

## ON SOME PROPERTIES OF (FC)-SEQUENCES OF IDEALS IN LOCAL RINGS

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ABSTRACT. The paper characterizes the length of maximal sequences satisfying conditions (i) and (ii) of (FC)-sequences, and proves some properties of (FC)-sequences, such as a bound on their lengths. As a consequence we get some results for mixed multiplicities and multiplicities of Rees rings of equimultiple ideals. We also prove that if  $I$  is an ideal of positive height and  $x_1, x_2, \dots, x_p$  is an arbitrary maximal sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences, then  $(x_1, x_2, \dots, x_p)$  is a reduction of  $I$ .

### 1. INTRODUCTION

Throughout this paper, let  $(A, \mathfrak{m})$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ , infinite residue field  $k = A/\mathfrak{m}$ , and Krull dimension  $\dim A = d > 0$ . Let  $J$  be an  $\mathfrak{m}$ -primary ideal and  $\{I_1, I_2, \dots, I_s\}$  a set of ideals of positive height. Then the *Bhattacharya function*

$$B(n, n_1, \dots, n_s) = l_A\left(\frac{J^n I_1^{n_1} \dots I_s^{n_s}}{J^{n+1} I_1^{n_1} \dots I_s^{n_s}}\right)$$

is a polynomial of degree  $d - 1$  for all large values of  $n, n_1, \dots, n_s$ . This polynomial is called the *Bhattacharya polynomial* of  $(J, I_1, \dots, I_s)$ . The terms of total degree  $d - 1$  in this polynomial have the form

$$\sum_{d_0 + d_1 + \dots + d_s = d - 1} e_A(J^{[d_0+1]}, I_1^{[d_1]}, \dots, I_s^{[d_s]}) \frac{n^{d_0} n_1^{d_1} \dots n_s^{d_s}}{d_0! d_1! \dots d_s!}.$$

Here  $e_A(J^{[d_0+1]}, I_1^{[d_1]}, \dots, I_s^{[d_s]})$  are non-negative integers not all zero, called the *mixed multiplicity* of  $(J, I_1, \dots, I_s)$  of the *type*  $(d_0 + 1, d_1, \dots, d_s)$  [3].

Mixed multiplicities were first considered in [11] by Teissier and Risler for two  $\mathfrak{m}$ -primary ideals and in this case they can be interpreted as the multiplicity of *general elements*. Next, Rees in [9] showed the existence of joint reductions of  $d$   $\mathfrak{m}$ -primary ideals and the mixed multiplicity of those ideals is the multiplicity of the ideal generated by a *joint reduction*. A natural question is the relationship between mixed multiplicities of arbitrary ideals and Zariski-Samuel multiplicities. In answer to this question, Viet in [16] built a sequence of elements called an *(FC)-sequence*.

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**Definition** ([16]). Let  $U = \{I_1, I_2, \dots, I_s\}$  be a set of ideals of  $A$ ,  $I = I_1 I_2 \cdots I_s$ ,  $N := \bigcup_{n \geq 0} (0 : I^n)$ ,  $A^* = A/N$ ,  $I_i^* = I_i A^*$ ;  $i = 1, 2, \dots, s$ . Recall that an element  $x \in A$  satisfies the condition (FC) with respect to  $U$  if there exist an ideal  $I_i$  of  $U$  and an integer  $n'_i$  such that

(i)  $x \in I_i \setminus \mathfrak{m}I_i$  and

$$I_1^{*n_1} \cdots I_i^{*n_i} \cdots I_s^{*n_s} \cap (x^*) = I_1^{*n_1} \cdots I_{i-1}^{*n_{i-1}} I_i^{*n_i-1} I_{i+1}^{*n_{i+1}} \cdots I_s^{*n_s} x^*$$

for all  $n_i \geq n'_i$  and for all non-negative integers  $n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_s$ , where  $x^*$  is the image of  $x$  in  $A^*$ .

(ii)  $x$  is a filter-regular element with respect to  $I$ , i.e.,  $0 : x \subseteq \bigcup_{n \geq 0} (0 : I^n)$  (see [12]).

(iii)  $\dim A / \bigcup_{n \geq 0} [(x) : I^n] = \dim A/N - 1$ .

