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# ON MODULES OF FINITE PROJECTIVE DIMENSION OVER COMPLETE INTERSECTIONS

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ABSTRACT. Recently Avramov and Miller proved that over a local complete intersection ring $(R, m, k)$ in characteristic $p > 0$, a finitely generated module $M$ has finite projective dimension if for some $i > 0$ and for some $n > 0$, $\mathrm{Tor}_i^R(M, f_R^n) = 0 - f^n$ being the frobenius map repeated $n$ times. They used the notion of "complexity" and several related theorems. Here we offer a very simple proof of the above theorem without using "complexity" at all.

Recently Avramov and Miller [A–M] gave an important characterization for modules of finite projective dimension over complete intersections in positive characteristic. To describe their result we need the following set-up:

Let $(R, m)$ be a local ring of characteristic $p > 0$ with residue field $k = R/m$. Let $f : R \to R$, $f(x) = x^p$, denote the frobenius map and let $f^n$ denote the map $f$ repeated $n$-times. We denote by ${}^{f^n}R$ the bi-$R$-algebra $R$, having the structure of an $R$-algebra from the left by $f^n$ and from the right by identity. For any $R$-module $M$, $F_R^n(M)$ will stand for $M \otimes_R {}^{f^n}R$ and ${}^{f^n}N$ will denote the module $N$ viewed as an $R$-module via $f^n$[P – S]. Avramov and Miller proved the following:

**Theorem** ([A–M]). *Let $M$ be a finitely generated module over a local complete intersection ring $(R, m, k)$. If for some $i > 0$ and for some $n > 0$, $\mathrm{Tor}_i^R(M, {}^{f^n}R) = 0$, then $M$ is of finite projective dimension.*

Their proof used the notion of "complexity" and several related theorems. Here, we intend to provide a much simpler proof which does not use "complexity" at all. The only well-known theorems we exploit are the following: the theorem on flatness of the frobenius due to Kunz [K, 3.3] and Herzog's theorem on characterization of finite projective dimension [H, 3.1].

Historically, work on such problems started in 1969 with Kunz's theorem [K, 3.3] on equivalence of flatness of $f^n$ for all $n \geq 1$ with the regularity of the local ring $R$. Next, Peskine and Szpiro established the following [P–S, 1.7]: If a finitely generated $R$-module $M$ has finite projective dimension over $R$, $\mathrm{Tor}_i^R(M, f_R^n) = 0$ for all $i$, $n \geq 1$. Then Herzog [H, 3.1] proved the converse: If $M$ is finitely generated and $\mathrm{Tor}_i^R(M, f_R^n) = 0$ for all $i > 1$ and infinitely many $n$, $M$ has finite projective dimension over $R$. The theorem of Avramov and Miller stated above

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is a much desired extension of Herzog's theorem to complete intersections. For modules $M$ of finite length over a complete interection this author showed [D, 1.9] that $\ell(F_R^n(M)) \geq p^{nd}\ell(M)$ where $d$ denotes dimension of $R$ and sign of equility holds when $M$ is of finite projective dimension; in [M, 1.1] Miller proved the converse and deduced the following: If for some $i > 0$, $\lim_{n\to\infty} \ell(\operatorname{Tor}_i^R(M, f_R^n))/p^{nd} = 0$, then $M$ has finite projective dimension.

We accomplish our proof in the following steps.

**Step 0.** Without any loss of generality, we can assume that $R$ is complete and $R = S/\underline{x}$ where $S$ is a complete regular local ring of characteristic $p > 0$ and $\underline{x} = (x_1, \dots, x_r)$ is the ideal generated by an $S$-sequence $x_1, \dots, x_r$. Let $d$ be dimension of $R$ (henceforth dim); then $r + d = \dim S$. We know by Kunz's theorem that $f^n : S \to S : f^n(x) = x^{p^n}$ is a flat map $\forall n > 0$.

**Step 1.** Since $S \xrightarrow{f^n S} S$ is flat, $S/\underline{x} \xrightarrow{\tilde{f}^n} S/\underline{x}^{p^n}$ is flat (base change). Hence $R \xrightarrow{f^n R} R$ can be factored as

$$R = S/\underline{x} \xrightarrow{\tilde{f}^n} S/\underline{x}^{p^n} \xrightarrow{\eta_n} R = S/\underline{x}$$

where $\eta_n$ is the natural surjection.

Thus ${}^{f^n}R = \eta_n \cdot \tilde{f}^n$.

Let $M$ be a finitely generated $R$-module. Consider an exact sequence (a presentation of $M$)

$$R^{t_1} \xrightarrow{\phi} R^{t_0} \to M \to 0. \tag{1}$$

Apply $\otimes {}^{f^n}S$ and obtain an exact sequence

$$(S/\underline{x}^{p^n})^{t_1} \xrightarrow{\phi^{[p^n]}} (S/\underline{x}^{p^n})^{t_0} \to F_S^n(M) \to 0. \tag{2}$$

(1) and (2) imply that

$$M \otimes_R {}^{\tilde{f}^n}(S/\underline{x}^{p^n}) \simeq F_S^n(M). \tag{3}$$

Since $\tilde{f}^n$ is flat, we have

$$\operatorname{Tor}_i^R(M, {}^{f^n}R) = \operatorname{Tor}_i^{R_n}(F_S^n(M), S/\underline{x}), \tag{4}$$

where $R_n = S/\underline{x}^{p^n}$

**Step 2.** We want to show that $\operatorname{Tor}_1^R(M, {}^{f^n}R) = 0$ implies that $\operatorname{Tor}_i^R(M, {}^{f^n}R) = 0$ for $i \geq 1$. It is enough to show that $\operatorname{Tor}_2^R(M, {}^{f^n}R) = 0$. By flatness of $\tilde{f}^n$ we will be done by showing that

$$\operatorname{Tor}_1^{R_n}(F_S^n(M), S/\underline{x}) = 0 \qquad \text{implies} \qquad \operatorname{Tor}_2^{R_n}(F_S^n(M), S/\underline{x}) = 0.$$

