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ON APPROXIMATELY CONVEX FUNCTIONS

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ABSTRACT. A real valued function f defined on a real interval I is called (ε, δ)-convex if it satisfies

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)+\varepsilon t(1-t)|x-y|+\delta \quad \text{for } x, y \in I,\ t \in [0,1].$$

The main results of the paper offer various characterizations for (ε, δ)-convexity. One of the main results states that f is (ε, δ)-convex for some positive ε and δ if and only if f can be decomposed into the sum of a convex function, a function with bounded supremum norm, and a function with bounded Lipschitz-modulus. In the special case $\varepsilon = 0$, the results reduce to that of Hyers, Ulam, and Green obtained in 1952 concerning the so-called δ-convexity.

1. INTRODUCTION

The stability theory of functional inequalities started with the paper [HU52] of Hyers and Ulam who introduced the notion of δ-convex function: If D is a convex subset of a real linear space X and δ is a nonnegative number, then a function $f : D \to \mathbb{R}$ is called *δ-convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \delta \tag{1}$$

for all $x, y \in D$, $t \in [0, 1]$. The basic result obtained by Hyers and Ulam states that if the underlying space X is of finite dimension, then f can be written as $f = g + h$, where g is a convex function and h is a bounded function whose supremum norm is not larger than $k_n\delta$, where the positive constant k_n depends only on the dimension n of the underlying space X. Hyers and Ulam proved that $k_n \leq (n(n+3))/(4(n+1))$. Green [Gre52], Cholewa [Cho84] obtained much better estimations of k_n showing that asymptotically k_n is not bigger than $(\log_2(n))/2$. Laczkovich [Lac99] compared this constant to several other dimension-depending stability constants and proved that it is not less than $(\log_2(n/2))/4$. This result shows that there is no stability results for infinite dimensional spaces X. A counterexample in this direction was earlier constructed by Casini and Papini [CP93]. The stability aspects of δ-convexity are discussed by Ger [Ger94].

If $t = 1/2$ and (1) holds for all $x, y \in D$, then f is called a *δ-Jensen-convex function.* There is no analogous decomposition for δ-Jensen-convex functions by

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the counterexample given by Cholewa [Cho84]. However, one can get Bernstein-Doetsch type regularity theorems which show that δ-Jensen-convexity and local upper boundedness imply 2δ-convexity. This result is due to Bernstein and Doetsch [BD15] for $\delta = 0$, and to Ng and Nikodem [NN93] in the case $\delta \geq 0$. For some recent extensions of these results to more general convexity concepts, see [Pál00]. For locally upper bounded δ-Jensen-convex functions one can obtain the existence of an analogous stability constant j_n (defined similarly as k_n above). The sharp value of this stability constant has recently been found by Dilworth, Howard, and Roberts [DHR99] who have shown that

$$j_n = \frac{1}{2}\Big([\log_2(n)] + 1 + \frac{n}{2^{[\log_2(n)]}}\Big) \leq 1 + \frac{1}{2}\log_2(n)$$

is the best possible value for j_n. (Here $[\cdot]$ denotes the integer-part function.) The connection between δ-Jensen-convexity and δ-$\mathbb{Q}$-convexity has been investigated by Mrowiec [Mro01].

If $D \subset \mathbb{R}$ and (1) is supposed to be valid for all $x, y \in D$ except a set of 2-dimensional Lebesgue measure zero, then one can speak about *almost δ-convexity*. Results in this direction are due to Kuczma [Kuc70] (the case $\delta = 0$) and Ger [Ger88] (the case $\delta \geq 0$).

Roughly speaking, the content of the Hyers–Ulam theorem can be expressed in the following way: A function f is a perturbation of a convex function by a bounded function if and only if, for some δ, it satisfies (1). Therefore, the presence of the error term δ refers to a bounded perturbation. In what follows, we consider more general perturbations of convex functions. We assume that X is a normed space and we take, as a perturbation, functions of the form $h + \ell$, where h is bounded and ℓ has a bounded Lipschitz-modulus. The investigation of such functions leads to the following stability notion: Let $\varepsilon, \delta \geq 0$ be constants and let D be a subset of X. A function $f : D \to \mathbb{R}$ is called *(ε, δ)-convex on D* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x - y\| + \delta \tag{2}$$

for $x, y \in D$, $t \in [0, 1]$. It is not difficult to see that if X is a normed space, f is of the form $f = g + h + \ell$, where g is convex, h is bounded with supremum norm not greater than $\delta/2$, and the Lipschitz-modulus of ℓ defined by