A sequence  $x_1, x_2, \dots, x_t$  of  $A$  is said to be an (FC)-sequence with respect to  $U$  if  $\bar{x}_{i+1}$  is an element satisfying the condition (FC) with respect to  $\bar{U} = \{\bar{I}_1, \bar{I}_2, \dots, \bar{I}_s\}$  for each  $i = 0, 1, \dots, t-1$ , where  $\bar{A} = A/(x_1, x_2, \dots, x_i)$ ,  $\bar{x}_{i+1}$  is the image of  $x_{i+1}$  in  $\bar{A}$ ,  $\bar{I}_1 = I_1 \bar{A}, \dots, \bar{I}_s = I_s \bar{A}$ .

These elements satisfy the properties of filter-regular elements in [12] and elements of a complete reduction in [9]. That is why in [16] the author chose the name “(FC)”.

The results of [16] showed that (FC)-sequences carry important information on mixed multiplicities. One of interesting results of [16] is the following theorem.

**Theorem** ([16, Theorem 4.1]). *Let  $A$  be a Noetherian local ring and  $J$  an  $\mathfrak{m}$ -primary ideal. Let  $I$  be an ideal of positive height  $h$ . Suppose that  $x_1, x_2, \dots, x_q$  is a maximal (FC)-sequence in  $I$  with respect to  $U = \{J, I\}$ . Then*

(i)  $e(J^{[d-i]}, I^{[i]}) = e(J; A/(x_1, x_2, \dots, x_i))$  for all  $i \leq h-1$ .

(ii)  $e(J^{[d-i]}, I^{[i]}) = e(J; A/\bigcup_{n \geq 0} [(x_1, x_2, \dots, x_i) : I^n])$  for  $h \leq i \leq q$ .

(iii)  $e(J^{[d-i]}, I^{[i]}) \neq 0$  if and only if  $i \leq q$ .

It follows readily from this theorem that for any  $\mathfrak{m}$ -primary ideal  $J$ , the length of maximal (FC)-sequences in  $I$  with respect to  $U = \{J, I\}$  is an invariant and this invariant is the largest nonvanishing degree of mixed multiplicities  $e(J^{[d-i]}, I^{[i]})$ .

Based on the fact that maximal (FC)-sequences in  $I$  with respect to  $U$  have the same length (see Lemma 2.1(i)) and maximal sequences in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences have also the same length (see Proposition 2.2), we can state the first main result of this paper as follows:

**Theorem 2.3.** *Let  $J$  be  $\mathfrak{m}$ -primary. Let  $I$  be an ideal of positive height  $ht(I)$  and analytic spread  $\ell(I)$ . Suppose that  $p$  is the length of maximal sequences in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U = \{J, I\}$  and  $q$  is the length of maximal (FC)-sequences in  $I$  with respect to  $U$ . Then*

(i)  $p = \dim R(I) / \bigcup_{n \geq 0} [J^{n+1} R(I) : J^n R(I)]$ .

(ii)  $ht(I) - 1 \leq q \leq p - 1 \leq \ell(I) - 1$ .

(iii) Let  $x_1, x_2, \dots, x_i$  be a sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$ , then  $x_1, x_2, \dots, x_i$  is an (FC)-sequence if and only if  $i \leq q$ .

At this point we remark that if  $I$  is equimultiple, then as immediate consequences of Theorem 2.3 we have  $ht(I) - 1 = q = p - 1 = \ell(I) - 1$ . Moreover, we also get

some interesting results for mixed multiplicities and multiplicities of Rees rings of equimultiple ideals (see Theorem 3.1 and Theorem 3.2, Section 3).

Finally, we get the second main result of this paper as follows:

**Theorem 3.3.** *Let  $J$  be  $\mathfrak{m}$ -primary. Let  $I$  be an ideal of positive height. Set  $p = \dim R(I)/\bigcup_{n \geq 0} [J^{n+1}R(I) : J^n R(I)]$ . Suppose that  $x_1, x_2, \dots, x_p$  is a sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $\{J, I\}$ . Then  $(x_1, x_2, \dots, x_p)$  is a reduction of  $I$ .*

## 2. THE LENGTH OF MAXIMAL (FC)-SEQUENCES

From [16, Theorem 4.1] we see that the length of maximal (FC)-sequences is connected closely with the degrees of vanishing and non-vanishing of mixed multiplicities. In this section we give bounds for this length and characterizations for the length of maximal sequences satisfying conditions (i) and (ii) of (FC)-sequences.

