Abstract. Let $T(N)$ and $T(M)$ be two nest algebras. A Jordan isomorphism $\phi$ from $T(N)$ onto $T(M)$ is a bijective linear map such that $\phi(T^2) = \phi(T)^2$ for every $T \in T(N)$. In this note, we prove that every Jordan isomorphism of nest algebras is of the form $T \to STS^{-1}$ or $T \to ST^*S^{-1}$ and then is, in fact, an isomorphism or an anti-isomorphism.

The motivation for this paper is the work by J. Arazy and B. Solel. In [1], J. Arazy and B. Solel proved that every surjective isometry $\alpha$ of nest algebras is of the form $T \to UTU^{-1}$ or $T \to UT^*U^{-1}$ provided that $\alpha(I) = I$, where $U$ is a unitary operator. This is an elegant characterization. As we observed, they in fact first proved that such an isometry is a Jordan isomorphism and then completed their job. Let $T(N)$ and $T(M)$ be two nest algebras. A Jordan isomorphism $\phi$ from $T(N)$ onto $T(M)$ is a bijective linear map such that $\phi(T^2) = \phi(T)^2$ for every $T \in T(N)$. The aim of the present paper is to characterize Jordan isomorphisms of nest algebras. Our main result is that every Jordan isomorphism of nest algebras is of the form $T \to STS^{-1}$ or $T \to ST^*S^{-1}$ and then is, in fact, either an isomorphism or an anti-isomorphism. The same result was concluded in [6] for Jordan isomorphisms from a ring onto an integral domain. Clearly a nest algebra is not an integral domain and a Jordan isomorphism is not isometric; we must use different techniques. This leads us to study nilpotent Jordan ideals, which is the main subject of this paper.

Throughout, $H$ is a complex Hilbert space, $B(H)$ is the algebra of all linear bounded operators on $H$, $N$ and $M$ are nests of projections on $H$, $T(N)$ and $T(M)$ are the nest algebras associated with $N$ and $M$ respectively, and $\phi$ is a Jordan isomorphism from $T(N)$ onto $T(M)$. For $N \in \mathcal{N}$, we use $N^\perp$ to denote $I - N$. For more information concerning nest algebras, we refer readers to [3].

We begin with two lemmas. The first is due to [6] and the second is well-known.

**Lemma 1.** For any $A, B, C \in T(N)$, the following hold:
1. $\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$.
2. $\phi(ABA) = \phi(A)\phi(B)\phi(A)$.
3. $\phi(ABC + CBA) = \phi(A)\phi(B)\phi(C) + \phi(C)\phi(B)\phi(A)$.

**Lemma 2.** We have $T(N)' = \mathbb{C}I$, where $T(N)'$ is the commutant of $T(N)$ and $\mathbb{C}$ is the set of complex numbers.
Proposition 3. We have $\phi(I) = I$.

Proof. By Lemma 1, for every $T \in \mathcal{T}(N)$, we have that

\[\begin{align*}
2\phi(T) &= \phi(TI + IT) = \phi(T)\phi(I) + \phi(I)\phi(T) \\
\end{align*}\]

and

\[\begin{align*}
\phi(T) &= \phi(TTI) = \phi(I)\phi(T)\phi(I).
\end{align*}\]

Since $\phi(I)$ is an idempotent, by (1) and (2), we have that $\phi(T)\phi(I) = \phi(T) = \phi(T)\phi(I)$. Therefore $\phi(I) \in \mathcal{T}M'$, and then there exists a scalar $\lambda$ such that $\phi(I) = \lambda I$. Thus the result follows from the fact that $\phi(I)$ is an idempotent and $\phi(I) \neq 0$.

Proposition 4. Suppose that $T$ and $S$ are in $\mathcal{T}(N)$ such that $TS = ST = 0$. Then $\phi(T)\phi(S) = \phi(S)\phi(T) = 0$.

