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EQUIDISTRIBUTION OF HECKE EIGENFORMS ON THE MODULAR SURFACE

WENZHI LUO

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ABSTRACT. For the orthonormal basis of Hecke eigenforms in $S_{2k}(\Gamma(1))$, one can associate with it a probability measure $d\mu_k$ on the modular surface $X = \Gamma(1)\backslash\mathbf{H}$. We establish that this new measure tends weakly to the invariant measure on X as k tends to infinity, and obtain a sharp estimate for the rate of convergence.

1. INTRODUCTION

Let $\{f_{j,k}\}_{1\leq j\leq J_k}$ be the orthonormal basis of Hecke eigenforms in $S_{2k}(\Gamma(1))$, the space of holomorphic cusp forms of weight $2k$ with respect to the modular group $\Gamma(1)$. Thus

$$J_k = \dim_{\mathbf{C}} S_{2k}(\Gamma(1)) = \begin{cases} [k/6]-1, & \text{if } k \equiv 1(\mathrm{mod} 6), \\ [k/6], & \text{if } k \not\equiv 1(\mathrm{mod} 6). \end{cases}$$

One expects that the following equidistribution law holds for $f_{j,k}$ as the weight $k \to \infty$: for any measurable subset A on the modular surface $X = \Gamma(1)\backslash\mathbf{H}$, we have

$$\lim_{k\to\infty} \max_{1\leq j\leq J_k} \left| \int_A y^{2k}|f_{j,k}(z)|^2 d\mu - \int_A d\mu \right| = 0, \tag{1}$$

where

$$d\mu = \frac{1}{\mathrm{area}(X)} \frac{dxdy}{y^2}$$

is the normalized invariant measure on X.

The above statement can be viewed as a natural analogue of the unique quantum ergodicity conjecture, as formulated by Rudnick and Sarnak [2], for holomorphic cusp forms. While conjecture (1) seems out of reach at the present, one would like to establish this conjecture on the average. In [1] the variance over short intervals of k has been studied.

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For each integer $k \geq 8$, which we are assuming throughout the paper, we associate with it a probability measure on X,

$$d\mu_k = \frac{\sum_{j=1}^{\dim S_{2k}(\Gamma(1))} y^{2k}|f_{j,k}(z)|^2}{\dim S_{2k}(\Gamma(1))} d\mu. \tag{2}$$

The purpose of this simple note is to establish the following.

Theorem. *For any measurable subset A on the modular surface X and any $\epsilon > 0$, we have*

$$\int_A d\mu_k = \int_A d\mu + O_\epsilon(k^{-1/2+\epsilon}) \tag{3}$$

uniformly for all A on X.

Remark 1. What seems new here, at least for the *non-compact* modular surface, is the sharp decay rate $k^{-1/2+\epsilon}$. The stronger statement that

$$\sum_{j=1}^{J_k} \left| \int_A y^{2k}|f_{j,k}(z)|^2 d\mu - \int_A d\mu \right|^2 \ll k^\epsilon$$

seems out of reach at the present.

Remark 2. One notes that $d\mu_k$ is actually independent of the choice of orthonormal basis.

Remark 3. Our proof is direct, without employing the theory of L-functions or imposing any conditions on A (other than the measurability). The key ingredient is the holomorphic (automorphic) kernel $h_{k,m}(z,\ z')$ (see §2) for the Hecke operator $T_k(m)$.

2. Holomorphic kernel $h_{k,m}(z,\ z')$ for $T_k(m)$

The Hecke operator $T_k(m)$ $(m \geq 1)$ acts on cusp form $f \in S_{2k}(\Gamma(1))$ by

$$T_k(m)f = \frac{1}{m} \sum_{ad=m} a^{2k} \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right).$$

It was shown by Zagier (see [3]) that $T_k(m)$ can be represented by the holomorphic automorphic kernel $C_k^{-1} m^{2k-1} h_{k,m}(z,\ z')$ (C_k defined in (7) below),

$$h_{k,m}(z, z') = \sum_{ad-bc=m} (czz' + dz' + az + b)^{-2k}, \tag{4}$$

where the sum is taken over all integer matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant m, in the sense that

$$\left\langle f,\ C_k^{-1} m^{2k-1} h_{k,m}(\cdot\,, -\overline{z'}) \right\rangle_k = (T_k(m)f)(z') \tag{5}$$

for any $f \in S_{2k}(\Gamma(1))$, where $\langle\ ,\ \rangle_k$ is the (normalized) Petersson inner product on $S_{2k}(\Gamma(1))$.