$$\mathrm{Lip}(\ell) := \inf\big\{L \mid |\ell(x) - \ell(y)| \leq L\|x - y\| \text{ for } x, y \in D\big\}$$

is not greater than $\varepsilon/2$, then f satisfies (2). The main result of this paper (see Corollary 4 below) is the more surprising statement that if D is one dimensional (i.e., D is a real interval) and f satisfies (2), then f admits a decomposition described above. Thus, one can observe, that the error terms $\varepsilon t(1-t)\|x - y\|$ and δ in (2) correspond to perturbations by Lipschitz and by bounded functions, respectively.

Most of the results obtained in this paper essentially use the fact that D is one dimensional. Thus, the analogous problem for the higher dimensional setting remains unsolved.

2. Main results

Throughout the rest of this paper, we assume that I is a nonempty open real interval of $\mathbb{R}$. Our first result offers a number of equivalent conditions for (ε, δ)-convexity.

Theorem 1. *Let $f : I \to \mathbb{R}$ and ε, δ be nonnegative numbers. Then the following conditions are pairwise equivalent:*

(i) *f is (ε, δ)-convex on I, i.e.,*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon t(1-t)|x-y| + \delta \tag{3}$$

for $x, y \in I$, $t \in [0,1]$.

(ii) *For $x, u, y \in I$ with $x < u < y$,*

$$\frac{f(x) + \delta - f(u)}{x-u} \le \frac{f(y) + \delta - f(u)}{y-u} + \varepsilon. \tag{4}$$

(iii) *There exists a function $p : I \to \mathbb{R}$ such that, for $x, u \in I$,*

$$f(u) + p(u)(x-u) \le f(x) + \frac{\varepsilon}{2}|x-u| + \delta. \tag{5}$$

(iv) *If $x_1, \ldots, x_n \in I$, $t_1, \ldots, t_n \ge 0$, $t_1 + \cdots + t_n = 1$ and $u := t_1x_1 + \cdots + t_nx_n$, then*

$$f(u) \le t_1f(x_1) + \cdots + t_nf(x_n) + \frac{\varepsilon}{2}\Big(t_1|x_1-u| + \cdots + t_n|x_n-u|\Big) + \delta. \tag{6}$$

Proof. (i)⇒(ii): Assume that f is (ε, δ)-convex and let $x < u < y$ be arbitrary elements of I. Choose $t \in [0,1]$ such that $u = tx + (1-t)y$, that is let $t = (y-u)/(y-x)$. Then applying the (ε, δ)-convexity of f, we get

$$f(u) \le \frac{y-u}{y-x}f(x) + \frac{u-x}{y-x}f(y) + \varepsilon\frac{(y-u)(u-x)}{y-x} + \delta,$$

which is equivalent to (4).

(ii)⇒(iii): Assume that (ii) holds and define

$$p(u) := \sup_{x \in I,\, x<u}\Big(\frac{f(x)+\delta-f(u)}{x-u} - \frac{\varepsilon}{2}\Big) \qquad \text{for} \quad u \in I.$$

Then, due to (ii), we have

$$\frac{f(x)+\delta-f(u)}{x-u} - \frac{\varepsilon}{2} \le p(u) \le \frac{f(y)+\delta-f(u)}{y-u} + \frac{\varepsilon}{2} \tag{7}$$

for all $x < u < y$ in I. The left-hand side inequality in (7) yields (5) in the case $x < u$, and analogously, the right-hand side inequality reduces to (5) in the case $x > u$. The case $x = u$ is obvious.

(iii)⇒(iv): To deduce (iv) from (iii), let $x_1, \ldots, x_n \in I$, $t_1, \ldots, t_n \ge 0$, $t_1 + \cdots + t_n = 1$ and $u := t_1x_1 + \cdots + t_nx_n$. Then, substituting x by x_i in (5), multiplying this inequality by t_i, and adding up the inequalities so obtained, we get

$$\begin{aligned} f(u) &= \sum_{i=1}^n t_i[f(u) + p(u)(x_i - u)] \\ &\le \sum_{i=1}^n t_i\Big(f(x_i) + \frac{\varepsilon}{2}|x_i-u| + \delta\Big) = \sum_{i=1}^n t_if(x_i) + \frac{\varepsilon}{2}\sum_{i=1}^n t_i|x_i-u| + \delta, \end{aligned}$$

which is the desired inequality (6).