Proof. By Lemma 1(1), we have that

\[\begin{align*}
\phi(T)\phi(S) + \phi(S)\phi(T) &= \phi(TS + ST) = 0.
\end{align*}\]

For every $A \in \mathcal{T}(N)$, by Lemma 1(3),

\[\begin{align*}
\phi(T)\phi(S)\phi(A) + \phi(A)\phi(S)\phi(T) &= \phi(TSA + AST) = 0.
\end{align*}\]

Combining (3) and (4) yields

\[\begin{align*}
\phi(T)\phi(S)\phi(A) - \phi(A)\phi(T)\phi(S) &= 0.
\end{align*}\]

Since $A$ is arbitrary, $\phi(T)\phi(S) \in \mathcal{T}(M)'$. Hence

\[\begin{align*}
\phi(T)\phi(S) &= \lambda I
\end{align*}\]

for some scalar $\lambda$. Thus

\[\begin{align*}
0 &= \phi(TST) = \phi(T)\phi(S)\phi(T) = \lambda \phi(T).
\end{align*}\]

Equalities (5) and (6) force $\phi(T)\phi(S) = 0$ and then $\phi(S)\phi(T) = 0$.

Let $S$ be a subset of a Banach algebra $A$. If $AB = BA = 0$ for any $A, B \in S$, we say that $S$ is nilpotent. Proposition 4 shows that a Jordan isomorphism preserves nilpotent sets. Let $N_0 = \{N \in N: 0 < N < I\}$ and $M_0 = \{M \in M: 0 < M < I\}$. For $N \in N_0$, let $I(N) = \{NTN^\perp: T \in \mathcal{T}(N)\}$. Then $I(N)$ is a nilpotent subset of $\mathcal{T}(N)$. Moreover, we will show that such $I(N)$ is maximal in the sense that $I(N)$ is not properly contained in any other nilpotent subset of $\mathcal{T}(N)$.

Lemma 5. If $S$ is a nilpotent subset of $\mathcal{T}(N)$ such that $S \supset I(N)$ for some $N \in N_0$, then $S = I(N)$.

Proof. Suppose that $S \in \mathcal{S}$. Then for every $T \in \mathcal{T}(N)$, we have

\[\begin{align*}
SNTN^\perp &= 0 \quad \text{and} \quad NTN^\perp S = 0.
\end{align*}\]

Since $S \in \mathcal{T}(N)$,

\[\begin{align*}
NSNTN^\perp &= 0 \quad \text{and} \quad NTN^\perp SN^\perp = 0.
\end{align*}\]

Since $T$ is arbitrary, $NSN = N^\perp SN^\perp = 0$. Hence $S = NSN^\perp \in I(N)$.

$I(N)$ is also an ideal, but in general, a Jordan isomorphism does not preserve ideals. For our purpose, we need the following weaker concept [9].
Definition 6. Let $\mathcal{J}$ be a subspace of $T(N)$. $\mathcal{J}$ is called a J-ideal (Jordan ideal) if $AT + TA \in \mathcal{J}$ for every $A \in \mathcal{J}$ and $T \in T(N)$.

By Lemma 1, Jordan isomorphisms preserve J-ideals. By Lemma 5, $\mathcal{I}(N)$ is a maximal nilpotent J-ideal. We will show that the ideals $\mathcal{I}(N)$ are a model for maximal nilpotent J-ideals. In what follows, the rank one operator $e \otimes f$ is defined by $(e \otimes f)x = (x, f)e$. For $N \in \mathcal{N}$, define $N_\perp = \sup\{P \in \mathcal{N} : P < N\}$. It is well known that $e \otimes f$ belongs to $\mathcal{I}(N)$ if and only if there is an element $N$ in $\mathcal{N}$ such that $e \in NH$ and $f \in N^\perp H$.

Theorem 7. Suppose that $\mathcal{J}$ is a maximal nilpotent ideal of $\mathcal{I}(N)$. Then there exists an element $N$ in $\mathcal{N}_0$ such that $\mathcal{J} = \mathcal{I}(N)$.

Proof. Define

$$N = \inf\{L \in \mathcal{N} : L^+ \mathcal{J} = \{0\}\},$$

$$M = \sup\{L \in \mathcal{N} : \mathcal{J}L = \{0\}\}.$$

We first prove that $N \leq M$. Otherwise $N > M$. Then we can take $T, S \in \mathcal{J}$ and vectors $e, f$ such that $e \otimes f \in \mathcal{I}(N)$ and $Te \otimes fS \neq 0$ as follows. If $M = N_\perp$, by the definition of $N$ and $M$, there exist $e \in (N - M)H, f \in M^+H$ and $T, S \in \mathcal{J}$ such that $Te \neq 0 \neq S^*f$. If $M \neq N_\perp$, then there is an element $P \in \mathcal{N}$ such that $M < P < N$. By the definition of $N$ and $M$, there exist $e \in (P - M)H, f \in (N - P)H$ and $T, S \in \mathcal{J}$ such that $Te \neq 0 \neq S^*f$.

