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THE COMPOSITION OF PROJECTIONS ONTO CLOSED CONVEX SETS IN HILBERT SPACE IS ASYMPTOTICALLY REGULAR

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Abstract. The composition of finitely many projections onto closed convex sets in Hilbert space arises naturally in the area of projection algorithms. We show that this composition is asymptotically regular, thus proving the so-called "zero displacement conjecture" of Bauschke, Borwein and Lewis. The proof relies on a rich mix of results from monotone operator theory, fixed point theory, and convex analysis.

1. The problem

We assume that

$$\boxed{X \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle,}$$

and that

$$\boxed{C_1,\dots,C_N \text{ are closed convex nonempty sets in } X,}$$

with

$$\boxed{\text{corresponding projections } P_1,\dots,P_N.}$$

Our aim is to show that the composition $P_N P_{N-1}\cdots P_1$ is *asymptotically regular* [7], i.e.,

$$(P_N P_{N-1}\cdots P_1)^k x - (P_N P_{N-1}\cdots P_1)^{k+1} x \to 0, \qquad \text{for every } x \in X.$$

This in turn will imply the "zero displacement conjecture", formulated in [4].

We briefly sketch the origin and the interest in this conjecture. (The reader is referred to [4, 2] and [9] and the various references therein for further comments.) Numerous problems in mathematics and physical sciences can be recast as a *convex feasibility problem*: find $x \in \bigcap_{n=1}^N C_n$. A well-known result due to Bregman states that the sequence of iterates $(P_N \cdots P_1)^k x$ converges weakly to a point in the intersection *provided it is nonempty* [5]. However, in applications it may not be a priori clear whether or not the intersection is nonempty. Hence, one wishes clarity about the asymptotic behaviour of the sequence in the inconsistent case. A complete analysis has been carried out for $N = 2$ or when each set C_n is an affine subspace. Nonetheless, even for $N = 3$, some nagging questions on the behavior of

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the sequence remain. We believe that the asymptotic regularity established here is an important step towards a complete understanding of the sequence $(P_N \cdots P_1)^k x$.

It will be extremely convenient to work in the product Hilbert space X^N, equipped with the induced inner product

$$\langle x, y\rangle = \sum_{n=1}^{N} \langle x_n, y_n\rangle, \quad \text{for all } x = (x_n)_{n=1}^N \text{ and } y = (y_n)_{n=1}^N \text{ in } X^N.$$

We use standard notation and results from convex analysis and monotone operator theory; see, for instance, [11] and [13].

2. Preliminaries

We define the *"diagonal"*

$$\boxed{\Delta := \{(\xi, \xi, \ldots, \xi) \in X^N : \xi \in X\},}$$

and the *right-shift operator*

$$\boxed{R : X^N \to X^N : (x_1, x_2, \ldots, x_N) \mapsto (x_N, x_1, \ldots, x_{N-1}).}$$

Clearly, R is an isometry and the conjugate (or adjoint) of R is the *left-shift operator*

$$\boxed{L : X^N \to X^N : (x_1, x_2, \ldots, x_N) \mapsto (x_2, x_3, \ldots, x_N, x_1).}$$

Now set

$$\boxed{M = I - R.}$$

Proposition 2.1. *M is maximal monotone.*

Proof. Clearly, M is a continuous linear operator from X^N to itself. For $x \in X^N$, we have $\langle x, Rx\rangle \le \|x\|\|Rx\| = \|x\|\|x\| = \langle x, x\rangle$, and hence $\langle x, Mx\rangle = \langle x, (I-R)x\rangle \ge 0$. By linearity of M, the operator M is monotone. Continuity now implies maximal monotonicity; see, for instance, [12, Corollary 2.6]. □

Fact 2.2. *Suppose A is a continuous linear monotone operator from a Banach space E to its dual E^*. Let $S = \frac{1}{2}A + \frac{1}{2}A^*$ be the symmetric part of A, where A^* denotes the conjugate operator of A. Further, let $q(x) = \frac{1}{2}\langle x, Ax\rangle$, for all $x \in E$. Then:*

(i) *q is convex and Gâteaux differentiable with $\nabla q = S$.*
(ii) *$q^* \circ S = q$, where q^* is the conjugate function of q from convex analysis.*
(iii) *$\operatorname{ran} S \subseteq \operatorname{dom} q^* \subseteq \operatorname{cl} \operatorname{ran} S$.*

Proof. For (i) and (ii), see [3, Theorem 3.6.(i)].