We know $S/\underline{x}^{p^n}$ has a filtration such that successive quotients are isomorphic to $S/\underline{x}$. We have the following exact sequences:

$$0 \to K_1 \to S/\underline{x}^{p^n} \to S/\underline{x} \to 0,$$

$$0 \to K_2 \to K_1 \to S/\underline{x} \to 0,$$

$$0 \to K_{t_n} \to K_{t_{n-1}} \to S/\underline{x} \to 0$$

where $K_{t_n} = S/\underline{x}$. Since $\operatorname{Tor}_1^{R_n}(F_S^n(M), S/\underline{x}) = 0$, we obtain by going up along the above exact sequences successively that $\operatorname{Tor}_1^{R_n}(F_S^n(M), K_1) = 0$. This implies that $\operatorname{Tor}_2^{R_n}(F_S^n(M), S/\underline{x}) = 0$. Hence $\operatorname{Tor}_2^R(M, {}^{f^n}R) = 0$.

**Step 3.** We want to show that $\operatorname{Tor}_1^R(M, {}^{f^n}R) = 0$ implies that $\operatorname{Tor}_1^R(M, {}^{f^{n+1}}R) = 0$. Hence, we need to show that $\operatorname{Tor}_1^{R_{n+1}}(F_S^{n+1}(M), S/\underline{x}) = 0$, provided that $\operatorname{Tor}_1^{R_n}(F_S^n(M), S/\underline{x}) = 0$. Recall from Step 2 that since $\operatorname{Tor}_1^{R_n}(F_S^n(M), S/\underline{x}) = 0$, $\operatorname{Tor}_i^{R_n}(F_S^n(M), S/\underline{x}) = 0$ for $i \geq 1$ and from Step 1 that $R_n \xrightarrow{\tilde{f}} R_{n+1}$ is flat. Hence (tensoring the above equation by ${}^fR_{n+1}$),

$$\operatorname{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/\underline{x}^p) = 0 \quad \text{for } i \geq 1. \tag{5}$$

Consider the short exact sequences

$$0 \to S/(x_1, x_2^p, \dots, x_r^p) \to S/(x_1^{p+1}, x_2^p, \dots, x_n^p) \to S/\underline{x}^p \to 0 \tag{6}$$

and

$$0 \to S/\underline{x}^p \to S/(x_1^{p+1}, x_2^p, \dots, x_r^p) \to S/(x_1, x_2^p, \dots, x_r^p) \to 0. \tag{7}$$

Applying (5), from (6) we obtain

$$\begin{aligned} &\operatorname{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1, x_2^p, \dots, x_r^p)) \\ &\qquad \simeq \operatorname{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1^{p+1}, x_2^p, \dots, x_r^p)) \quad \text{for } i \geq 1, \end{aligned} \tag{8}$$

and applying (5), from (7) we obtain

$$\begin{aligned} &\operatorname{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1^{p+1}, x_2^p, \dots, x_r^p)) \\ &\qquad \hookrightarrow \operatorname{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1, x_2^p, \dots, x_r^p)) \quad \text{for } i \geq 1 \end{aligned} \tag{9}$$

which is an isomorphism for $i \geq 2$ and an injection for $i = 1$.

Note that the composition of the two maps (8) and (9) is the one induced by mutiplication by $x_1^p$.

Combining (6), (7), (8) and (9) we observe that

$$S/(x_1, x_2^p, \dots, x_r^p) \xrightarrow{x_1^p} S/(x_1, x_2^p, \dots, x_r^p)$$

is the 0-map and it induces an injection between

$$\operatorname{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1, x_2^p, \dots, x_r^p)) \hookrightarrow \operatorname{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1, x_2^p, \dots, x_r^p))$$

for $i \geq 1$.

Thus $\operatorname{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1, x_2^p, \dots, x_r^p)) = 0$. Repeating this process $(r-1)$ times we obtain

$$\operatorname{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(\underline{x}) = 0 \quad \text{for } i \geq 1.$$

**Step 4.** Steps 1, 2, 3 and the well-known theorem due to Herzog [H, 3.1] prove the assertion stated in the theorem at the beginning.

*Remark.* With notation as above, over a complete intersection $R = S/\underline{x}$, for a finitely generated module $M$, we have $\operatorname{Pd}_R M < \infty$ if and only if $\operatorname{Pd}_{R_n} F_S^n(M) < \infty$.

The proof follows easily from Step 1.

## References


[A–M] L. Avramov and C. Miller, *Frobenius powers of complete intersections*, Math. Research Letters **8**, nos. 1 and 2, 2001, 225–232. MR **2002b:**13022

[D] S. P. Dutta, *Frobenius and Multiplicities*, J. Algebra **85** (1983), 424–448. MR **85f:**13022

[H] J. Herzog, *Ringe de Charakteristik p und Frobeniusfunktoren*, Math Z. **140** (1974), 67–78. MR **50:**4569

[K] E. Kunz, *Characterization of regular local rings for characteristic $p$*, Amer. J. Math. **91** (1969), 772–784. MR **40:**5609

[M] C. Miller, *A Frobenius characterization of finite projective dimension over complete intersections*, Math. Z. **233** (2000), 127–136. MR **2001a:**13037

[P–S] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale*, I.H.E.S. Publ. Math. **42** (1973), 47–119. MR **51:**10330



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