We begin by establishing the following lemma.

**Lemma 2.1.** *Let  $J$  be an  $\mathfrak{m}$ -primary ideal and let  $I$  be an ideal of positive height. Set  $U = \{J, I\}$ . Then*

- (i) *Maximal (FC)-sequences in  $I$  with respect to  $U$  have the same length.*
- (ii) *Let  $e_A(J^{[d-i]}, I^{[i]}) \neq 0$  and  $i > 0$ . Suppose that  $x$  is an element in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$ . Then  $x$  also satisfies the condition (iii) of (FC)-sequences (i.e.,  $x$  satisfies the condition (FC)) with respect to  $U$ .*

*Proof.* The proof of (i): By [16, Theorem 4.1], it follows that the length of any maximal (FC)-sequence in  $I$  with respect to  $U$  is equal to

$$\max \{i | e_A(J^{[d-i]}, I^{[i]}) \neq 0\}.$$

So the length of maximal (FC)-sequences in  $I$  with respect to  $U$  is an invariant.

The proof of (ii): Set  $N := \bigcup_{n \geq 0} (0 : I^n)$ ;  $A^* = A/N$ ,  $J^* = JA^*$ ,  $I^* = IA^*$ . We denote by  $B(m, n)$  the Bhattacharya polynomial of function  $l_{A^*}(\frac{J^{*m} I^{*n}}{J^{*m+1} I^{*n}})$ . By [2] we have  $\deg B(m, n) = d - 1$ . Let  $x \in I$  be an element satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$ . If  $x^*$  is the image of  $x$  in  $A^*$ , then  $x^*$  is not contained in any prime ideal belonging to  $\text{Ass } A^*$  (see condition (ii) of (FC)). Hence  $x^*$  is a non-zero-divisor in  $A^*$ . Set

$$B = A^*/x^*A^*, \mathfrak{b} = J^*B, \mathfrak{c} = I^*B.$$

For all large  $m$  and  $n$ , we have

$$\begin{aligned} l_B\left(\frac{\mathfrak{b}^m \mathfrak{c}^n}{\mathfrak{b}^m \mathfrak{c}^n}\right) &= l_{A^*}\left(\frac{J^{*m} I^{*n} + (x^*)}{J^{*m+1} I^{*n} + (x^*)}\right) = l_{A^*}\left(\frac{J^{*m} I^{*n}}{J^{*m+1} I^{*n} + (x^*) \cap J^{*m} I^{*n}}\right) \\ &= l_{A^*}\left(\frac{J^{*m} I^{*n}}{J^{*m+1} I^{*n} + x^* J^{*m} I^{*n-1}}\right) \text{ (see condition (i) of (FC))} \\ &= l_{A^*}\left(\frac{J^{*m} I^{*n}}{J^{*m+1} I^{*n}}\right) - l_{A^*}\left(\frac{J^{*m+1} I^{*n} + x^* J^{*m} I^{*n-1}}{J^{*m+1} I^{*n}}\right) \\ &= l_{A^*}\left(\frac{J^{*m} I^{*n}}{J^{*m+1} I^{*n}}\right) - l_{A^*}\left(\frac{x^* J^{*m} I^{*n-1}}{J^{*m+1} I^{*n} \cap (x^*) \cap J^{*m} I^{*n}}\right) \\ &= l_{A^*}\left(\frac{J^{*m} I^{*n}}{J^{*m+1} I^{*n}}\right) - l_{A^*}\left(\frac{x^* J^{*m} I^{*n-1}}{x^* J^{*m+1} I^{*n-1}}\right). \end{aligned}$$

Since  $x^*$  is a non-zero-divisor in  $A^*$ , it follows that

$$l_{A^*}\left(\frac{J^{*m}I^{*n-1}}{J^{*m+1}I^{*n-1}}\right) = l_{A^*}\left(\frac{x^*J^{*m}I^{*n-1}}{x^*J^{*m+1}I^{*n-1}}\right).$$

