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JORDAN ISOMORPHISMS OF NEST ALGEBRAS

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Abstract. Let $\mathcal{T}(N)$ and $\mathcal{T}(\mathcal{M})$ be two nest algebras. A Jordan isomorphism ϕ from $\mathcal{T}(N)$ onto $\mathcal{T}(\mathcal{M})$ is a bijective linear map such that $\phi(T^2) = \phi(T)^2$ for every $T \in \mathcal{T}(\mathcal{N})$. In this note, we prove that every Jordan isomorphism of nest algebras is of the form $T \to STS^{-1}$ or $T \to ST^*S^{-1}$ and then is, in fact, an isomorphism or an anti-isomorphism.

The motivation for this paper is the work by J. Arazy and B. Solel. In [1], J. Arazy and B. Solel proved that every surjective isometry α of nest algebras is of the form $T \to UTU^{-1}$ or $T \to UT^*U^{-1}$ provided that $\alpha(I) = I$, where U is a unitary operator. This is an elegant characterization. As we observed, they in fact first proved that such an isometry is a Jordan isomorphism and then completed their job. Let $\mathcal{T}(N)$ and $\mathcal{T}(\mathcal{M})$ be two nest algebras. A Jordan isomorphism ϕ from $\mathcal{T}(N)$ onto $\mathcal{T}(\mathcal{M})$ is a bijective linear map such that $\phi(T^2) = \phi(T)^2$ for every $T \in \mathcal{T}(\mathcal{N})$. The aim of the present paper is to characterize Jordan isomorphisms of nest algebras. Our main result is that every Jordan isomorphism of nest algebras is of the form $T \to STS^{-1}$ or $T \to ST^*S^{-1}$ and then is, in fact, either an isomorphism or an anti-isomorphism. The same result was concluded in [6] for Jordan isomorphisms from a ring onto an integral domain. Clearly a nest algebra is not an integral domain and a Jordan isompohism is not isometric; we must use different techniques. This leads us to study nilpotent Jordan ideals, which is the main subject of this paper.

Throughout, $\mathcal{H}$ is a complex Hilbert space, $B(\mathcal{H})$ is the algebra of all linear bounded operators on $\mathcal{H}$, $\mathcal{N}$ and $\mathcal{M}$ are nests of projections on $\mathcal{H}$, $\mathcal{T}(N)$ and $\mathcal{T}(\mathcal{M})$ are the nest algebras associated with $\mathcal{N}$ and $\mathcal{M}$ respectively, and ϕ is a Jordan isomorphism from $\mathcal{T}(\mathcal{N})$ onto $\mathcal{T}(\mathcal{M})$. For $N \in \mathcal{N}$, we use $N^\perp$ to denote $I - N$. For more information concerning nest algebras, we refer readers to [3].

We begin with two lemmas. The first is due to [6] and the second is well-known.

Lemma 1. *For any $A, B, C \in \mathcal{T}(\mathcal{N})$, the following hold:*
(1) $\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$.
(2) $\phi(ABA) = \phi(A)\phi(B)\phi(A)$.
(3) $\phi(ABC + CBA) = \phi(A)\phi(B)\phi(C) + \phi(C)\phi(B)\phi(A)$.

Lemma 2. *We have $\mathcal{T}(\mathcal{N})' = \mathbb{C}I$, where $\mathcal{T}(\mathcal{N})'$ is the commutant of $\mathcal{T}(\mathcal{N})$ and $\mathbb{C}$ is the set of complex numbers.*

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Proposition 3. *We have $\phi(I) = I$.*

Proof. By Lemma 1, for every $T \in \mathcal{T}(\mathcal{N})$, we have that

$$2\phi(T) = \phi(TI + IT) = \phi(T)\phi(I) + \phi(I)\phi(T) \tag{1}$$

and

$$\phi(T) = \phi(ITI) = \phi(I)\phi(T)\phi(I). \tag{2}$$

Since $\phi(I)$ is an idempotent, by (1) and (2), we have that $\phi(T)\phi(I) = \phi(T) = \phi(I)\phi(T)$. Therefore $\phi(I) \in \mathcal{T}(\mathcal{M})'$, and then there exists a scalar λ such that $\phi(I) = \lambda I$. Thus the result follows from the fact that $\phi(I)$ is an idempotent and $\phi(I) \neq 0$.