(iv)⇒(i): Taking $x_1 = x$, $x_2 = y$, $t_1 = t$, and $t_2 = 1-t$ in condition (iv), one can immediately see that the inequality (6) reduces to (3). □

Remark 1. The equivalence of the conditions (i)-(iv) established in Theorem 1 reduces to well-known characterizations of convexity (cf. Hardy-Littlewood-Pólya [HLP34], Roberts-Varberg [RV73], and Kuczma [Kuc85]).

Motivated by condition (iii) of Theorem 1, a function $f : I \to \mathbb{R}$ is called (ε, δ)-*subdifferentiable at a point* $u \in I$, if there exists a real number $p = p(u)$ such that (5) holds for all $x \in I$. The (ε, δ)-*subgradient of* f *at* u is defined as the set of all values $p(u)$ such that (5) is valid for $x \in I$ and is denoted by $\partial_{\varepsilon,\delta} f(u)$. Clearly,

$$\begin{aligned}\partial_{\varepsilon,\delta} f(u) &= \Big\{p \in \mathbb{R} \mid \frac{f(x) + \delta - f(u)}{x-u} - \frac{\varepsilon}{2} \le p \le \frac{f(y) + \delta - f(u)}{y-u} + \frac{\varepsilon}{2},\ x < u < y\Big\} \\ &= \Big[\sup_{x\in I,\, x<u} \frac{f(x) + \delta - f(u)}{x-u} - \frac{\varepsilon}{2}, \inf_{y\in I,\, y>u} \frac{f(y) + \delta - f(u)}{y-u} + \frac{\varepsilon}{2}\Big].\end{aligned}$$

In the case if f is (ε, δ)-subdifferentiable at each point of I and if a nondecreasing function $p : I \to \mathbb{R}$ can be chosen such that (5) holds, then we say that f is *nondecreasingly* (ε, δ)-*subdifferentiable on* I.

The equivalence of conditions (i) and (iii) in Theorem 1 now can be expressed via the following result.

Corollary 1. *Let* $f : I \to \mathbb{R}$ *and* ε, δ *be nonnegative numbers. Then* f *is* (ε, δ)-*convex on* I *if and only if it is* (ε, δ)-*subdifferentiable on* I.

In our next result we obtain the decomposition of (ε, δ)-convex functions in terms of an $(\varepsilon, 0)$-convex function and a function whose supremum norm is not larger than $\delta/2$.

Theorem 2. *Let* $f : I \to \mathbb{R}$ *and* ε, δ *be nonnegative numbers. Then* f *is* (ε, δ)-*convex on* I *if and only if there exists an* $(\varepsilon, 0)$-*convex function* $\varphi : I \to \mathbb{R}$ *such that* $\|f - \varphi\| = \sup_I |f - \varphi| \le \delta/2$.

Proof. Assume that there exists an $(\varepsilon, 0)$-convex function φ such that $\|f-\varphi\| \le \delta/2$. Denote by h the difference $f - \varphi$. Then we have

$$\varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y) + \varepsilon t(1-t)|x-y|$$

and

$$\begin{aligned}h(tx + (1-t)y) &\le th(x) + (1-t)h(y) \\ &\qquad + |h(tx + (1-t)y)| + t|h(x)| + (1-t)|h(y)| \\ &\le th(x) + (1-t)h(y) + \delta\end{aligned}$$

for all $x, y \in I$ and $t \in [0, 1]$. Summing up these two inequalities, we get that f is (ε, δ)-convex.

To prove the converse, assume that f is (ε, δ)-convex and apply Theorem 1. Then there exists a function $p : I \to \mathbb{R}$ such that (5) holds for all $x, u \in I$. Define

$$\varphi(x) := \sup_{u\in I}\Big(f(u) + p(u)(x-u) - \frac{\varepsilon}{2}|x-u| - \frac{\delta}{2}\Big) \qquad \text{for} \quad x \in I.$$

Then, by (5), we have that $\varphi(x) \le f(x) + \delta/2$ for all $x \in I$. On the other hand, clearly $f(x) - \delta/2 \le \varphi(x)$. Thus, we get that $\|f - \varphi\| \le \delta/2$.