Hence

$$\begin{aligned} (*) \quad l_B\left(\frac{\mathfrak{b}^m\mathfrak{c}^n}{\mathfrak{b}^{m+1}\mathfrak{c}^n}\right) &= l_{A^*}\left(\frac{J^{*m}I^{*n}}{J^{*m+1}I^{*n}}\right) - l_{A^*}\left(\frac{J^{*m}I^{*n-1}}{J^{*m+1}I^{*n-1}}\right) \\ &= B(m, n) - B(m, n-1) \end{aligned}$$

for all large  $m$  and  $n$ . By [16, Proposition 3.1] we have

$$e_{A^*}(J^{*[d-i]}, I^{*[i]}) = e_A(J^{[d-i]}, I^{[i]}).$$

So  $e_{A^*}(J^{*[d-i]}, I^{*[i]}) \neq 0$ . Note that  $i > 0$ . By (\*), it follows that  $l_B\left(\frac{\mathfrak{b}^m\mathfrak{c}^n}{\mathfrak{b}^{m+1}\mathfrak{c}^n}\right)$  is a polynomial of degree  $d-2$  for all large  $m$  and  $n$ . Thus,

$$\dim A / \bigcup_{n \geq 0} [(x) : I^n] = d-1 = \dim A^* - 1$$

and  $x$  satisfies condition (iii) of (FC)-sequences.  $\square$

By (\*), there exists a positive integer  $z$  such that

$$\begin{aligned} l_B\left(\frac{\mathfrak{b}^m\mathfrak{c}^n}{\mathfrak{b}^{m+1}\mathfrak{c}^n}\right) &= l_{A^*}\left(\frac{J^{*m}I^{*n}}{J^{*m+1}I^{*n}}\right) - l_{A^*}\left(\frac{J^{*m}I^{*n-1}}{J^{*m+1}I^{*n-1}}\right) \\ &= B(m, n) - B(m, n-1) \end{aligned}$$

for all  $m, n \geq z$ . Fix an integer  $v \geq z$ , set  $m = v$  and  $Q(n) = B(v, n)$ . Let  $Q^*(n)$  denote the polynomial in  $n$  of function  $l_B\left(\frac{\mathfrak{b}^v\mathfrak{c}^n}{\mathfrak{b}^{v+1}\mathfrak{c}^n}\right)$ . From this it follows that

$$(**) \quad Q(n) - Q(n-1) = Q^*(n).$$

Motivated by this fact, we introduce the following result.

**Proposition 2.2.** *Let  $J$  be an  $\mathfrak{m}$ -primary ideal and let  $I$  be an ideal of positive height. Set  $U = \{J, I\}$ . Then maximal sequences in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$  have the same length and this length is equal to  $\deg Q(n) + 1$ .*

*Notes.* (a)  $J$  is an  $\mathfrak{m}$ -primary ideal, then  $JI$  is nilpotent if and only if  $I$  is nilpotent. (b) By [16, Remark 1], it follows that if  $x_1, x_2, \dots, x_f$  is a sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $\{J, I\}$ , then  $x_1, x_2, \dots, x_f$  is a maximal sequence if and only if  $\bar{I} = I[A/(x_1, x_2, \dots, x_f)]$  is nilpotent.