Proposition 4. *Suppose that T and S are in $\mathcal{T}(\mathcal{N})$ such that $TS = ST = 0$. Then $\phi(T)\phi(S) = \phi(S)\phi(T) = 0$.*

Proof. By Lemma 1(1), we have that

$$\phi(T)\phi(S) + \phi(S)\phi(T) = \phi(TS + ST) = 0. \tag{3}$$

For every $A \in \mathcal{T}(\mathcal{N})$, by Lemma 1(3),

$$\phi(T)\phi(S)\phi(A) + \phi(A)\phi(S)\phi(T) = \phi(TSA + AST) = 0. \tag{4}$$

Combining (3) and (4) yields

$$\phi(T)\phi(S)\phi(A) - \phi(A)\phi(T)\phi(S) = 0.$$

Since A is arbitrary, $\phi(T)\phi(S) \in \mathcal{T}(\mathcal{M})'$. Hence

$$\phi(T)\phi(S) = \lambda I \tag{5}$$

for some scalar λ. Thus

$$0 = \phi(TST) = \phi(T)\phi(S)\phi(T) = \lambda\phi(T). \tag{6}$$

Equalities (5) and (6) force $\phi(T)\phi(S) = 0$ and then $\phi(S)\phi(T) = 0$.

Let $\mathcal{S}$ be a subset of a Banach algebra $\mathcal{A}$. If $AB = BA = 0$ for any $A, B \in \mathcal{S}$, we say that $\mathcal{S}$ is nilpotent. Proposition 4 shows that a Jordan isomorphism preserves nilpotent sets. Let $\mathcal{N}_0 = \{N \in \mathcal{N} : 0 < N < I\}$ and $\mathcal{M}_0 = \{M \in \mathcal{M} : 0 < M < I\}$. For $N \in \mathcal{N}_0$, let $\mathcal{I}(N) = \{NTN^\perp : T \in \mathcal{T}(\mathcal{N})\}$. Then $\mathcal{I}(N)$ is a nilpotent subset of $\mathcal{T}(\mathcal{N})$. Moreover, we will show that such $\mathcal{I}(N)$ is maximal in the sense that $\mathcal{I}(N)$ is not properly contained in any other nilpotent subset of $\mathcal{T}(\mathcal{N})$.

Lemma 5. *If $\mathcal{S}$ is a nilpotent subset of $\mathcal{T}(\mathcal{N})$ such that $\mathcal{S} \supset \mathcal{I}(N)$ for some $N \in \mathcal{N}_0$, then $\mathcal{S} = \mathcal{I}(N)$.*

Proof. Suppose that $S \in \mathcal{S}$. Then for every $T \in \mathcal{T}(\mathcal{N})$, we have

$$SNTN^\perp = 0 \ \text{ and } NTN^\perp S = 0.$$

Since $S \in \mathcal{T}(\mathcal{N})$,

$$NSNTN^\perp = 0 \ \text{ and } NTN^\perp SN^\perp = 0.$$

Since T is arbitrary, $NSN = N^\perp SN^\perp = 0$. Hence $S = NSN^\perp \in \mathcal{I}(N)$.

$\mathcal{I}(N)$ is also an ideal, but in general, a Jordan isomorphism does not preserve ideals. For our purpose, we need the following weaker concept [6].

Definition 6. Let $\mathcal{J}$ be a subspace of $\mathcal{T}(\mathcal{N})$. $\mathcal{J}$ is called a J-ideal (Jordan ideal) if $AT + TA \in \mathcal{J}$ for every $A \in \mathcal{J}$ and $T \in \mathcal{T}(\mathcal{N})$.

By Lemma 1, Jordan isomorphisms preserve J-ideals. By Lemma 5, $\mathcal{I}(N)$ is a maximal nilpotent J-ideal. We will show that the ideals $\mathcal{I}(N)$ are a model for maximal nilpotent J-ideals. In what follows, the rank one operator $e \otimes f$ is defined by $(e \otimes f)x = (x, f)e$. For $N \in \mathcal{N}$, define $N_- = \sup\{P \in \mathcal{N} : P < N\}$. It is well known that $e \otimes f$ belongs to $\mathcal{T}(\mathcal{N})$ if and only if there is an element N in $\mathcal{N}$ such that $e \in N\mathcal{H}$ and $f \in N_-^{\perp}\mathcal{H}$.