To complete the proof, it remains to show that φ is $(\varepsilon, 0)$-convex. For, let $x, y \in I$ and $t \in [0, 1]$ be fixed. Let $c > 0$ be an arbitrary (small) positive number. Then, by the definition of $\varphi(tx+(1-t)y)$, there exists $u \in I$ such that

$$\varphi(tx+(1-t)y)-c < f(u)+p(u)(tx+(1-t)y-u)-\frac{\varepsilon}{2}|tx+(1-t)y-u|-\frac{\delta}{2}.$$

Thus,

$$\begin{aligned}
\varphi(tx&+(1-t)y)-c \\
&< t\Big(f(u)+p(u)(x-u)-\frac{\varepsilon}{2}|x-u|-\frac{\delta}{2}\Big) \\
&\qquad +(1-t)\Big(f(u)+p(u)(y-u)-\frac{\varepsilon}{2}|y-u|-\frac{\delta}{2}\Big) \\
&\quad +\frac{\varepsilon}{2}t\Big(|x-u|-|tx+(1-t)y-u|\Big) \\
&\qquad +\frac{\varepsilon}{2}(1-t)\Big(|y-u|-|tx+(1-t)y-u|\Big) \\
&\leq t\varphi(x)+(1-t)\varphi(y)+\frac{\varepsilon}{2}t(1-t)|x-y|+\frac{\varepsilon}{2}(1-t)t|x-y|.
\end{aligned}$$

Taking the limit $c \to 0$, it follows from the above inequality that φ is $(\varepsilon, 0)$-convex. □

Remark 2. The case $\varepsilon = 0$ of the above theorem is the stability theorem of convex functions obtained by Hyers and Ulam [HU52] (see also Green [Gre52]). Their results, however, concern also the higher but finite dimensional case.

Corollary 2. *Let $f : I \to \mathbb{R}$ and ε, δ be nonnegative numbers. Then f is (ε, δ)-subdifferentiable at each point of I if and only if there exists an $(\varepsilon, 0)$-subdifferentiable function $\varphi : I \to \mathbb{R}$ such that $\|f-\varphi\| \leq \delta/2$. In addition, for all $u \in I$,*

$$\partial_{\varepsilon,0}\varphi(u) \subset \partial_{\varepsilon,\delta} f(u).$$

Proof. By Corollary 1, f is (ε, δ)-subdifferentiable on I if and only if it is (ε, δ)-convex on I. This property, by the previous theorem, is equivalent to the condition that there exists an $(\varepsilon, 0)$-convex function φ such that $\|f-\varphi\| \leq \delta/2$. Applying Corollary 1 again, we see that this latter property is equivalent to the condition that there exists an $(\varepsilon, 0)$-subdifferentiable function φ such that $\|f-\varphi\| \leq \delta/2$.

To see the inclusion between the subdifferentials, let $u \in I$ and $p \in \partial_{\varepsilon,0}\varphi(u)$. Then

$$\varphi(u)+p(x-u) \leq \varphi(x)+\frac{\varepsilon}{2}|x-u| \qquad \text{for} \quad x \in I.$$

Using $\|f-\varphi\| \leq \delta/2$, we get, for all $x \in I$, that

$$f(u)+p(x-u) \leq \varphi(u)+p(x-u)+\frac{\delta}{2} \leq \varphi(x)+\frac{\varepsilon}{2}|x-u|+\frac{\delta}{2} \leq f(x)+\frac{\varepsilon}{2}|x-u|+\delta,$$

i.e., $p \in \partial_{\varepsilon,\delta} f(u)$. □

Now we intend to give characterizations of $(\varepsilon, 0)$-subdifferentiable functions. First we investigate the properties of the $(\varepsilon, 0)$-subdifferential. For, introduce the following notion: A *function $p : I \to \mathbb{R}$ is called ε-nondecreasing on I* if

$$p(x) \leq p(y)+\varepsilon$$

holds for all $x \leq y$ in I. A *set-valued function $P : I \to 2^{\mathbb{R}}$ is called ε-nondecreasing* if any selection function p of P is ε-nondecreasing, which is equivalent to requiring that $u \leq v+\varepsilon$ whenever $u \in P(x)$, $v \in P(y)$, and $x \leq y$.

The connection between ε-nondecreasing and nondecreasing functions is described in the next result.

Theorem 3. *Let I be an open interval of $\mathbb{R}$, $p : I \to \mathbb{R}$, and ε be a nonnegative number. Then p is ε-nondecreasing if and only if there exists a nondecreasing function $q : I \to \mathbb{R}$ such that $\|p - q\| \le \varepsilon/2$.*

Proof. Assume that q is nondecreasing such that $\|p - q\| \le \varepsilon/2$. Then for $x \le y$, we have

$$p(x) \le q(x) + |p(x) - q(x)| \le q(y) + \frac{\varepsilon}{2} \le p(y) + \frac{\varepsilon}{2} + |p(y) - q(y)| \le p(y) + \varepsilon.$$

Thus, p is ε-nondecreasing.