(iii): this is in [1, Proposition 12.3.6.(iii)]; for the reader's convenience, we repeat the argument here. Note first that (ii) implies the inclusion $\operatorname{ran} S \subseteq \operatorname{dom} q^*$. Next, fix $x^* \in \operatorname{dom} q^*$ and let $f = q - x^*$. Then

$$q^*(x^*) = \sup_{x\in E}\big[\langle x^*, x\rangle - q(x)\big] = -\inf_{x\in E}\big[q(x) - \langle x^*, x\rangle\big] = -\inf_{x\in E} f(x).$$

Fix $\epsilon > 0$. By [11, Lemma 3.22], there exists some $x \in E$ such that $\|\nabla f(x)\| = \|Sx - x^*\| < \epsilon$. Since ϵ was chosen arbitrarily, it follows that $x^* \in \operatorname{cl} \operatorname{ran} S$. □

We now let S be the *symmetric part of* M, i.e.,

$$\boxed{S = \tfrac{1}{2}M + \tfrac{1}{2}M^* = I - \tfrac{1}{2}R - \tfrac{1}{2}L}$$

and

$$\boxed{q : X^N \to \,]-\infty, +\infty] : x \mapsto \tfrac{1}{2}\langle x, Mx\rangle.}$$

Proposition 2.3. $\operatorname{ran} S = \operatorname{dom} q^* = \Delta^\perp$.

Proof. In light of Proposition 2.1 and Fact 2.2(iii), we have $\operatorname{ran} S \subseteq \operatorname{dom} q^* \subseteq \operatorname{cl}\operatorname{ran} S$. Thus it suffices to show that $\operatorname{ran} S = \Delta^\perp$. It is easy to see that $y = (y_n)_{n=1}^N \in \Delta^\perp$ if and only if $\sum_{n=1}^N y_n = 0$. Using this, it is readily verified that $\operatorname{ran} S \subseteq \Delta^\perp$. The reverse inclusion takes a little more work: pick $y = (y_n)_{n=1}^N \in \Delta^\perp$. We need to show that $y \in \operatorname{ran} S$. For $1 \leq n \leq N-1$, set

$$z_n = y_1 + 2y_2 + 3y_3 + \cdots + ny_n.$$

Next, set $x_N = 0$ and define $x = (x_n)_{n=1}^N$ by the backwards recursion

$$(n+1)x_n - nx_{n+1} = z_n, \quad \text{for } 1 \leq n \leq N-1.$$

Then $y_1 = z_1 = 2x_1 - x_2$ and thus

$$-x_N + 2x_1 - x_2 = y_1.$$

On the other hand, $ny_n = z_n - z_{n-1} = (n+1)x_n - nx_{n+1} - nx_{n-1} + (n-1)x_n = 2nx_n - nx_{n+1} - nx_{n-1}$, or

$$-x_{n-1} + 2x_n - x_{n+1} = y_n, \quad \text{for } 2 \leq n \leq N-1.$$

Altogether, $\sum_{n=1}^{N-1} y_n = x_1 + x_{N-1} - 2x_N$. This implies, using $\sum_{n=1}^N y_n = 0$ or equivalently $\sum_{n=1}^{N-1} y_n = -y_N$,

$$-x_{N-1} + 2x_N - x_1 = y_N.$$

The last three displayed equations are equivalent to $y = S(2x)$. ☐

The following fact on the approximation of the sum of two monotone operators will be crucial.