Theorem 7. *Suppose that $\mathcal{J}$ is a maximal nilpotent ideal of $\mathcal{T}(\mathcal{N})$. Then there exists an element N in $\mathcal{N}_0$ such that $\mathcal{J} = \mathcal{I}(N)$.*

Proof. Define

$$N = \inf\{L \in \mathcal{N} : L^{\perp}\mathcal{J} = \{0\}\},$$
$$M = \sup\{L \in \mathcal{N} : \mathcal{J}L = \{0\}\}.$$

We first prove that $N \leq M$. Otherwise $N > M$. Then we can take $T, S \in \mathcal{J}$ and vectors e, f such that $e \otimes f \in \mathcal{T}(\mathcal{N})$ and $Te \otimes fS \neq 0$ as follows. If $M = N_-$, by the definition of N and M, there exist $e \in (N - M)\mathcal{H}, f \in M^{\perp}\mathcal{H}$ and $T, S \in \mathcal{J}$ such that $Te \neq 0 \neq S^*f$. If $M \neq N_-$, then there is an element P in $\mathcal{N}$ such that $M < P < N$. By the definition of N and M, there exist $e \in (P - M)\mathcal{H}$, $f \in (N - P)\mathcal{H}$ and $T, S \in \mathcal{J}$ such that $Te \neq 0 \neq S^*f$.

Since J is a J-ideal, $A = Te \otimes f + e \otimes fT \in \mathcal{J}$. Thus $AS = 0$. But

$$AS = Te \otimes fS + e \otimes fTS = Te \otimes fS \neq 0.$$

Therefore $N \leq M$, and then

$$\mathcal{J} = (N + N^{\perp})\mathcal{J}(N + N^{\perp}) = N\mathcal{J}N^{\perp} \subset \mathcal{I}(N).$$

By the maximality, we have that $\mathcal{J} = \mathcal{I}(N)$.

Since $\mathcal{I}(N)$ $(N \in \mathcal{N}_0)$ is a maximal nilpotent J-ideal in $\mathcal{T}(\mathcal{N})$, $\phi(\mathcal{I}(N))$ is also a maximal nilpotent J-ideal in $\mathcal{T}(\mathcal{M})$. By Theorem 7, there is only one element $\widehat{N} \in \mathcal{M}_0$ such that $\phi(\mathcal{I}(N)) = \mathcal{I}(\widehat{N})$. Define a map $\widehat{\phi}$ from $\mathcal{N}_0$ to $\mathcal{M}_0$ by $\widehat{\phi}(N) = \widehat{N}$ for $N \in N_0$ such that $\phi(\mathcal{I}(N) = \mathcal{I}(\widehat{N})$. Then $\phi(NTN^{\perp}) = \widehat{N}\phi(NTN^{\perp})\widehat{N}^{\perp}$ for every $T \in \mathcal{T}(\mathcal{N})$ and $N \in \mathcal{N}_0$.

Proposition 8. *The map $\widehat{\phi}$ is bijective.*

Proof. First we show that $\widehat{\phi}$ is injective. For otherwise, there are $P < Q$ (in $\mathcal{N}_0$) such that $\phi(\mathcal{I}(P)) = \phi(\mathcal{I}(Q)) = \mathcal{I}(\widehat{P})$. Choose non-zero vectors $x \in P\mathcal{H}$, $y \in (Q - P)\mathcal{H}$ and $z \in Q^{\perp}\mathcal{H}$. Clearly $\phi(x \otimes y)$ and $\phi(y \otimes z)$ are both in $\mathcal{I}(\widehat{P})$ and hence

$$\phi(x \otimes y)\phi(y \otimes z) = \phi(y \otimes z)\phi(x \otimes y) = 0.$$

Applying Proposition 4 to ϕ^{-1},

$$(x \otimes y)(y \otimes z) = 0,$$

but

$$(x \otimes y)(y \otimes z) = \|y\|^2 x \otimes z \neq 0.$$

Considering ϕ^{-1} instead of ϕ, for every element $M \in \mathcal{M}_0$, $\phi^{-1}(\mathcal{I}(M))$ is a maximal nilpotent J-ideal in $\mathcal{T}(\mathcal{N})$. Hence there is an element N in $\mathcal{N}_0$ such that

$\mathcal{I}(N) = \phi^{-1}(\mathcal{I}(M))$. Thus $\phi(\mathcal{I}(N)) = \mathcal{I}(M)$ and hence $M = \widehat{N}$. That is to say, $\widehat{\phi}$ is surjective.