Conversely, assume that p is ε-nondecreasing and define

$$q(x) := \sup_{v\in I,\, v\le x} \Big(p(v) - \frac{\varepsilon}{2}\Big) \qquad \text{for} \quad x \in I.$$

Then q is obviously nondecreasing. By its definition, we have that

$$p(x) - \frac{\varepsilon}{2} \le q(x).$$

On the other hand, using that p is ε-nondecreasing, $p(v) \le p(x) + \varepsilon$ for all $v \le x$, whence

$$q(x) = \sup_{v\in I,\, v\le x} \Big(p(v) - \frac{\varepsilon}{2}\Big) \le p(x) + \frac{\varepsilon}{2}.$$

The two inequalities obtained yield that $\|p - q\| \le \varepsilon/2$. □

Corollary 3. *Let $\varphi : I \to \mathbb{R}$ be an $(\varepsilon, 0)$-convex function on I, where ε is a nonnegative number. Then φ is nondecreasingly $(2\varepsilon, 0)$-subdifferentiable on I.*

Proof. By Corollary 1, φ is $(\varepsilon, 0)$-subdifferentiable on I, that is there exists a function $p : I \to \mathbb{R}$ such that

$$\varphi(u) + p(u)(x - u) \le \varphi(x) + \frac{\varepsilon}{2}|x - u| \qquad \text{for} \quad x, u \in I. \tag{8}$$

Interchanging x and u and adding up the two inequalities, we get

$$(p(u) - p(x))(x - u) \le \varepsilon|x - u| \qquad \text{for} \quad x, u \in I.$$

If $x < u$, then this inequality yields

$$p(x) - p(u) \le \varepsilon,$$

whence it follows that p is ε-nondecreasing. By the previous result, there exists a nondecreasing function $q : I \to \mathbb{R}$ such that $\|p - q\| \le \varepsilon/2$. Thus, using (8), we get

$$\begin{aligned}
\varphi(u) + q(u)(x - u) &\le \varphi(u) + p(u)(x - u) + |(p(u) - q(u))(x - u)| \\
&\le \varphi(u) + p(u)(x - u) + \frac{\varepsilon}{2}|x - u| \\
&\le \varphi(x) + \varepsilon|x - u|
\end{aligned}$$

for all $x, u \in I$. Hence φ is nondecreasingly $(2\varepsilon, 0)$-subdifferentiable. □

Now we characterize nondecreasingly $(\varepsilon, 0)$-subdifferentiable functions.

Theorem 4. *Let $\varphi : I \to \mathbb{R}$ and ε be a nonnegative number. Then φ is nondecreasingly $(\varepsilon, 0)$-subdifferentiable on I if and only if there exists a convex function $g : I \to \mathbb{R}$ such that $Lip(\varphi - g) \le \varepsilon/2$.*

Proof. Assume that $\varphi = g + \ell$, where g is convex and $\mathrm{Lip}(\ell) \le \varepsilon/2$. Then, by Corollary 3 (with $\varepsilon = 0$) applied to g, there exists a nondecreasing function $q : I \to \mathbb{R}$ such that

$$g(u) + q(u)(x - u) \le g(x) \qquad \text{for} \quad x, u \in I.$$

The function ℓ also satisfies

$$\ell(u) \le \ell(x) + \frac{\varepsilon}{2}|x - u| \qquad \text{for} \quad x, u \in I.$$

Adding up these inequalities, we get that φ satisfies

$$\varphi(u) + q(u)(x - u) \le \varphi(x) + \frac{\varepsilon}{2}|x - u| \qquad \text{for} \quad x, u \in I,$$

i.e., it is nondecreasingly $(\varepsilon, 0)$-subdifferentiable.

Conversely, assume that φ is nondecreasingly $(\varepsilon, 0)$-subdifferentiable. Then, we have that

$$q(u)(x - u) \le \varphi(x) - \varphi(u) + \frac{\varepsilon}{2}|x - u| \qquad \text{for} \quad x, u \in I. \tag{9}$$

Define now $g : I \to \mathbb{R}$ as the primitive function of q, that is, let $g(x) := \int_{x_0}^{x} q$, where x_0 is an arbitrarily fixed element of I. Then, q being nondecreasing, we get that g is a convex function. To complete the proof of the theorem, we show that the Lipschitz modulus of the function $\ell := \varphi - g$ is not greater than $\varepsilon/2$.