*Proof of Proposition 2.2.* Since  $\text{ht}(I) > 0$ , there exists an element  $x_1 \in I$  satisfying conditions (i) and (ii) of (FC)-sequences (see [16, Remark 1]). Let  $x_1, x_2, \dots, x_p$  be a maximal sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$ . Set  $t = \deg Q(n)$ . Since  $\text{ht}(I) > 0$ ,  $\deg Q(n) \geq 0$ . We shall begin by showing that  $p = t + 1$ . Set  $N := \bigcup_{n \geq 0} (0 : I^n)$ ;  $A^* = A/N$ ,  $J^* = JA^*$ ,  $I^* = IA^*$ . Let  $x_1^*$  denote the image of  $x_1$  in  $A^*$ . Set  $B = A^*/x_1^*A^*$ ,  $\mathfrak{b} = J^*B$ ,  $\mathfrak{c} = I^*B$ . We will denote by  $x'_i$  the image of  $x_i$  in  $B$  for  $i = 2, \dots, t$ . The proof is by induction on  $t$ . For  $t = 0$ , by (\*\*\*) we have  $Q^*(n) = 0$ . From this it follows that  $\mathfrak{b}\mathfrak{c}$  is nilpotent. Hence  $\mathfrak{c}$  is nilpotent by Note (a). By Note (b), this implies  $p = 1$ . So  $p = \deg Q(n) + 1$ . Suppose that the result has been proved for  $t-1 \geq 0$ , we need to show that the result is true for  $t$ . Since  $\deg Q^*(n) = t-1$  and  $x'_2, \dots, x'_p$  is also a maximal sequence

in  $\mathfrak{c}$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $\{\mathfrak{b}, \mathfrak{c}\}$ . By inductive assumption, it follows that  $p - 1 = \deg Q^*(n) + 1 = t$ . Thus,  $p = t + 1$ . The induction is complete. By the result just obtained, it follows that the length of maximal sequences in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$  is an invariant and this invariant is equal to  $\deg Q(n) + 1$ .  $\square$

Let  $I$  be an ideal of positive height. We call  $\ell(I) := \dim \bigoplus_{n \geq 0} (I^n / \mathfrak{m}I^n)$  the analytic spread of  $I$ . Then our first main result is the following theorem.

**Theorem 2.3.** *Let  $J$  be  $\mathfrak{m}$ -primary. Let  $I$  be an ideal of positive height  $ht(I)$  and analytic spread  $\ell(I)$ . Suppose that  $q$  is the length of maximal (FC)-sequences in  $I$  with respect to  $U = \{J, I\}$  and  $p$  is the length of maximal sequences in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$ . Then*

- (i)  $p = \dim R(I) / \bigcup_{n \geq 0} [J^{n+1}R(I) : J^n R(I)]$ .
- (ii)  $ht(I) - 1 \leq q \leq p - 1 \leq \ell(I) - 1$ .
- (iii) *Let  $x_1, x_2, \dots, x_i$  be a sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$ , then  $x_1, x_2, \dots, x_i$  is an (FC)-sequence if and only if  $i \leq q$ .*

*Proof.* The proof of (i). Let  $B(m, n)$  denote the Bhattacharya polynomial of function

$l_A(\frac{J^m I^n}{J^{m+1} I^n})$ . We assume that  $B(m, n) = l_A(\frac{J^m I^n}{J^{m+1} I^n})$  for all  $m, n \geq z$ . Fix an integer  $v \geq z$ , set  $m = v$  and  $Q(n) = B(v, n)$ . Set

$$M_v = \frac{J^v R(I)}{J^{v+1} R(I)},$$

it is easily seen that  $M_v$  is a finitely generated graded  $R(I)$ -module (see [4]). Evidently, the component of degree  $n$  of  $M_v$  is  $(M_v)_n = J^v I^n / J^{v+1} I^n$ . Consequently,

$$l_A[(M_v)_n] = l_A(\frac{J^v I^n}{J^{v+1} I^n}) = Q(n)$$

is a polynomial in  $n$  having degree  $\dim M_v - 1$  for all  $n > v$ . It can be verified that  $\dim M_v = \dim [R(I) / J^{v+1} R(I) : J^v R(I)]$ . By Proposition 2.2 we have  $p = \deg Q(n) + 1$ . Hence

$$p = \dim [R(I) / J^{v+1} R(I) : J^v R(I)] \text{ for all large } v.$$

Since

$$J^{t+1} R(I) : J^t R(I) \subseteq J^{t+2} R(I) : J^{t+1} R(I)$$

for all  $t$ ,  $\dim R(I) / \bigcup_{n \geq 0} [J^{n+1} R(I) : J^n R(I)] = \dim [R(I) / J^{v+1} R(I) : J^v R(I)]$  for all large  $v$ . Thus,  $p = \dim R(I) / \bigcup_{n \geq 0} [J^{n+1} R(I) : J^n R(I)]$ .