Since $\mathcal{J}$ is a J-ideal, $A = Te \otimes f + e \otimes fT \in \mathcal{J}$. Thus $AS = 0$. But

$$AS = Te \otimes fS + e \otimes fTS = Te \otimes fS \neq 0.$$

Therefore $N \leq M$, and then

$$\mathcal{J} = (N + N^\perp)\mathcal{J}(N + N^\perp) = N\mathcal{J}N^\perp \subset \mathcal{I}(N).$$

By the maximality, we have that $\mathcal{J} = \mathcal{I}(N)$.

Since $\mathcal{I}(N) (N \in \mathcal{N}_0)$ is a maximal nilpotent J-ideal in $\mathcal{T}(N)$, $\phi(\mathcal{I}(N))$ is also a maximal nilpotent J-ideal in $\mathcal{T}(M)$. By Theorem 7, there is only one element $\hat{N} \in \mathcal{M}_0$ such that $\phi(\mathcal{I}(N)) = \mathcal{I}(\hat{N})$. Define a map $\tilde{\phi}$ from $\mathcal{N}_0$ to $\mathcal{M}_0$ by $\tilde{\phi}(N) = \hat{N}$ for $N \in \mathcal{N}_0$ such that $\phi(\mathcal{I}(N)) = \mathcal{I}(\hat{N})$. Then $\phi(NTN^\perp) = \hat{N}\phi(NTN^\perp)\hat{N}^\perp$ for every $T \in \mathcal{T}(N)$ and $N \in \mathcal{N}_0$.

Proposition 8. The map $\tilde{\phi}$ is bijective.

Proof. First we show that $\tilde{\phi}$ is injective. For otherwise, there are $P < Q$ (in $\mathcal{N}_0$) such that $\phi(\mathcal{I}(P)) = \phi(\mathcal{I}(Q)) = \mathcal{I}(\hat{P})$. Choose non-zero vectors $x \in PH, y \in (Q - P)H$ and $z \in Q^\perp H$. Clearly $\phi(x \otimes y)$ and $\phi(y \otimes z)$ are both in $\mathcal{I}(\hat{P})$ and hence

$$\phi(x \otimes y)\phi(y \otimes z) = \phi(y \otimes z)\phi(x \otimes y) = 0.$$

Applying Proposition 4 to $\phi^{-1}$,

$$(x \otimes y)(y \otimes z) = 0,$$

but

$$(x \otimes y)(y \otimes z) = \|y\|^2 x \otimes z \neq 0.$$

Considering $\phi^{-1}$ instead of $\phi$, for every element $M \in \mathcal{M}_0$, $\phi^{-1}(\mathcal{I}(M))$ is a maximal nilpotent J-ideal in $\mathcal{T}(N)$. Hence there is an element $N$ in $\mathcal{N}_0$ such that
\( \mathcal{I}(N) = \phi^{-1}(\mathcal{I}(M)) \). Thus \( \phi(\mathcal{I}(N)) = \mathcal{I}(M) \) and hence \( M = \hat{N} \). That is to say, \( \hat{\phi} \) is surjective.

Now we want to identify \( \phi(N) \). For that, we need Lemma 9. It seems to be known, but we cannot find a reference.

**Lemma 9.** Suppose that \( S_1 \in B(\mathcal{H}_1) \) and \( S_2 \in B(\mathcal{H}_2) \) are idempotent operators. If \( S_1 T + TS_2 = T \) for every \( T \in B(\mathcal{H}_2, \mathcal{H}_1) \), then either \( S_1 = I \) and \( S_2 = 0 \) or \( S_1 = 0 \) and \( S_2 = I \).

**Proof.** Fix a non-zero vector \( y \) in \( \mathcal{H}_2 \). Then for every \( x \) in \( \mathcal{H}_1 \), we have
\[
S_1 x \otimes y + x \otimes y S_2 = x \otimes y.
\]
This implies that \( S_1 = \lambda I \) for some scalar \( \lambda \). Thus the result is immediate from the fact that \( S_1 \) is an idempotent.

**Theorem 10.** Let \( \hat{N} = \hat{\phi}(N) \) for \( N \in N_0 \). Then exactly one of the following holds:

(I) For all \( N \in N \), \( \phi(N) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \) on \( \mathcal{H} = \hat{N} \mathcal{H} \oplus \hat{N}^\perp \mathcal{H} \).