Now we want to identify $\phi(\mathcal{N})$. For that, we need Lemma 9. It seems to be known, but we cannot find a reference.

Lemma 9. *Suppose that $S_1 \in B(\mathcal{H}_1)$ and $S_2 \in B(\mathcal{H}_2)$ are idempotent operators. If $S_1T + TS_2 = T$ for every $T \in B(\mathcal{H}_2, \mathcal{H}_1)$, then either $S_1 = I$ and $S_2 = 0$ or $S_1 = 0$ and $S_2 = I$.*

Proof. Fix a non-zero vector y in $\mathcal{H}_2$. Then for every x in $\mathcal{H}_1$, we have

$$S_1x \otimes y + x \otimes yS_2 = x \otimes y.$$

This implies that $S_1 = \lambda I$ for some scalar λ. Thus the result is immediate from the fact that S_1 is an idempotent.

Theorem 10. *Let $\widehat{N} = \widehat{\phi}(N)$ for $N \in \mathcal{N}_0$. Then exactly one of the following holds:*
(I) *For all $N \in \mathcal{N}$, $\phi(N) = \left[\begin{smallmatrix} I & * \\ 0 & 0 \end{smallmatrix}\right]$ on $\mathcal{H} = \widehat{N}\mathcal{H} \oplus \widehat{N}^{\perp}\mathcal{H}$.*
(II) *For all $N \in \mathcal{N}$, $\phi(N) = \left[\begin{smallmatrix} 0 & * \\ 0 & I \end{smallmatrix}\right]$ on $\mathcal{H} = \widehat{N}\mathcal{H} \oplus \widehat{N}^{\perp}\mathcal{H}$.*

Proof. We first prove that for every $N \in \mathcal{N}_0$, one of the following holds:
(a) $\phi(N) = \left[\begin{smallmatrix} I & * \\ 0 & 0 \end{smallmatrix}\right]$ on $\mathcal{H} = \widehat{N}\mathcal{H} \oplus \widehat{N}^{\perp}\mathcal{H}$.
(b) $\phi(N) = \left[\begin{smallmatrix} 0 & * \\ 0 & I \end{smallmatrix}\right]$ on $\mathcal{H} = \widehat{N}\mathcal{H} \oplus \widehat{N}^{\perp}\mathcal{H}$.
For every $T = NTN^{\perp}$, by Lemma 1(1)

$$\phi(T) = \phi(NT + TN) = \phi(N)\phi(T) + \phi(T)\phi(N). \tag{7}$$

Suppose that $\phi(N) = \left[\begin{smallmatrix} S_1 & * \\ 0 & S_2 \end{smallmatrix}\right]$ on $\mathcal{H} = \widehat{N}\mathcal{H} \oplus \widehat{N}^{\perp}\mathcal{H}$. Then S_1 and S_2 are idempotent. Since $\phi(T) = \widehat{N}\phi(NTN^{\perp})\widehat{N}^{\perp}$, by (7) we have

$$\phi(T) = S_1\phi(T) + \phi(T)S_2.$$

Since $\phi(\mathcal{I}(N)) = \mathcal{I}(\widehat{N})$, by Lemma 9, either $S_1 = I$ and $S_2 = 0$ which implies (a) holds, or $S_1 = 0$ and $S_2 = I$ which implies (b) holds.

Suppose that there are N_1 and N_2 in $\mathcal{N}_0$ such that $\phi(N_1) = \left[\begin{smallmatrix} I & * \\ 0 & 0 \end{smallmatrix}\right]$ on $\mathcal{H} = \widehat{N}_1\mathcal{H} \oplus \widehat{N}_1^{\perp}\mathcal{H}$ and $\phi(N_2) = \left[\begin{smallmatrix} 0 & * \\ 0 & I \end{smallmatrix}\right]$ on $\mathcal{H} = \widehat{N}_2\mathcal{H} \oplus \widehat{N}_2^{\perp}\mathcal{H}$. We consider two cases and reach a contradition.