The proof of (ii): We write the terms of total degree  $d - 1$  in the Bhattacharya polynomial  $B(m, n)$  of the Bhattacharya function  $l_A(\frac{J^m I^n}{J^{m+1} I^n})$  as

$$\sum_{i=0}^{d-1} e_A(J^{[d-i]}, I^{[i]}) \frac{m^{d-i-1} n^i}{i!(d-i-1)!}.$$

Since  $\sqrt{J^{v+1} R(I)} = \sqrt{\mathfrak{m}R(I)}$ , it follows that

$$\dim [R(I) / J^{v+1} R(I)] = \dim [R(I) / \mathfrak{m}R(I)] = \ell(I).$$

Further, we have

$$\dim [R(I) / J^{v+1} R(I) : J^v R(I)] \leq \dim [R(I) / J^{v+1} R(I)].$$

Using the results just obtained we get  $p \leq \ell(I)$ . Since

$$e(J^{[d-q]}, I^{[q]}) \neq 0 \text{ and } e(J^{[d-i]}, I^{[i]}) = 0$$

for all  $i > q$ ,

$$q = \deg \sum_{i=0}^{d-1} e_A(J^{[d-i]}, I^{[i]}) \frac{v^{d-i-1} n^i}{i!(d-i-1)!} \leq \deg Q(n).$$

So we get  $q \leq p - 1 = \dim R(I) / \bigcup_{n \geq 0} [J^{n+1}R(I) : J^n R(I)] - 1 \leq \ell(I) - 1$ . By [16, Theorem 4.1] we have  $q \geq \text{ht}(I) - 1$ . Thus, (ii) of Theorem 2.3 have been proved.

The proof of (iii): We first will prove by induction on  $q$  that if  $i \leq q$  and  $x_1, x_2, \dots, x_i$  is a sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$ , then  $x_1, x_2, \dots, x_i$  is also an (FC)-sequence. If  $q = 0$ , in this case the result is trivial. For suppose the result has been proved for  $q - 1 \geq 0$ . As the next step, we claim that the result is true for  $q$ . On the one hand  $q > 0$  and on the other hand  $e(J^{[d-q]}, I^{[q]}) \neq 0$ . By Lemma 2.1(ii), it follows that  $x_1$  satisfies condition (iii) of (FC)-sequences with respect to  $U$ . Hence by [16, Proposition 3.3], we get  $e(J^{[d-q]}, I^{[q]}) = e_{\bar{A}}(\bar{J}^{[d-q]}, \bar{I}^{[q-1]})$ , where  $\bar{A} = A/(x_1)$  and  $\bar{J} = J\bar{A}$ ,  $\bar{I} = I\bar{A}$ . From this equality we have  $e_{\bar{A}}(\bar{J}^{[d-q]}, \bar{I}^{[q-1]}) \neq 0$ . It can be verified that  $q - 1$  is the length of maximal (FC)-sequences in  $\bar{I}$  with respect to  $\{\bar{J}, \bar{I}\}$ . Moreover,  $\bar{x}_2, \dots, \bar{x}_i$  is a sequence satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $\{\bar{J}, \bar{I}\}$ , where  $\bar{x}_j$  the image of  $x_j$  in  $\bar{A}$ ,  $j = 2, \dots, i$ . Since  $i - 1 \leq q - 1$ , by our inductive assumption applied to  $(q - 1)$  that  $\bar{x}_2, \dots, \bar{x}_i$  is a sequence satisfying conditions of (FC)-sequences with respect to  $\{\bar{J}, \bar{I}\}$ . Since  $x_1$  satisfies the condition (FC) with respect to  $U$  and  $\bar{x}_2, \bar{x}_3, \dots, \bar{x}_i$  is a sequence satisfying conditions of (FC)-sequences with respect to  $\{\bar{J}, \bar{I}\}$ , it follows that  $x_1, x_2, \dots, x_i$  is a sequence satisfying conditions of (FC)-sequences with respect to  $U$ . The induction is complete. So we have proved that if  $i \leq q$  and  $x_1, x_2, \dots, x_i$  is a sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$ , then  $x_1, x_2, \dots, x_i$  is an (FC)-sequence. Since  $q$  is the length of maximal (FC)-sequences in  $I$  with respect to  $U$ . Hence, using the result just obtained we get (iii). Theorem 2.3 has been proved.  $\square$

### 3. MAXIMAL (FC)-SEQUENCES AND REDUCTIONS

First, we will give some results on (FC)-sequences and mixed multiplicities of equimultiple ideals.