(II) For all \( N \in N \), \( \phi(N) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) on \( \mathcal{H} = \hat{N} \mathcal{H} \oplus \hat{N}^\perp \mathcal{H} \).

**Proof.** We first prove that for every \( N \in N_0 \), one of the following holds:

(a) \( \phi(N) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \) on \( \mathcal{H} = \hat{N} \mathcal{H} \oplus \hat{N}^\perp \mathcal{H} \).

(b) \( \phi(N) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) on \( \mathcal{H} = \hat{N} \mathcal{H} \oplus \hat{N}^\perp \mathcal{H} \).

For every \( T = NTN^\perp \), by Lemma 1(1)
\[
\phi(T) = \phi(NT + TN) = \phi(N)\phi(T) + \phi(T)\phi(N).
\]
Suppose that \( \phi(N) = \begin{bmatrix} \frac{S_1}{2} & 0 \\ 0 & 0 \end{bmatrix} \) on \( \mathcal{H} = \hat{N} \mathcal{H} \oplus \hat{N}^\perp \mathcal{H} \). Then \( S_1 \) and \( S_2 \) are idempotent.

Since \( \phi(T) = \hat{N}\phi(NTN^\perp)\hat{N}^\perp \), by (7) we have
\[
\phi(T) = S_1\phi(T) + \phi(T)S_2.
\]
Since \( \phi(\mathcal{I}(N)) = \mathcal{I}(\hat{N}) \), by Lemma 9, either \( S_1 = I \) and \( S_2 = 0 \) which implies (a) holds, or \( S_1 = 0 \) and \( S_2 = I \) which implies (b) holds.

Suppose that there are \( N_1 \) and \( N_2 \) in \( N_0 \) such that \( \phi(N_1) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \) on \( \mathcal{H} = \hat{N}_1 \mathcal{H} \oplus \hat{N}_1^\perp \mathcal{H} \) and \( \phi(N_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) on \( \mathcal{H} = \hat{N}_2 \mathcal{H} \oplus \hat{N}_2^\perp \mathcal{H} \). We consider two cases and reach a contradiction.

**Case 1.** \( N_1 < N_2 \). Then \( N_1N_2^\perp = N_2^\perp N_1 = 0 \) and hence \( \phi(N_1)\phi(N_2^\perp) = \phi(N_2^\perp)\phi(N_1) = 0 \) by Proposition 4. But \( \phi(N_2^\perp) = I - \phi(N_2) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \) on \( \mathcal{H} = \hat{N}_2 \mathcal{H} \oplus \hat{N}_2^\perp \mathcal{H} \). By a simple computation, if \( \hat{N}_1 \leq \hat{N}_2 \), then \( \phi(N_1)\phi(N_2^\perp) \neq 0 \). If \( \hat{N}_1 > \hat{N}_2 \) (up to now, we don’t know whether \( \hat{\phi} \) is order-preserving, i.e. \( \hat{N}_1 < \hat{N}_2 \)), then \( \phi(N_2^\perp)\phi(N_1) \neq 0 \). This is a contradiction.

**Case 2.** \( N_1 > N_2 \). Similarly we can reach a contradiction.

**Remark 11.** If Theorem 10(I) holds, then \( \phi(N)\hat{N} = \hat{N} \) and \( \hat{N}\phi(N) = \phi(N) \) for every \( N \in N_0 \), which implies that \( \hat{N} \) is the projection onto the range of \( \phi(N) \).

Hence the range of \( \phi(N) \) is invariant for \( T(M) \) for every \( N \in N \), therefore for every \( T \in T(M) \) we have that \( \phi(T)\phi(N) = \phi(N)\phi(T)\phi(N) \) and
\[
\phi(NTN^\perp) = \phi(NNTN^\perp N^\perp + N^\perp NTTN^\perp N) = \phi(N)\phi(NTN^\perp)\phi(N^\perp) + \phi(N^\perp)\phi(NTN^\perp)\phi(N) = \phi(N)\phi(NTN^\perp)\phi(N^\perp).
\]
Hence, since $\phi(NTN) = \phi(N)\phi(T)\phi(N)$ and $\phi(N^\perp TN^\perp) = \phi(N^\perp)\phi(T)\phi(N^\perp)$, we have that