Case 1. $N_1 < N_2$. Then $N_1N_2^{\perp} = N_2^{\perp}N_1 = 0$ and hence $\phi(N_1)\phi(N_2^{\perp}) = \phi(N_2^{\perp})\phi(N_1) = 0$ by Proposition 4. But $\phi(N_2^{\perp}) = I - \phi(N_2) = \left[\begin{smallmatrix} I & * \\ 0 & 0 \end{smallmatrix}\right]$ on $\mathcal{H} = \widehat{N}_2\mathcal{H} \oplus \widehat{N}_2^{\perp}\mathcal{H}$. By a simple computation, if $\widehat{N}_1 \leq \widehat{N}_2$, then $\phi(N_1)\phi(N_2^{\perp}) \neq 0$. If $\widehat{N}_1 > \widehat{N}_2$ (up to now, we don't know whether $\widehat{\phi}$ is order-preserving, i.e. $\widehat{N}_1 < \widehat{N}_2$), then $\phi(N_2^{\perp})\phi(N_1) \neq 0$. This is a contradiction.

Case 2. $N_1 > N_2$. Similarly we can reach a contradiction.

Remark 11. If Theorem 10(I) holds, then $\phi(N)\widehat{N} = \widehat{N}$ and $\widehat{N}\phi(N) = \phi(N)$ for every $N \in \mathcal{N}_0$, which implies that $\widehat{N}$ is the projection onto the range of $\phi(N)$. Hence the range of $\phi(N)$ is invariant for $\mathcal{T}(\mathcal{M})$ for every $N \in \mathcal{N}$, therefore for every $T \in \mathcal{T}(\mathcal{N})$ we have that $\phi(T)\phi(N) = \phi(N)\phi(T)\phi(N)$ and

$$\begin{aligned}\phi(NTN^{\perp}) =& \phi(NNTN^{\perp}N^{\perp} + N^{\perp}NTN^{\perp}N) \\ =& \phi(N)\phi(NTN^{\perp})\phi(N^{\perp}) + \phi(N^{\perp})\phi(NTN^{\perp})\phi(N) \\ =& \phi(N)\phi(NTN^{\perp})\phi(N^{\perp}).\end{aligned}$$

Hence, since $\phi(NTN) = \phi(N)\phi(T)\phi(N)$ and $\phi(N^\perp TN^\perp) = \phi(N^\perp)\phi(T)\phi(N^\perp)$, we have that

$$\phi(NTN^\perp) = \phi(N)\phi(T)\phi(N^\perp).$$

Moreover, in this case $\widehat{\phi}$ is order-preserving. Indeed, let $P < Q$ (in $\mathcal{N}_0$) and $\widehat{P} = \widehat{\phi}(P)$ and $\widehat{Q} = \widehat{\phi}(Q)$. Choose x, y, z as in Proposition 8. Let $T = x \otimes y$ and $S = y \otimes z$. Then $TS \neq 0$. Since

$$\phi(S)\phi(T) = \phi(Q)\phi(S)\phi(Q^\perp)\phi(P)\phi(T)\phi(P^\perp) = 0,$$

by Proposition 4,

$$\phi(T)\phi(S) \neq 0. \tag{8}$$

But $\phi(T)\phi(S) = \phi(T)\widehat{P}^\perp\widehat{Q}\phi(S)$, so (8) implies that $\widehat{P}^\perp\widehat{Q} \neq 0$ and hence $\widehat{P} < \widehat{Q}$.

Similarly, if Theorem 10(II) holds, then $\phi(T)\phi(N^\perp) = \phi(N^\perp)\phi(T)\phi(N^\perp)$ and

$$\phi(NTN^\perp) = \phi(N^\perp)\phi(NTN^\perp)\phi(N) = \phi(N^\perp)\phi(T)\phi(N).$$

Moreover $\widehat{\phi}$ is anti-order-preserving, i.e. if $P < Q$ (in $\mathcal{N}_0$), then $\widehat{P} > \widehat{Q}$.

In the foregoing, we say that ϕ is order preserving if $\widehat{\phi}$ is order preserving and ϕ is anti-order preserving if $\widehat{\phi}$ is anti-order preserving.

Lemma 12. *We have* $\phi(\mathcal{N}') = \phi(\mathcal{N})'$.

Proof. Suppose that D is in $\mathcal{N}'$. Then $DN = ND$ for every $N \in \mathcal{N}$.