It is well-known that if  $\ell(I)$  is the analytic spread of  $I$ , then  $\text{ht}(I) \leq \ell(I)$ . In the case  $\text{ht}(I) = \ell(I)$ , the ideal  $I$  is called *equimultiple*.

Let  $J$  be  $\mathfrak{m}$ -primary. Let  $I$  be an equimultiple ideal of positive height. If  $q$  is the length of maximal (FC)-sequences in  $I$  with respect to  $U = \{J, I\}$  and  $p$  is the length of maximal sequences in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences, then by Theorem 2.3 we have  $\text{ht}(I) - 1 = q = p - 1 = \ell(I) - 1$ . As a consequence of Theorem 2.3 and [16, Theorem 4.1], we immediately get the following result.

**Theorem 3.1.** *Let  $J$  be  $\mathfrak{m}$ -primary. Let  $I$  be an equimultiple ideal of positive height  $\text{ht}(I) = h$ . Suppose that  $q$  is the length of maximal (FC)-sequences in  $I$  with respect to  $U = \{J, I\}$ . Then*

- (i)  $q = \text{ht}(I) - 1$ .
- (ii) *The length of maximal sequences in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$  is equal to  $h$ .*

- (iii)  $e(J^{[d-i]}, I^{[i]}) = 0$  if and only if  $i \geq ht(I)$ .
- (iv) If  $x_1, x_2, \dots, x_{h-1}$  is a sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $U$ , then  $x_1, x_2, \dots, x_{h-1}$  is also an (FC)-sequence and  $e(J^{[d-i]}, I^{[i]}) = e(J; A/(x_1, x_2, \dots, x_i))$  for all  $i \leq h - 1$ .

From Theorem 3.1(iv) and [16, Theorem 4.2] we immediately get the following theorem for multiplicities of Rees rings of equimultiple ideals.

**Theorem 3.2.** *Let  $R := \bigoplus_{n \geq 0} I^n$  be the Rees algebra of an equimultiple ideal  $I$  of positive height  $ht(I) = h$ . Let  $J$  be an  $\mathfrak{m}$ -primary ideal of  $A$ . Suppose that  $x_1, x_2, \dots, x_{h-1}$  is an arbitrary sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $\{J, I\}$ . Then*

$$e((J, It); R) = \sum_{i=0}^{h-1} e(J; A/(x_1, \dots, x_i)) \text{ and } e(R) = \sum_{i=0}^{h-1} e(A/(x_1, \dots, x_i)).$$

An ideal  $\mathfrak{J}$  is called a *reduction* of  $I$  if  $\mathfrak{J} \subseteq I$  and  $I^{n+1} = \mathfrak{J}I^n$  for some non-negative integer  $n$ .

We proceed now to establish the second main result of this paper as follows.

**Theorem 3.3.** *Let  $J$  be  $\mathfrak{m}$ -primary. Let  $I$  be an ideal of positive height. Set  $p = \dim R(I)/\bigcup_{n \geq 0} [J^{n+1}R(I) : J^n R(I)]$ . Suppose that  $x_1, x_2, \dots, x_p$  is a sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $\{J, I\}$ . Then  $(x_1, x_2, \dots, x_p)$  is a reduction of  $I$ .*