The series in (4) is absolutely convergent and $h_{k,m}(z,\ z')$ as a function of each variable z or z' is a cusp form in $S_{2k}(\Gamma(1))$, and we have the identity

$$C_k^{-1} m^{2k-1} h_{k,m}(z,\ z') = \sum_{j=1}^{J_k} \lambda_{j,k}(m) f_{j,k}(z) f_{j,k}(z'), \tag{6}$$

where $\lambda_{j,k}(m)$ is the Hecke eigenvalue of $f_{j,k}$ under $T_k(m)$ and

$$C_k = \frac{3(-1)^k}{2^{(2k-3)}(2k-1)}. \tag{7}$$

In particular, for $m = 1$ and $z' = z$, we obtain

$$C_k^{-1} h_{k,1}(z,\ -\overline{z}) = \sum_{j=1}^{J_k} |f_{j,k}(z)|^2. \tag{8}$$

The kernel $h_{k,m}(z,\ z')$ was used in [3] to give a new proof of Eichler-Selberg trace formula on $\Gamma(1)$. This note is motivated by [3].

3. Proof of the theorem

Throughout we denote $\Gamma = \Gamma(1)$ and let χ_A denote the characteristic function of A on X. One can extend it (with the same notation) to $\mathbf{H}$ as a Γ-invariant function. We have

$$\begin{aligned} \int_A d\mu_k &= \frac{1}{J_k} \int_X \chi_A(z) (\sum_{j=1}^{J_k} y^{2k} |f_{j,k}(z)|^2) d\mu \\ &= \frac{1}{J_k C_k} \int_X \chi_A(z) h_{k,1}(z, -\overline{z}) y^{2k} d\mu \\ &= \frac{1}{J_k C_k} \int_X \chi_A(z) \left(\sum_{ad-bc=1} \frac{y^{2k}}{(c|z|^2 + d\overline{z} - az - b)^{2k}} \right) d\mu. \end{aligned} \tag{9}$$

Since replacing z by γz ($\gamma \in \Gamma$) in each term of the sum in (9) amounts to replacing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{by} \quad \gamma^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma,$$

we may decompose the sum into Γ-invariant pieces with $a + d = t$, $t \in \mathbf{Z}$. Thus,

(10)

$$\int_A d\mu_k = \sum_{t=-\infty}^{\infty} \frac{1}{J_k C_k} \int_X \chi_A(z) \left(\sum_{ad-bc=1,\ a+d=t} \frac{y^{2k}}{(c|z|^2 + d\overline{z} - az - b)^{2k}} \right) d\mu.$$

There is a bijection between the integral matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant 1 and trace t, and the set of integral binary quadratic forms g with discriminant $\operatorname{disc}(g) = t^2 - 4$. The bijection is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g(u,\ v) = cu^2 + (d-a)uv - bv^2, \tag{11}$$

$$g(u,\ v) = \alpha u^2 + \beta uv + \gamma v^2 \mapsto \begin{pmatrix} (t-\beta)/2 & -\gamma \\ \alpha & (t+\beta)/2 \end{pmatrix}. \tag{12}$$

For $g(u,\ v) = \alpha u^2 + \beta uv + \gamma v^2$ and $z = x + iy$, set

$$R_g(z,\ t) = \frac{y^{2k}}{(\alpha(x^2+y^2) + \beta x + \gamma - ity)^{2k}}. \tag{13}$$

Then for $\gamma \in \Gamma$ we have (here we identify the quadratic form g with the symmetric 2×2 matrix representing g)

$$R_{\gamma^T g \gamma}(z,\ t) = R_g(\gamma z,\ t), \tag{14}$$

and (10) can be written as

$$\int_A d\mu_k = \sum_{t=-\infty}^{\infty} \frac{1}{J_k C_k} \int_X \chi_A(z) \left(\sum_{\mathrm{disc}(g)=t^2-4} R_g(z,\ t) \right) d\mu, \tag{15}$$

where the sum is taken over all forms of discriminant $t^2 - 4$.