$$\phi(NTN^\perp) = \phi(N)\phi(T)\phi(N^\perp).$$

Moreover, in this case $\hat{\phi}$ is order-preserving. Indeed, let $P < Q$ (in $N_0$) and $\hat{P} = \hat{\phi}(P)$ and $\hat{Q} = \hat{\phi}(Q)$. Choose $x, y, z$ as in Proposition 8. Let $T = x \otimes y$ and $S = y \otimes z$. Then $TS \neq 0$. Since

$$\phi(S)\phi(T) = \phi(Q)\phi(S)\phi(Q^\perp)\phi(P)\phi(T)\phi(P^\perp) = 0,$$

by Proposition 4,

$$\phi(T)\phi(S) \neq 0.$$  

But $\phi(T)\phi(S) = \phi(T)\hat{P}^\perp\hat{Q}\phi(S)$, so (8) implies that $\hat{P}^\perp\hat{Q} \neq 0$ and hence $\hat{P} < \hat{Q}$.

Similarly, if Theorem 10(II) holds, then $\phi(T)\phi(N^\perp) = \phi(N^\perp)\phi(T)\phi(N^\perp)$ and

$$\phi(NTN^\perp) = \phi(N^\perp)\phi(NTN^\perp)\phi(N) = \phi(N^\perp)\phi(T)\phi(N).$$

Moreover $\hat{\phi}$ is anti-order-preserving, i.e. if $P < Q$ (in $N_0$), then $\hat{P} > \hat{Q}$.

In the foregoing, we say that $\phi$ is order preserving if $\hat{\phi}$ is order preserving and $\phi$ is anti-order preserving if $\hat{\phi}$ is anti-order preserving.

**Lemma 12.** We have $\phi(N') = \phi(N)'$.

**Proof.** Suppose that $D$ is in $N'$. Then $DN = ND$ for every $N \in N$.

If $\phi$ is order-preserving, then

$$\phi(N)\phi(D) = \phi(N)\phi(NDN + N^\perp DN^\perp) = \phi(N)(\phi(D)\phi(N) + \phi(N^\perp)\phi(D)\phi(N^\perp)) = \phi(N)\phi(D)\phi(N) = \phi(D)\phi(N).$$

So $\phi(N') \subset \phi(N)'$. On the other hand, suppose that $T$ is in $\phi(N)'$. Then $T\phi(N) = \phi(N)T$ for every $N \in N$ and hence $T \in T(M)$. Therefore, there is $D \in T(N)$ such that $T = \phi(D)$. Considering $\phi^{-1}$, we have

$$ND = N\phi^{-1}(\phi(N)T\phi(N) + \phi(N^\perp)T\phi(N^\perp)) = N(\phi^{-1}(T)N + N^\perp\phi^{-1}(T)N^\perp) = NDN = DN,$$

which implies $D \in N'$ and hence $T \in \phi(N')$.

If $\phi$ is anti-order preserving, the proof is similar.

Let $\Omega$ be the subspace spanned by $N'$ and $\{I(N) : N \in N_0\}$. It is easy to verify that $\Omega$ is in fact an algebra and it contains all rank-1 operators in $T(N)$. Moreover, using the argument of Lemma 3.11 in [1], we have that:

**Lemma 13.** Suppose that $N_0 \neq \emptyset$. If $\phi$ is order-preserving, then the restriction of $\phi$ to $\Omega$ is multiplicative. If $\phi$ is anti-order preserving, then the restriction of $\phi$ to $\Omega$ is anti-multiplicative.

**Lemma 14.** Suppose that $N_0 \neq \emptyset$. Let $\mathcal{G}$ be a maximal abelian $*$-subalgebra of $T(N)$. Then $\mathcal{G}$ and $\phi(\mathcal{G})$ are both maximal abelian subalgebras of $B(H)$. 

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Proof. Since $G$ is a $*$-subalgebra of $T(N)$, it commutes with each $N \in \mathcal{N}$. By the maximality, $N \subset G$. Suppose that $T \in B(H)$ such that $T$ commutes with $G$. Then $T$ commutes with $N$ and then $T \in T(N)$. Hence $T \in G$ and $G$ is maximal in $B(H)$.

Since $N' \subset G$, $G = G' \subset N'$. By Lemma 13, $\phi(G)$ is an abelian subalgebra. Suppose that $X$ belongs to $\phi(G)'$; then $X \in \phi(N)'$. By Lemma 12, $X = \phi(D)$ for some $D \in N'$ and hence $X \in \phi(\Omega)$. By Lemma 13, the restriction of $\phi^{-1}$ to $\phi(\Omega)$ is multiplicative or anti-multiplicative. Therefore, $D$ commutes with $G$ and hence $D \in G$. Thus $X \in \phi(G)$ and $\phi(G)$ is maximal abelian in $B(H)$.