*Proof.* Let  $x_1, x_2, \dots, x_p$  be a sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $\{J, I\}$ . Since

$$\dim R(I)/\bigcup_{n \geq 0} [J^{n+1}R(I) : J^n R(I)] = p,$$

$x_1, x_2, \dots, x_p$  is a maximal sequence by Theorem 2.3(i). By Note (b), this implies that  $I[A/(x_1, x_2, \dots, x_p)]$  is nilpotent. Hence  $\sqrt{I} = \sqrt{(x_1, x_2, \dots, x_p)}$ . Next, the proof is by induction on  $i$  that  $(x_1, x_2, \dots, x_i) \cap I^{n+1} = (x_1, x_2, \dots, x_i)I^n$  for all large  $n$  and all  $i \leq p$ . If  $i = 0$ , in this case the result is trivial. For suppose the result has been proved for  $i - 1 \geq 0$ . As the next step, we claim that the result is true for  $i$ . Set  $I' := \bigcup_{n \geq 0} [(x_1, x_2, \dots, x_{i-1}) : I^n]$ . Denote by  $x'_i$  the image of  $x_i$  in  $A/(x_1, x_2, \dots, x_{i-1})$ . Since  $x'_i$  is an element satisfying condition (i) of (FC)-sequences, it can be verified that  $(I^{n+1} + I') \cap (I', x_i) = (x_i I^n + I')$  for all large  $n$ . From this it follows that

$$\begin{aligned} I^{n+1} \cap (I', x_i) &= I^{n+1} \cap (I^{n+1} + I') \cap (I', x_i) \\ &= I^{n+1} \cap (x_i I^n + I') = x_i I^n + I' \cap I^{n+1} \end{aligned}$$

for all large  $n$ . Therefore,

$$(1) \quad I^{n+1} \cap (I', x_i) = x_i I^n + I' \cap I^{n+1}$$

for all large  $n$ . By the Artin-Rees lemma, then there exists an integer  $t$  such that  $I^n \cap I' = I^{n-t}(I^t \cap I')$  for all  $n \geq t$ . Hence, as  $I^n I' \subseteq (x_1, x_2, \dots, x_{i-1})$  for large  $n$ , we get  $I^n \cap I' = I^{n-t}(I^t \cap I') \subseteq I^n \cap I^{n-t} I' \subseteq I^n \cap (x_1, x_2, \dots, x_{i-1})$  for all large  $n$ . Consequently,

$$(2) \quad I^n \cap I' = I^n \cap (x_1, x_2, \dots, x_{i-1})$$

for large  $n$ . By our inductive assumption applied to  $(i - 1)$  we get

$$(x_1, x_2, \dots, x_{i-1}) \cap I^{n+1} = (x_1, x_2, \dots, x_{i-1})I^n$$

for all large  $n$ . Combining this fact with (2), we have  $I^{n+1} \cap I' = (x_1, x_2, \dots, x_{i-1})I^n$  for all large  $n$ . Hence by (1), we get  $I^{n+1} \cap (I', x_i) = (x_1, x_2, \dots, x_{i-1})I^n + x_i I^n = (x_1, x_2, \dots, x_i)I^n$  for all large  $n$ . It follows readily from this that

$$(3) \quad I^{n+1} \cap (x_1, x_2, \dots, x_i) = (x_1, x_2, \dots, x_i)I^n$$

for all large  $n$ . The induction is complete. Since  $\sqrt{I} = \sqrt{(x_1, x_2, \dots, x_p)}$ , it follows that  $I^n \subseteq (x_1, x_2, \dots, x_p)$  for all large  $n$ . Hence

$I^{n+1} = I^{n+1} \cap (x_1, x_2, \dots, x_p) = (x_1, x_2, \dots, x_p)I^n$  for all large  $n$ . Consequently,  $(x_1, x_2, \dots, x_p)$  is a reduction of  $I$ . The proof of Theorem 3.3 is complete.  $\square$

*Remark 3.4.* Let  $I, J, p$  and  $x_1, x_2, \dots, x_p$  as in Theorem 3.3. Then combining (2) with (3), we get  $I^{n+1} \cap (\bigcup_{j \geq 0} [(x_1, x_2, \dots, x_i) : I^j]) = (x_1, x_2, \dots, x_i)I^n$  for all large  $n$  and all  $i \leq p$ .

*Remark 3.5.* From Theorem 3.1(ii) and Theorem 3.3 it follows that if  $I$  is an equimultiple ideal of positive height  $h$  and  $x_1, x_2, \dots, x_h$  is a sequence in  $I$  satisfying conditions (i) and (ii) of (FC)-sequences with respect to  $\{J, I\}$ , then  $(x_1, x_2, \dots, x_h)$  is a reduction of  $I$ .

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