For each discriminant $D = t^2 - 4$ and a quadratic form g of discriminant D, we let Γ_g denote the isotropy group of elements leaving g fixed, and observe that

$$\begin{aligned} \sum_{\mathrm{disc}(g)=D} R_g(z,\ t) &= \sum_{\mathrm{disc}(g)=D, \bmod \Gamma} \ \sum_{\gamma \in \Gamma_g \backslash \Gamma} R_{\gamma^T g \gamma}(z,\ t) \\ &= \sum_{\mathrm{disc}(g)=D, \bmod \Gamma} \ \sum_{\gamma \in \Gamma_g \backslash \Gamma} R_g(\gamma z,\ t), \end{aligned} \tag{16}$$

where mod Γ means the sum is taken over a set of representatives for classes of quadratic forms with discriminant D. For $D \neq 0$, recall the class number $h(D)$ is finite, and thus we obtain

$$\int_X \chi_A(z) \left(\sum_{\mathrm{disc}(g)=D} R_g(z,\ t) \right) d\mu = \sum_{\mathrm{disc}(g)=D, \bmod \Gamma} \int_{X_g} \chi_A(z) R_g(z,\ t) d\mu, \tag{17}$$

where

$$X_g = \bigcup_{\gamma \in \Gamma_g \backslash \Gamma} \gamma X \tag{18}$$

is a fundamental domain for the action of Γ_g on $\mathbf{H}$ with X identified with a fundamental domain of Γ.

We distinguish the following three cases:

Case 1. $D = t^2 - 4 < 0$.

In this case $|\Gamma_g| = 1,\ 2,$ or 3 and $t = 0,$ or ± 1. For a quadratic form

$$g(u,\ v) = \alpha u^2 + \beta uv + \gamma v^2$$

with discriminant D, we have

$$\begin{aligned} \int_{X_g} \chi_A(z) R_g(z,\ t) d\mu &= \frac{1}{|\Gamma_g|} \int_{\mathbf{H}} \chi_A(z) R_g(z,\ t) d\mu \\ &= \frac{1}{|\Gamma_g|} \int_{\mathbf{H}} \chi_A((2z-\beta)/(2|\alpha|)) \frac{y^{2k}}{(|z|^2 \pm\ ity - D/4)^{2k}} d\mu, \end{aligned}$$

where in the last step, we made the substitution $z \mapsto (2z - \beta)/(2|\alpha|)$, and the sign '$\pm$' is '$-$' or '$+$' according to whether $\alpha > 0$ or not.

Since

$$\frac{2y}{|(|z|^2 - ity - D/4)|} \leq 1,$$

we deduce that, with $D = t^2 - 4$ and $B_k = \{z = x+iy : \ |x| \le k^{-1/2+\epsilon}, \ 0 \le y \le 3\}$,

$$\begin{aligned} &\sum_{D<0} \sum_{\mathrm{disc}(g)=D, \mathrm{mod}\,\Gamma} \frac{1}{J_k C_k} \int_{X_g} \chi_A(z) R_g(z,\ t) d\mu \\ \ll\ & \int_{\mathbf{H}} \frac{(2y)^{2k}}{|(|z|^2 - ity - D/4)|^{2k}} d\mu \\ \ll\ & k^{-1/2+\epsilon} + \sup_{z \notin B_k} \frac{(2y)^{2k-2}}{|(|z|^2 - ity - D/4)|^{2k-2}} \times \int_{\mathbf{H}} \frac{(2y)^2}{|(|z|^2 - ity - D/4)|^2} d\mu \\ \ll\ & k^{-1/2+\epsilon} + \left(1 - k^{-1+\epsilon}\right)^k \ll k^{-1/2+\epsilon}. \end{aligned}$$

In the last step above, the double integral converges and we estimate the function

$$\frac{(2y)^2}{|(|z|^2 - ity - D/4)|^2}$$

in the cases $\{z = x + iy : \ y > 3\}$ and $\{z = x + iy : \ |x| > k^{-1/2+\epsilon}, \ 0 \le y \le 3\}$ separately.

Case 2. $D = t^2 - 4 = 0$.

In this case, we can take as a system of representatives the forms g_r ($r \in \mathbf{Z}$), where $g_r(u,\ v) = rv^2$. Γ_{g_r} is equal to Γ for $r = 0$, and is equal to

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\}$$

for $r \ne 0$. In the last case, in view of (16), we infer that

$$\begin{aligned} \int_X \chi_A(z) (\sum_{\mathrm{disc}(g)=0} R_g(z,\ t)) d\mu = & \int_X \chi_A(z) R_{g_0}(z,\ t) d\mu \\ & + \int_{X_\infty} \chi_A(z) (\sum_{r \ne 0} R_{g_r}(z,\ t)) d\mu, \end{aligned} \tag{19}$$

where X_∞ is a fundamental domain for Γ_∞, say the strip between 0 and 1.