For, let $x < y$, $x, y \in I$ be arbitrary. Let $t_0 = x < t_1 < \cdots < t_n = y$ be an arbitrary division of the interval $[x, y]$. Substituting $x := t_{i-1}$, $u := t_i$ for $i = 1, \dots, n$ into (9) and adding the inequalities obtained, we get

$$\begin{aligned}\sum_{i=1}^{n} q(t_i)(t_{i-1} - t_i) &\le \sum_{i=1}^{n}\Big(\varphi(t_{i-1}) - \varphi(t_i) + \frac{\varepsilon}{2}(t_i - t_{i-1})\Big)\\ &= \varphi(t_0) - \varphi(t_n) + \frac{\varepsilon}{2}(t_n - t_0)\\ &= \varphi(x) - \varphi(y) + \frac{\varepsilon}{2}(y - x).\end{aligned}$$

The left-hand side of this inequality is arbitrarily close to $-\int_x^y q = g(x) - g(y)$ if $\max_{1\le i\le n}(t_i - t_{i-1})$ is small enough. Therefore, we obtain

$$g(x) - g(y) \le \varphi(x) - \varphi(y) + \frac{\varepsilon}{2}(y - x),$$

that is,

$$\ell(y) - \ell(x) \le \frac{\varepsilon}{2}(y - x). \tag{10}$$

On the other hand, substituting $x := t_i$, $u := t_{i-1}$ for $i = 1, \dots, n$ into (9) and adding these inequalities, we get

$$\begin{aligned}\sum_{i=1}^{n} q(t_{i-1})(t_i - t_{i-1}) &\le \sum_{i=1}^{n}\Big(\varphi(t_i) - \varphi(t_{i-1}) + \frac{\varepsilon}{2}(t_i - t_{i-1})\Big)\\ &= \varphi(t_n) - \varphi(t_0) + \frac{\varepsilon}{2}(t_n - t_0)\\ &= \varphi(y) - \varphi(x) + \frac{\varepsilon}{2}(y - x),\end{aligned}$$

whence, by taking the limit, we get that

$$g(y) - g(x) = \int_x^y q \le \varphi(y) - \varphi(x) + \frac{\varepsilon}{2}(y - x).$$

This inequality yields

$$\ell(x) - \ell(y) \le \frac{\varepsilon}{2}(y - x),$$

which, together with (10), results that

$$|\ell(x) - \ell(y)| \le \frac{\varepsilon}{2}(y - x)$$

for all $x < y$ in I. Hence, the Lipschitz modulus of ℓ is not greater than $\varepsilon/2$. □

In view of Theorem 2, in order to obtain the desired decomposition of (ε, δ)-convex functions, it suffices to show that $(\varepsilon, 0)$-convex functions can be decomposed in terms of a convex function and a function with small Lipschitz modulus.

Theorem 5. *Let $\varphi : I \to \mathbb{R}$ and ε be a nonnegative number. If there exists a convex function $g : I \to \mathbb{R}$ such that Lip$(\varphi - g) \le \varepsilon/2$, then φ is $(\varepsilon, 0)$-convex on I.*

Conversely, if φ is $(\varepsilon, 0)$-convex on I, then there exists a convex function $g : I \to \mathbb{R}$ such that Lip$(\varphi - g) \le \varepsilon$.

Proof. Assume that there exists a convex function $g : I \to \mathbb{R}$ such that, for $\ell := \varphi - g$, we have $\mathrm{Lip}(\ell) \le \varepsilon/2$. Then we have

$$g(tx + (1-t)y) \le tg(x) + (1-t)g(y)$$

and

$$\begin{aligned}
&\ell(tx + (1-t)y)\\
&\quad\le\ t\ell(x) + (1-t)\ell(y)\\
&\qquad\qquad + t\Big|\ell(x) - \ell(tx + (1-t)y)\Big| + (1-t)\Big|\ell(y) - \ell(tx + (1-t)y)\Big|\\
&\quad\le\ t\ell(x) + (1-t)\ell(y) + t\frac{\varepsilon}{2}(1-t)|x-y| + (1-t)\frac{\varepsilon}{2}t|x-y|\\
&\quad=\ t\ell(x) + (1-t)\ell(y) + \varepsilon t(1-t)|x-y|
\end{aligned}$$

for all $x, y \in I$ and $t \in [0,1]$. Summing up these two inequalities, we get that φ is $(\varepsilon, 0)$-convex.