If ϕ is order-preserving, then

$$\begin{aligned}\phi(N)\phi(D) &= \phi(N)\phi(NDN + N^\perp DN^\perp)\\ &=\phi(N)(\phi(N)\phi(D)\phi(N) + \phi(N^\perp)\phi(D)\phi(N^\perp))\\ &=\phi(N)\phi(D)\phi(N) = \phi(D)\phi(N).\end{aligned}$$

So $\phi(\mathcal{N}') \subset \phi(\mathcal{N})'$. On the other hand, suppose that T is in $\phi(\mathcal{N})'$. Then $T\phi(N) = \phi(N)T$ for every $N \in \mathcal{N}$ and hence $T \in \mathcal{T}(\mathcal{M})$. Therefore, there is $D \in \mathcal{T}(\mathcal{N})$ such that $T = \phi(D)$. Considering ϕ^{-1}, we have

$$\begin{aligned}ND &= N\phi^{-1}(\phi(N)T\phi(N) + \phi(N^\perp)T\phi(N^\perp))\\ &=N(N\phi^{-1}(T)N + N^\perp\phi^{-1}(T)N^\perp)\\ &=NDN = DN,\end{aligned}$$

which implies $D \in \mathcal{N}'$ and hence $T \in \phi(\mathcal{N}')$.

If ϕ is anti-order preserving, the proof is similar.

Let Ω be the subspace spanned by $\mathcal{N}'$ and $\{\mathcal{I}(N) : N \in \mathcal{N}_0\}$. It is easy to verify that Ω is in fact an algebra and it contains all rank-1 operators in $\mathcal{T}(\mathcal{N})$. Moreover, using the argument of Lemma 3.11 in [1], we have that:

Lemma 13. *Suppose that* $\mathcal{N}_0 \neq \emptyset$. *If* ϕ *is order-preserving, then the restriction of* ϕ *to* Ω *is multiplicative. If* ϕ *is anti-order preserving, then the restriction of* ϕ *to* Ω *is anti-multiplicative.*

Lemma 14. *Suppose that* $\mathcal{N}_0 \neq \emptyset$. *Let* $\mathcal{G}$ *be a maximal abelian* $*$*-subalgebra of* $\mathcal{T}(\mathcal{N})$. *Then* $\mathcal{G}$ *and* $\phi(\mathcal{G})$ *are both maximal abelian subalgebras of* $B(\mathcal{H})$.

Proof. Since $\mathcal{G}$ is a $*$-subalgebra of $\mathcal{T}(\mathcal{N})$, it commutes with each $N \in \mathcal{N}$. By the maximality, $\mathcal{N} \subset \mathcal{G}$. Suppose that $T \in B(\mathcal{H})$ such that T commutes with $\mathcal{G}$. Then T commutes with $\mathcal{N}$ and then $T \in \mathcal{T}(\mathcal{N})$. Hence $T \in \mathcal{G}$ and $\mathcal{G}$ is maximal in $B(\mathcal{H})$.

Since $\mathcal{N} \subset \mathcal{G}$, $\mathcal{G} = \mathcal{G}' \subset \mathcal{N}'$. By Lemma 13, $\phi(\mathcal{G})$ is an abelian subalgebra. Suppose that X belongs to $\phi(\mathcal{G})'$; then $X \in \phi(\mathcal{N})'$. By Lemma 12, $X = \phi(D)$ for some $D \in \mathcal{N}'$ and hence $X \in \phi(\Omega)$. By Lemma 13, the restriction of ϕ^{-1} to $\phi(\Omega)$ is multiplicative or anti-muliplicative. Therefore, D commutes with $\mathcal{G}$ and hence $D \in \mathcal{G}$. Thus $X \in \phi(\mathcal{G})$ and $\phi(\mathcal{G})$ is maximal abelian in $B(\mathcal{H})$.

Theorem 15. *Suppose that ϕ is a Jordan isomorphism from a nest algebra $\mathcal{T}(\mathcal{N})$ onto a nest algebra $\mathcal{T}(\mathcal{M})$. Then there is an invertible operator S such that either $\phi(T) = STS^{-1}$ or $\phi(T) = ST^*S^{-1}$ for every $T \in \mathcal{T}(\mathcal{N})$.*

Proof. First we consider the exceptional case where the nest $\mathcal{N}$ is the trivial nest $\{0, I\}$. By Proposition 8, $\mathcal{M}$ is also trivial and so ϕ is a Jordan automorphism of $B(\mathcal{H})$. Since $B(\mathcal{H})$ is prime ring, it follows from [9] that ϕ is either an algebraic automorphism or an anti-automorphism. It is well known that automorphisms of $B(\mathcal{H})$ are spatial. This establishes Therorem 15 in this case.

