an essential extension of left $A\#H$-modules.

Now we have the functorial isomorphisms of left $A\#H$-modules $A\#H \otimes_A M \simeq A \otimes_{A^H} M$, and $A\#H \otimes_A N \simeq A \otimes_{A^H} N$, so $A \otimes_{A^H} M \subset A \otimes_{A^H} N$ is an essential extension of left $A\#H$-modules. But the induced functor $A \otimes_{A^H} - : A^H - \mod \to A\#H - \mod$ is also an equivalence, since $A/A^H$ is a faithfully flat Galois extension, and therefore we have that $M \subset A^H N$ is an essential extension.
b) The proof is similar to the proof of the Fisher-Montgomery theorem, as given in [15], but we sketch it for the convenience of the reader. Let $0 \neq N$ be an ideal of $A$ such that $N^2 = 0$. Then $I = \text{r.ann}_A(N)$ is an essential left ideal of $A$. By a), we get that $I$ is an essential $A^H$-submodule of $A$, so $I^H = I \cap A^H$ is an essential left ideal of $A^H$. Let

$$J = (I : H) = \{a \in I \mid h \cdot a \in I \ \forall h \in H\}.$$  

Since $I^H \subseteq J \subseteq I$, it follows that $J^H = I^H$, and since the extension is Galois, we get that $J = I^H A$, by [12, Corollary 8.3.10, p.138]. But now

$$\text{l.ann}_A(J) = \text{l.ann}_A(I^H) \supseteq \text{l.ann}_A(I) \supseteq N \neq 0.$$  

Thus $L = \text{l.ann}_A(J)$ is a nonzero $H$-stable ideal of $A$, which is again generated (as a left ideal) by its intersection with $A^H$. Hence we have that $L^H \cap I^H \neq 0$. Finally, $J \cap L$ is a nonzero $H$-stable ideal of $A$ with $(J \cap L)^2 = 0$, a contradiction to the fact that $A$ is $H$-semiprime.

The following is a corollary of the proof of Theorem 3 and is presumably known.

**Corollary 4.** If $G$ is a finite group acting as automorphisms on the $k$-algebra $R$, $R^G$ denotes the subalgebra of invariants, and $R/R^G$ is a Galois extension having an element of trace 1, then the following hold:

a) If $M \subset_R N$ is an essential extension of left $R$-modules, then $M \subset N$ is also an essential extension as left $R^G$-modules.

b) If $R^G$ is semiprime, then $R$ is semiprime.

d) follows from b), because $H$-stable ideals of $A$ are generated by their intersection with $A^H$.

Proof. As in the proof of part a) of Theorem 3 we want to prove that

$$R \ast kG \otimes_R M \subset R \ast kG \otimes_R N$$

is an essential extension of $R \ast kG$-modules. But since this is an essential extension of left $(R \ast kG) \mathbb{#}kG$-modules, i.e. a gr-essential extension of graded $R \ast kG$-modules, the claim follows easily from [14, 1.2.8, p. 9]. The rest of the proof is the same.

A direct consequence of Theorem 3 is the following.

**Corollary 5.** Let $H$ be a finite-dimensional Hopf algebra such that $H^*$ is pointed, and $A \mathbb{#}_\sigma H$ a crossed product with invertible cocycle $\sigma$. If $A$ is $H$-semiprime and $A \mathbb{#}_\sigma H$ is a left nonsingular ring, then $A \mathbb{#}_\sigma H$ is also semiprime.

Proof. Denote the left weak action of $H$ on $A$ by $h \cdot a$ for $a \in A$ and $h \in H$.

We remark first that if $A$ is $H$-semiprime, then $A \mathbb{#}_\sigma H$ is $H^*$-semiprime. Indeed, if $N$ is a nonzero $H^*$-stable ideal of $A \mathbb{#}_\sigma H$ with $N^2 = 0$, then $N \cap A$ generates $N$ as a left ideal (again by [12, Corollary 8.3.10, p.138]), so $N \cap A \neq 0$, and clearly $(N \cap A)^2 = 0$. Moreover, $N \cap A$ is an $H^*$-stable ideal of $A$, since for $n \in N \cap A$ and $h \in H$, we have by [12, 7.2.3 and 7.2.7] that

$$h \cdot n = \sum (1 \mathbb{#} h_1)(n \mathbb{#} 1)(\sigma^{-1}(S(h_3), h_4) \mathbb{#} S(h_2)).$$

This provides the desired contradiction.

Now apply part b) of Theorem 3 to finish the proof.
Remark 6. The relationship between the above Corollary and the question mentioned in the beginning of this note is as follows: the condition “$A \# H$ is nonsingular” is needed as a replacement for the condition “$H$ is semisimple” (the latter would imply that $H = kG$ for some finite group $G$, and this would bring us back to the graded case). This can be seen by taking $A$ to be trivial. Then $H$ nonsingular is equivalent to $H$ semisimple, or $H$ semiprime, because $H$ is Frobenius (see [10, 13.2, p.362]).

A simple example where $A \# H$ is nonsingular (in fact, even simple) without $H$ being semisimple is as follows: let $F$ be the field with two elements, $E = F(X)$ the field of rational fractions in the indeterminate $X$, and let $k$ be the subfield $F(X^2)$. Let $\delta = \frac{d}{dX} \in \text{Der}(E)$, and denote by $L$ the 1-dimensional Lie algebra generated by $\delta$. Consider the action of $L$ on $E$ as derivations. Then the invariants of this action are just $k$ itself, and the Hopf algebra is $H = U(L) \simeq H^*$. The extension is Galois. This follows immediately from a result in [5] which may also be found in [12, 8.3.7, p. 137], or it can be seen directly after a short computation. Now, since the extension is Galois, the smash product $E \# H$ is isomorphic to $\text{End}_k(E)$, which is a simple ring.

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