Fact 2.4 (Brézis-Haraux)**.** *Suppose E is a Hilbert space, and S_1 and S_2 are two monotone operators from E to 2^{E^*} such that $S_1 + S_2$ is maximal monotone. Suppose further that* $\operatorname{dom} S_1 \subseteq \operatorname{dom} S_2$*, and for every* $x_2 \in \operatorname{dom} S_2$ *and* $y_2^* \in \operatorname{ran} S_2$*, we have*

$$\sup_{w^* \in S_2(w)} \langle w - x_2, y_2^* - w^* \rangle < +\infty.$$

Then:

(i) $\operatorname{cl}\operatorname{ran}(S_1 + S_2) = \operatorname{cl}(\operatorname{ran}(S_1) + \operatorname{ran}(S_2))$, *and*
(ii) $\operatorname{int}\operatorname{ran}(S_1 + S_2) = \operatorname{int}(\operatorname{ran}(S_1) + \operatorname{ran}(S_2))$.

Proof. See [6] as well as the new approach in [13, Section 19]. ☐

We conclude this section with an immensely useful result from fixed point theory. Recall that a map T is called *strongly nonexpansive* [8], if it is nonexpansive, and $(x_k - y_k) - (Tx_k - Ty_k) \to 0$ whenever $(x_k - y_k)$ is bounded and $\|x_k - y_k\| - \|Tx_k - Ty_k\| \to 0$ for sequences $(x_k), (y_k)$.

Fact 2.5 (Bruck-Reich)**.** *Suppose E is a Hilbert space, and $T : E \to E$ is strongly nonexpansive. Then there exists a vector $v \in E$ such that $T^k x - T^{k+1}x \to v$. In fact, v is the unique element of minimum norm in* $\operatorname{cl}\operatorname{ran}(I - T)$.

Proof. This follows from the considerably more general [8, Corollary 1.5]. (The function $\frac{1}{2}\|\cdot\|^2$ is Fréchet differentiable in Hilbert space, and E is a sunny nonexpansive retract of itself (via the identity); hence Bruck and Reich's assumptions in [8, Corollary 1.5] are indeed satisfied.) □

3. Main result

Theorem 3.1. *The composition $(P_N P_{N-1} \cdots P_1)$ is asymptotically regular.*

Proof. We work mostly in the product space X^N, in which we set $C = C_1 \times \cdots \times C_N$. Separability of the set C readily implies that P_C, the projection onto C, is separable as well: $P_C = P_1 \times \cdots \times P_N$. Denote the subdifferential map of the indicator function of C by N_C. We now proceed in several steps.

`Step 1:` N_C is maximal monotone.

This is a consequence of Rockafellar's maximal monotonicity theorem; see, for instance, [11, Theorem 3.24].

`Step 2:` $N_C + M$ is maximal monotone.

By `Step 1`, N_C is maximal monotone. On the other hand, M is maximal monotone (Proposition 2.1) with $\operatorname{dom} M = X^N$. Altogether, by Rockafellar's sum theorem [13, Section 20], $N_C + M$ is maximal monotone.

`Step 3:` $\sup_{w \in X^N} \langle w - x, My - Mw\rangle < +\infty$, for all $x, y \in X^N$.

Using Proposition 2.3 and $(My + M^*x)/2 \in \Delta^\perp$, we indeed have

$$\begin{aligned}\sup_w \langle w - x, My - Mw\rangle &= -\langle x, My\rangle + 2\sup_w \left[\langle w, \tfrac{1}{2}My + \tfrac{1}{2}M^*x\rangle - q(w)\right] \\ &= -\langle x, My\rangle + 2q^*(\tfrac{1}{2}My + \tfrac{1}{2}M^*x) \\ &< +\infty.\end{aligned}$$

`Step 4:` $\operatorname{cl}\operatorname{ran}(M + N_C) = \operatorname{cl}(\operatorname{ran}(M) + \operatorname{ran}(N_C))$.

Let $S_1 = N_C$ and $S_2 = M$. Then S_1 and S_2 are both maximal monotone (`Step 1` and Proposition 2.1), and so is their sum (`Step 2`). Also, $\operatorname{dom} S_1 \subseteq \operatorname{dom} S_2$. Now the desired equality follows from `Step 3` and Fact 2.4.

`Step 5:` $0 \in \operatorname{cl}\operatorname{ran}(M + N_C)$.