In the following, we assume that the nest $\mathcal{N}$ is not trivial (i.e. $\mathcal{N}_0 \neq \emptyset$). Let Ω be as above. By Lemma 13, we only need to consider two cases.

Case 1. The restriction of ϕ to Ω is multiplicative. Let $\mathcal{G}$ be a maximal abelian $*$-subalgebra of $\mathcal{T}(\mathcal{N})$. Then $\mathcal{G}$ and $\phi(\mathcal{G})$ are maximal abelian in $B(\mathcal{H})$. Hence $\phi(\mathcal{G})$ is norm-closed since the norm closure of $\phi(\mathcal{G})$ is abelian and contains $\phi(\mathcal{G})$. Let φ be the restriction of ϕ to $\mathcal{G}$. Then φ is an isomorphism from the Banach space $\mathcal{G}$ onto $\phi(\mathcal{G})$. Thus for each $D \in \mathcal{G}$,

$$\sigma(D) = \sigma_{\mathcal{G}}(D) = \sigma_{\varphi(\mathcal{G})}(\varphi(D)),$$

where $\sigma(D)$ is the spectrum of D in $B(\mathcal{H})$ and $\sigma_{\mathcal{G}}(D)$ is the spectrum of D in $\mathcal{G}$. Since D is normal, $\|\varphi(D)\| \geq \|D\|$. That is, φ^{-1} is contractive. Hence, by the Open Mapping Theorem, φ is bounded.

Let $\mathcal{U}$ be the set of all unitaries in $\mathcal{G}$. Then $\varphi(\mathcal{U})$ is a bounded abelian group of operators. By a result of Dixmier [4] (also see Corollary 17.2 [3]), there is an invertible operator T such that $T\varphi(\mathcal{G})T^{-1}$ is a group of unitaries. Since $\mathcal{G}$ is spanned by $\mathcal{U}$, it follows that $T\varphi(\mathcal{G})T^{-1}$ is spanned by the abelian unitary group $T\varphi(\mathcal{U})T^{-1}$. Hence $T\varphi(\mathcal{G})T^{-1}$ is an abelian von Neumann algebra. Clearly, it is maximal abelian in $T\mathcal{T}(\mathcal{M})T^{-1}$.

For $M \in \mathcal{M}$, let P_{TM} be the orthogonal projection onto the range of TM. Let $\mathcal{P}_{\mathcal{M}} = \{P_{TM} : M \in \mathcal{M}\}$. Then $\mathcal{P}_{\mathcal{M}}$ is a nest of projections on $\mathcal{H}$ and $T\mathcal{T}(\mathcal{M})T^{-1} = \mathcal{T}(\mathcal{P}_{\mathcal{M}})$. By Lemma 14, $T\varphi(\mathcal{G})T^{-1}$ is a maximal abelian $*$-subalgebra. Thus $AdT \circ \varphi$ is an algebraic isomorphim between two maximal abelian $*$-subalgebras, where $AdT \circ \varphi$ means that $AdT \circ \varphi(D) = T\varphi(D)T^{-1}$. Therefore there is a unitary operator U such that $AdT \circ \varphi = AdU$ [5, Chapter III, Part 3 §2]. Let $S_1 = U^{-1}T$ and $\psi = AdS_1 \circ \phi$. Then ψ is a Jordan isomorphism from $\mathcal{T}(\mathcal{N})$ onto $S_1\mathcal{T}(\mathcal{N})S_1^{-1}$. Moreover the restriction of ψ to Ω is multiplicative and $\psi(D) = D$ for every $D \in \mathcal{G}$.

Since $S_1\mathcal{T}(\mathcal{N})S_1^{-1}$ is a nest algebra, by Remark 11, its corresponding nest is $\psi(\mathcal{N}) = \mathcal{N}$ and hence $S_1\mathcal{T}(\mathcal{N})S_1^{-1} = \mathcal{T}(\mathcal{N})$. Moreover $\psi(\mathcal{N}') = \psi(\mathcal{N})' = \mathcal{N}'$ and $\psi(\mathcal{I}(N)) = \mathcal{I}(N)$ for every $N \in \mathcal{N}_0$, so $\psi(\Omega) = \Omega$. Hence the restriction of ψ to Ω, still denoted by ψ, is an isomorphism onto Ω. Since Ω contains all rank-1 operators in $\mathcal{T}(\mathcal{N})$, by Theorem 4.1 of [7], there is an invertible operator S_2 such that $\psi(T) = S_2TS_2^{-1}$ for every $T \in \Omega$. Let $S = S_1^{-1}S_2$. Then for every $T \in \Omega$ we have that $\phi(T) = STS^{-1}$. In particular, for every $x \otimes y$ in $\mathcal{T}(\mathcal{N})$, we have that $\phi(x \otimes y) = Sx \otimes yS^{-1}$.