Conversely, assume that φ is $(\varepsilon, 0)$-convex on I. Then, by Corollary 3, it is also nondecreasingly $(2\varepsilon, 0)$-subdifferentiable. Applying Theorem 4 (with ε replaced by 2ε), we get that there exists a convex function $g : I \to \mathbb{R}$ satisfying the condition $\mathrm{Lip}(\varphi - g) \le \varepsilon$. □

Remark 3. There is a discrepancy in the above result, since it is not an if-and-only-if characterization of $(\varepsilon, 0)$-convex functions. We show, however, that the constants in the result cannot be improved. Without loss of generality, we may assume that $[-1, 1] \subset I$.

First let $\varphi(x) := -\varepsilon|x|/2$ for $x \in I$. Then, obviously, $\mathrm{Lip}(\varphi - 0) = \mathrm{Lip}(\varphi) = \varepsilon/2$. Therefore, by the first part of Theorem 5, φ is $(\varepsilon, 0)$-convex, i.e., it satisfies

$$\varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y) + \varepsilon t(1-t)|x-y| \qquad (x, y \in I,\ t \in [0,1]).$$

Putting $x := t-1$, $y := t$ into this inequality, we can see that it holds with equality. Thus, φ cannot be $(\bar{\varepsilon}, 0)$-convex for any $0 \le \bar{\varepsilon} < \varepsilon$.

On the other hand, we show that there exists an $(\varepsilon, 0)$-convex function $\varphi : I \to \mathbb{R}$ such that $\mathrm{Lip}(\varphi - g) \ge \varepsilon$ for all convex functions $g : I \to \mathbb{R}$. Define $\varphi : I \to \mathbb{R}$ by

$$\varphi(x) := \begin{cases} \varepsilon(1 - x^2)/2 & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

In order to prove the $(\varepsilon, 0)$-convexity of φ, due to Theorem 1, it suffices to check the inequality

$$\frac{\varphi(x)-\varphi(u)}{x-u} \le \frac{\varphi(y)-\varphi(u)}{y-u} + \varepsilon \qquad (x,u,y \in I,\ x<u<y). \tag{11}$$

We distinguish three cases according to the position of u with respect to the interval $[-1, 1]$.

If $u \le -1$, then $\dfrac{\varphi(x)-\varphi(u)}{x-u} = 0$ and

$$\frac{\varphi(y)-\varphi(u)}{y-u} \begin{cases} \le \dfrac{\varphi(y)-\varphi(-1)}{y+1} = \dfrac{\varepsilon(1-y^2)}{2(y+1)} = \dfrac{\varepsilon(1-y)}{2} \le \varepsilon & \text{if } |y|<1, \\ = 0 & \text{if } |y| \ge 1. \end{cases}$$

Thus, (11) trivially holds.

A similar argument shows (11) also in the case when $u \ge 1$.

The last case is when $|u| < 1$. Then, it is elementary to check the following inequalities and equalities:

$$\frac{\varphi(x)-\varphi(u)}{x-u} \le \frac{\varphi(u)-\varphi(-1)}{u-(-1)} = \frac{\varepsilon(1-u)}{2} = \frac{\varphi(u)-\varphi(1)}{u-1} + \varepsilon \le \frac{\varphi(y)-\varphi(u)}{y-u} + \varepsilon.$$

Therefore, (11) is also valid in this case.

Thus we have proved (11), i.e., the $(\varepsilon, 0)$-convexity of φ.

Finally, we show that $\mathrm{Lip}(\varphi - g) \ge \varepsilon$ for all convex functions $g : I \to \mathbb{R}$. Assume on the contrary, that $\lambda = \mathrm{Lip}(\varphi - g) < \varepsilon$ for some convex function g. Then the right (and also left) derivatives of $\ell = \varphi - g$ are between $-\lambda$ and λ, that is,

$$-\lambda \le \varphi'(t) - g^+(t) \le \lambda \qquad (|t| < 1).$$

Let $-1 < x < y < 1$ be arbitrary. Then, by the convexity of g, we have $g^+(x) \le g^+(y)$. Substituting $t = x$, and $t = y$ into the above inequality, we get

$$-\lambda \le \varphi'(y) - g^+(y) \le \varphi'(y) - g^+(x) \le \varphi'(y) - \varphi'(x) + \lambda = \varepsilon(x-y) + \lambda.$$

Taking the limits $x \to -1$ and $y \to 1$, this inequality reduces to

$$-\lambda \le -2\varepsilon + \lambda,$$

which yields $\varepsilon \le \lambda$, a contradiction.