Theorem 15. Suppose that $\phi$ is a Jordan isomorphism from a nest algebra $T(N)$ onto a nest algebra $T(M)$. Then there is an invertible operator $S$ such that either $\phi(T) = STS^{-1}$ or $\phi(T) = ST^*S^{-1}$ for every $T \in T(N)$.

Proof. First we consider the exceptional case where the nest $N$ is the trivial nest $\{0, I\}$. By Proposition 8, $M$ is also trivial and so $\phi$ is a Jordan automorphism of $B(H)$. Since $B(H)$ is prime ring, it follows from $[9]$ that $\phi$ is either an algebraic automorphism or an anti-automorphism. It is well known that automorphisms of $B(H)$ are spatial. This establishes theorem 15 in this case.

In the following, we assume that the nest $N$ is not trivial (i.e. $N_0 \neq \emptyset$). Let $\Omega$ be as above. By Lemma 13, we only need to consider two cases.

Case 1. The restriction of $\phi$ to $\Omega$ is multiplicative. Let $G$ be a maximal abelian $*$-subalgebra of $T(N)$. Then $G$ and $\phi(G)$ are maximal abelian in $B(H)$. Hence $\phi(G)$ is norm-closed since the norm closure of $\phi(G)$ is abelian and contains $\phi(G)$. Let $\varphi$ be the restriction of $\phi$ to $G$. Then $\varphi$ is an isomorphism from the Banach space $G$ onto $\phi(G)$. Thus for each $D \in G$,

$$\sigma(D) = \sigma_G(D) = \sigma_{\phi(G)}(\varphi(D)),$$

where $\sigma(D)$ is the spectrum of $D$ in $B(H)$ and $\sigma_G(D)$ is the spectrum of $D$ in $G$. Since $D$ is normal, $\|\varphi(D)\| \geq \|D\|$. That is, $\varphi^{-1}$ is contractive. Hence, by the Open Mapping Theorem, $\varphi$ is bounded.

Let $U$ be the set of all unitaries in $G$. Then $\varphi(U)$ is a bounded abelian group of operators. By a result of Dixmier $[11]$ (also see Corollary 17.2 $[3]$), there is an invertible operator $T$ such that $T\varphi(G)T^{-1}$ is a group of unitaries. Since $G$ is spanned by $U$, it follows that $T\varphi(G)T^{-1}$ is spanned by the abelian unitary group $T\varphi(U)T^{-1}$. Hence $T\varphi(G)T^{-1}$ is an abelian von Neumann algebra. Clearly, it is maximal abelian in $TT(M)T^{-1}$.

For $M \in M$, let $P_{TM}$ be the orthogonal projection onto the range of $TM$. Let $\mathcal{P}_M = \{P_{TM} : M \in M\}$. Then $\mathcal{P}_M$ is a nest of projections on $H$ and $TT(M)T^{-1} = T(\mathcal{P}_M)$. By Lemma 14, $T\varphi(G)T^{-1}$ is a maximal abelian $*$-subalgebra. Thus $AdT \circ \varphi$ is an algebraic isomorphism between two maximal abelian $*$-subalgebras, where $AdT \circ \varphi$ means that $AdT \circ \varphi(D) = T\varphi(D)T^{-1}$. Therefore there is a unitary operator $U$ such that $AdT \circ \varphi = AdU$ $[5]$ Chapter III, Part 3 $[2]$. Let $S_1 = U^{-1}T$ and $\psi = AdS_1 \circ \phi$. Then $\psi$ is a Jordan isomorphism from $T(N)$ onto $S_1T(N)S_1^{-1}$. Moreover the restriction of $\psi$ to $\Omega$ is multiplicative and $\psi(D) = D$ for every $D \in G$.