Suppose $T \in \mathcal{T}(\mathcal{N})$. If T is a scalar multiple of I, then clearly $\phi(T) = STS^{-1}$. So we assume that T is not a scalar multiple of I. Let $\mathcal{N}_1 = \{N \in \mathcal{N} : N \neq 0 \text{ and } N_- < I\}$. Let N be an arbitrary element in $\mathcal{N}_1$. Fix a non-zero vector y in $N_-^{\perp}\mathcal{H}$. Then $x \otimes y$ is in $\mathcal{T}(\mathcal{N})$ for every $x \in N\mathcal{H}$. Hence we have that

$$\begin{aligned} STx \otimes yS^{-1} + Sx \otimes yTS^{-1} &= \phi(Tx \otimes y + x \otimes yT) \\ &= \phi(T)Sx \otimes yS^{-1} + Sx \otimes yS^{-1}\phi(T), \end{aligned}$$

and then there is a scalar $\lambda(N)$ such that

$$\phi(T)Sx - STx = \lambda(N)Sx, \quad x \in N\mathcal{H}.$$

For N_1 and N_2 in $\mathcal{N}_1$, we have that

$$\lambda(N_1)Sx = \phi(T)Sx - STx = \lambda(N_2)Sx, \quad x \in (N_1\mathcal{H}) \cap (N_2\mathcal{H}),$$

and consequently $\lambda(N_1) = \lambda(N_2)$ since $N_1 < N_2$ or $N_1 \geq N_2$. Thus there is a scalar λ such that

$$\phi(T)Sx - STx = \lambda Sx$$

on $\{N\mathcal{H} : N \in \mathcal{N}_1\}$. But $\bigvee\{N\mathcal{H} : N \in \mathcal{N}_1\} = \mathcal{H}$, so

$$\phi(T) = STS^{-1} + \lambda.$$

Now we show that $\lambda = 0$. If $\lambda \neq 0$, for every rank-1 operator $x \otimes y \in \mathcal{T}(\mathcal{N})$, we have

$$\begin{aligned} STx \otimes yTS^{-1} &= \phi(Tx \otimes yT) = \phi(T)\phi(x \otimes y)\phi(T) \\ &= (\lambda + STS^{-1})Sx \otimes yS^{-1}(\lambda + STS^{-1}) \\ &= \lambda^2 Sx \otimes yS^{-1} + \lambda Sx \otimes yTS^{-1} + \lambda STx \otimes yS^{-1} + STx \otimes yTS^{-1}. \end{aligned}$$

Since $\lambda \neq 0$, we have that

$$Tx \otimes y = -x \otimes (\overline{\lambda} I + T^*)y.$$

By a similar argument as above, there is a scalar μ such that $ST = \mu S$ and hence $T = \mu I$ which contradicts the assumption. So $\lambda = 0$ and then $\phi(T) = STS^{-1}$.

Case 2. The restriction of ϕ to Ω is anti-multiplicative. Define $\Phi(T) = \phi(T)^*$. Then Φ is a Jordan isomorphism from $\mathcal{T}(\mathcal{N})$ onto $\mathcal{T}(\mathcal{M}^{\perp})$. Since the restriction of ϕ to Ω is anti-multiplicative, the restriction of Φ to Ω is multiplicative. By Case 1, there is an invertible operator S such that $\Phi(T) = STS^{-1}$. Thus $\phi(T) = (S^*)^{-1}T^*S^*$.

Remark 16. As a corollary of Theorem 15, we can conclude another result for Jordan isomorphisms of nest algebras: Every Jordan isomorphism between nest algebras is continuous.

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Note. After we submitted this paper we became aware of the recent paper [2] which proved that Jordan isomorphisms of triangular matrix algebras over a connected commutative ring are of the form stated above. So our result was covered by [2] for the special case in which the nest algebras under consideration are upper triangular matrix algebras over the complex numbers. In fact, [2] covers the present paper only in this case since the ring considered in [2] must contain no non-trivial idempotents and must be commutative.

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