As an immediate consequence of Theorems 2 and 5, we get the main result of this paper.

Corollary 4. *Let $f : I \to \mathbb{R}$ and ε, δ be nonnegative numbers. If f is of the form $f = g + \ell + h$, where $g : I \to \mathbb{R}$ is convex, $h : I \to \mathbb{R}$ is bounded with $\|h\| \le \delta/2$, and $\ell : I \to \mathbb{R}$ is Lipschitz with Lip$(\ell) \le \varepsilon/2$, then f is (ε, δ)-convex on I.*

Conversely, if f is (ε, δ)-convex on I, then there exist a convex function $g : I \to \mathbb{R}$, a bounded function $h : I \to \mathbb{R}$ with $\|h\| \le \delta/2$, and a Lipschitz function $\ell : I \to \mathbb{R}$ with Lip$(\ell) \le \varepsilon$ such that $f = g + h + \ell$.

Proof. Assume that f is of the form $f = g + h + \ell$. Then, by Theorem 5, $\varphi = g + \ell$ is an $(\varepsilon, 0)$-convex function. Since $\|f - \varphi\| \le \delta/2$, hence, by Theorem 2, f is (ε, δ)-convex.

Conversely, suppose that f is (ε, δ)-convex. Then, by Theorem 2, there exists an $(\varepsilon, 0)$-convex function $\varphi : I \to \mathbb{R}$ such that, with $h := f - \varphi$, we have $\|h\| \le \delta/2$.

Now, applying Theorem 5, there exists a convex function $g : I \to \mathbb{R}$ such that, with the notation $\ell := \varphi - g$, we have $\mathrm{Lip}(\ell) \le \varepsilon$. Thus we obtain

$$f = \varphi + h = g + \ell + h,$$

which is the desired decomposition. □

References

[BD15] F. Bernstein and G. Doetsch, *Zur Theorie der konvexen Funktionen*, Math. Annalen **76** (1915), 514–526.

[Cho84] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86. MR **86d:**39016

[CP93] E. Casini and P. L. Papini, *A counterexample to the infinity version of the Hyers-Ulam stability theorem*, Proc. Amer. Math. Soc. **118** (1993), 885–890. MR **93i:**26022

[DHR99] S. J. Dilworth, R. Howard, and J. W. Roberts, *Extremal approximately convex functions and estimating the size of convex hulls*, Advances Math. **148** (1999), 1–43. MR **2001c:**26015

[Ger88] R. Ger, *Almost approximately convex functions*, Math. Slovaca **38** (1988), no. 1, 61–78. MR **89m:**26020a

[Ger94] R. Ger, *Stability aspects of delta-convexity*, Stability of mappings of Hyers-Ulam type (Palm Harbor, FL), Hadronic Press, Palm Harbor, FL, 1994, pp. 99–109. MR **95i:**39026

[Gre52] J. W. Green, *Approximately convex functions*, Duke Math. J. **19** (1952), 499–504. MR **14:**254c

[HLP34] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934, (first edition), 1952 (second edition). MR **13:**727e

[HU52] D. H. Hyers and S. M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc. **3** (1952), 821–828. MR **14:**254b

[Kuc70] M. Kuczma, *Almost convex functions*, Colloq. Math. **21** (1970), 279–284. MR **41:**7043

[Kuc85] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe — Uniwersytet Śląski, Warszawa–Kraków–Katowice, 1985. MR **86i:**39008

[Lac99] M. Laczkovich, *The local stability of convexity, affinity and of the Jensen equation*, Aequationes Math. **58** (1999), 135–142. MR **2001d:**39028

[Mro01] J. Mrowiec, *Remark on approximately Jensen-convex functions*, C. R. Math. Rep. Acad. Sci. Canada **23** (2001), 16–21.

[NN93] C. T. Ng and K. Nikodem, *On approximately convex functions*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 103–108. MR **93f:**26006

[Pál00] Zs. Páles, *Bernstein–Doetsch-type results for general functional inequalities*, Rocznik Nauk.-Dydakt. Prace Mat. **17** (2000), 197–206, Dedicated to Professor Zenon Moszner on his 70th birthday. MR **2001k:**26015

[RV73] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York–London, 1973. MR **56:**1201

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