Since $S_1T(N)S_1^{-1}$ is a nest algebra, by Remark 11, its corresponding nest is $\psi(N) = N$ and hence $S_1T(N)S_1^{-1} = T(N)$. Moreover $\psi(N') = \psi(N)' = N'$ and $\psi(T(N)) = T(N)$ for every $N \in N_0$, so $\psi(\Omega) = \Omega$. Hence the restriction of $\psi$ to $\Omega$, still denoted by $\psi$, is an isomorphism onto $\Omega$. Since $\Omega$ contains all rank-1 operators in $T(N)$, by Theorem 4.1 of $[7]$, there is an invertible operator $S_2$ such that $\psi(T) = S_2TS_2^{-1}$ for every $T \in \Omega$. Let $S = S_1^{-1}S_2$. Then for every $T \in \Omega$ we
have that \( \phi(T) = STS^{-1} \). In particular, for every \( x \otimes y \) in \( T(N) \), we have that 
\[ \phi(x \otimes y) = Sx \otimes yS^{-1}. \]

Suppose \( T \in T(N) \). If \( T \) is a scalar multiple of \( I \), then clearly \( \phi(T) = STS^{-1} \).
So we assume that \( T \) is not a scalar multiple of \( I \). Let \( N_1 = \{ N \in N : N \neq 0 \text{ and } N_- < I \} \). Let \( N \) be an arbitrary element in \( N_1 \). Fix a non-zero vector \( y \) in \( N_{-1} \). Then \( x \otimes y \) is in \( T(N) \) for every \( x \in NH \). Hence we have that
\[
STx \otimes yS^{-1} + Sx \otimes yTS^{-1} = \phi(Tx \otimes y + x \otimes yT) \\
= \phi(T)Sx \otimes yS^{-1} + Sx \otimes yS^{-1}\phi(T),
\]
and then there is a scalar \( \lambda(N) \) such that
\[
\phi(T)Sx - STx = \lambda(N)Sx, \quad x \in NH.
\]
For \( N_1 \) and \( N_2 \) in \( N_1 \), we have that
\[
\lambda(N_1)Sx = \phi(T)Sx - STx = \lambda(N_2)Sx, \quad x \in (N_1H) \cap (N_2H),
\]
and consequently \( \lambda(N_1) = \lambda(N_2) \) since \( N_1 < N_2 \) or \( N_1 \geq N_2 \). Thus there is a scalar \( \lambda \) such that
\[
\phi(T)Sx - STx = \lambda Sx
\]
on \( \{ NH : N \in N_1 \} \). But \( \bigvee \{ NH : N \in N_1 \} = H \), so
\[
\phi(T) = STS^{-1} + \lambda.
\]
Now we show that \( \lambda = 0 \). If \( \lambda \neq 0 \), for every rank-1 operator \( x \otimes y \in T(N) \), we have
\[
STx \otimes yTS^{-1} = \phi(Tx \otimes yT) = \phi(T) \phi(x \otimes y) \phi(T) \\
= (\lambda + STS^{-1})Sx \otimes yS^{-1} (\lambda + STS^{-1}) \\
= \lambda^2 Sx \otimes yS^{-1} + \lambda Sx \otimes yTS^{-1} + \lambda STx \otimes yS^{-1} + STx \otimes yTS^{-1}.
\]
Since \( \lambda \neq 0 \), we have that
\[
Tx \otimes y = -x \otimes (\lambda I + T^*)y.
\]
By a similar argument as above, there is a scalar \( \mu \) such that \( ST = \mu S \) and hence \( T = \mu I \) which contradicts the assumption. So \( \lambda = 0 \) and then \( \phi(T) = STS^{-1} \).

**Case 2.** The restriction of \( \phi \) to \( \Omega \) is anti-multiplicative. Define \( \Phi(T) = \phi(T)^* \).
Then \( \Phi \) is a Jordan isomorphism from \( T(N) \) onto \( T(M^+) \). Since the restriction of \( \phi \) to \( \Omega \) is anti-multiplicative, the restriction of \( \Phi \) to \( \Omega \) is multiplicative. By Case 1, there is an invertible operator \( S \) such that \( \Phi(T) = STS^{-1} \). Thus \( \phi(T) = (S^*)^{-1}T^*S^* \).

**Remark 16.** As a corollary of Theorem 15, we can conclude another result for Jordan isomorphisms of nest algebras: Every Jordan isomorphism between nest algebras is continuous.

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Note. After we submitted this paper we became aware of the recent paper [2] which proved that Jordan isomorphisms of triangular matrix algebras over a connected commutative ring are of the form stated above. So our result was covered by [2] for the special case in which the nest algebras under consideration are upper triangular matrix algebras over the complex numbers. In fact, [2] covers the present paper only in this case since the ring considered in [2] must contain no non-trivial idempotents and must be commutative.

References


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