Clearly, $0 \in \operatorname{ran}(M) \cap \operatorname{ran}(N_C)$. Hence $0 \in \operatorname{ran}(M) + \operatorname{ran}(N_C)$. By `Step 4`, we obtain $0 \in \operatorname{cl}\operatorname{ran}(M + N_C)$.

`Step 6:` $(\forall \epsilon > 0)$ $(\exists b \in X^N)$ $(\exists x \in C)$: $\|b\| \le \epsilon$ and $x = P_C(Rx + b)$.

Indeed, fix $\epsilon > 0$. By `Step 5`, we obtain $x \in X^N$ and $b \in M(x) + N_C(x)$ with $\|b\| \le \epsilon$. Hence $x \in C$. Thus $b \in M(x) + N_C(x) \Leftrightarrow (Rx + b) - x \in N_C(x) \Leftrightarrow \sup\langle C - x, (Rx + b) - x\rangle \le 0 \Leftrightarrow x = P_C(Rx + b)$.

`Step 7:` $(\forall \epsilon > 0)$ $(\exists d \in X^N)$ $(\exists x \in C)$: $\|d\| \le \epsilon$ and $x = P_C(Rx) + d$.

This follows from `Step 6` and the fact that P_C is nonexpansive.

`Step 8:` $(\forall \epsilon > 0)$ $(\exists x \in C)$ $(\forall 1 \le n \le N)$:

$$\|P_{n-1} \cdots P_1 x_N - P_n P_{n-1} \cdots P_1 x_N - x_{n-1} + x_n\| \le (2n - 1)\epsilon,$$

where $x_0 = x_N$. Fix $\epsilon > 0$ and obtain x and d as in `Step 7`. Let $1 \leq n \leq N$. Then $x_n = P_n x_{n-1} + d_n$. Now P_n is nonexpansive, and $\|d_n\| \leq \|d\| \leq \epsilon$, hence

$$\begin{aligned}\|P_n P_{n-1} \cdots P_1 x_0 - x_n\| &\leq \|P_n P_{n-1} \cdots P_1 x_0 - P_n x_{n-1}\| + \|d_n\| \\ &\leq \|P_{n-1} \cdots P_1 x_0 - x_{n-1}\| + \epsilon.\end{aligned}$$

It follows by induction that, if $1 \leq n \leq N$, then

$$(n) \qquad \|P_n P_{n-1} \cdots P_1 x_0 - x_n\| \leq n\epsilon.$$

Adding (n) and $(n-1)$, followed by an application of the triangle inequality, completes the proof of `Step 8`.

`Step 9:` $(\forall \epsilon > 0)\ (\exists \xi \in X) : \|\xi - (P_N P_{N-1} \cdots P_1)(\xi)\| \leq N^2 \epsilon$.

For fixed $\epsilon > 0$, obtain $x \in C$ as in `Step 8`. For $1 \leq n \leq N$, set $e_n = P_{n-1} \cdots P_1 x_N - P_n P_{n-1} \cdots P_1 x_N - x_{n-1} + x_n$ so that $\|e_n\| \leq (2n-1)\epsilon$. Let $\xi = x_N$. On the one hand, $\sum_{n=1}^{N} e_n$ telescopes to $\xi - (P_N P_{N-1} \cdots P_1)(\xi)$. On the other hand, $\epsilon(1 + 3 + 5 + \cdots + (2N-1)) = N^2 \epsilon$. Altogether, `Step 9` follows.

`Last Step:` $P_N P_{N-1} \cdots P_1$ is asymptotically regular.

For brevity, denote this composition by T. It is well-known that projections are firmly nonexpansive; see, for instance, [10, Chapter 12]. Hence each P_n is strongly nonexpansive [8, Proposition 2.1], and so is the composition T [8, Proposition 1.1]. In view of `Step 9`, we have $0 \in \operatorname{cl} \operatorname{ran}(I - T)$. The asymptotic regularity of T now follows from Fact 2.5. ☐

Remark 3.2. It is easy to see that the "zero displacement conjecture" [4, Conjecture 5.3.6] is equivalent to the asymptotic regularity of $P_N \cdots P_1$.

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