First we have (recall $t = \pm 2$)

$$\frac{2}{J_k C_k} \int_X \chi_A(z) R_{g_0}(z,\ \pm 2) d\mu = \frac{2}{J_k C_k} \int_A \left(\frac{i}{2}\right)^{2k} d\mu = \mu(A) + O(k^{-1}). \tag{20}$$

Next we deduce that

$$\begin{aligned} &\frac{1}{J_k C_k} \int_{X_\infty} \chi_A(z) (\sum_{r \ne 0} R_{g_r}(z,\ t)) d\mu \\ &= \frac{1}{J_k C_k} \int_0^\infty \int_0^1 \chi_A(z) y^{2k-2} \sum_{r \ne 0} (r - ity)^{-2k} dx dy \\ &= \frac{1}{J_k C_k} \int_0^{k^{1/2-\epsilon}} \int_0^1 \chi_A(z) y^{2k-2} \sum_{r \ne 0} (r - ity)^{-2k} dx dy \\ &\quad + \frac{1}{J_k C_k} \int_{k^{1/2-\epsilon}}^\infty \int_0^1 \chi_A(z) y^{2k-2} \sum_{r \ne 0} (r - ity)^{-2k} dx dy. \end{aligned}$$

We infer that

$$\begin{aligned} & \frac{1}{J_k C_k}\int_0^{k^{1/2-\epsilon}}\int_0^1 \chi_A(z) y^{2k-2}\sum_{r\neq 0}(r-ity)^{-2k}dxdy \\ \ll & \max_{0\leq y\leq k^{1/2-\epsilon}}\left(\frac{(2y)^2}{1+(2y)^2}\right)^{k-1}\int_0^{k^{1/2-\epsilon}}(\sum_{r\neq 0}(r^2+(2y)^2)^{-1})dy \ll k^{-1}. \end{aligned}$$

On the other hand, since

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{r\in\mathbf{Z}}(r-z)^{-2},$$

and for $s \neq 0$,

$$\frac{d^{2k-2}}{ds^{2k-2}}\sinh^{-2}(\pi s y) = 4\sum_{n\geq 1} n(2\pi n y)^{2k-2}\exp(-2\pi n|s|y) \geq 0,$$

we have

$$\begin{aligned} & \frac{1}{J_k C_k}\int_{k^{1/2-\epsilon}}^{\infty}\int_0^1 \chi_A(z) y^{2k-2}(\sum_{r\neq 0}(r-ity)^{-2k})dxdy \\ = & \frac{1}{J_k C_k}\frac{i^{2k-2}}{(2k-1)!}\int_{k^{1/2-\epsilon}}^{\infty}\int_0^1 \chi_A(z)\ \frac{d^{2k-2}}{ds^{2k-2}}\Big|_{s=t}(\sum_{r\neq 0}(r-isy)^{-2})dxdy \\ = & \frac{1}{J_k C_k}\frac{i^{2k-2}}{(2k-1)!}\int_{k^{1/2-\epsilon}}^{\infty}\int_0^1 \chi_A(z)\ \frac{d^{2k-2}}{ds^{2k-2}}\Big|_{s=t}\left(\frac{1}{s^2y^2}-\frac{\pi^2}{\sinh^2(\pi s y)}\right)dxdy \\ \ll & k^{-1/2+\epsilon}+\frac{1}{(2k-1)!}\int_{k^{1/2-\epsilon}}^{\infty}(\sum_{n\geq 1} n(4\pi n y)^{2k-2}\exp(-4\pi n y))dy \\ \ll & k^{-1/2+\epsilon}+k^{-1}\sum_{n\geq 1}\Gamma(2k-1, 4\pi n k^{1/2-\epsilon}), \end{aligned}$$

where

$$\Gamma(m,x) = \frac{1}{\Gamma(m)}\int_x^{\infty} e^{-\xi}\xi^{m-1}d\xi = e^{-x}\left(\frac{x^{m-1}}{(m-1)!}+\frac{x^{m-2}}{(m-2)!}+\cdots+x+1\right)$$

is the incomplete Γ-function.

We deduce that

$$\begin{aligned} \sum_{n\geq 1}\Gamma(2k-1, 4\pi n k^{1/2-\epsilon}) &= \sum_{n\leq k^{1/2+2\epsilon}}\Gamma(2k-1, 4\pi n k^{1/2-\epsilon}) \\ &\quad + \sum_{n\geq k^{1/2+2\epsilon}}\Gamma(2k-1, 4\pi n k^{1/2-\epsilon}) \\ &\ll k^{1/2+2\epsilon}, \end{aligned}$$

in view of the estimates $0 < \Gamma(m,x) \leq 1$ for $x \geq 0$, and

$$\Gamma(m,x) \leq e^{-x}\frac{x^{m-1}}{(m-1)!}\left(1-\frac{m-1}{x}\right)^{-1}$$

provided $x > m-1$.

Thus

$$\sum_{D=0} \frac{1}{J_k C_k} \int_X \chi_A(z) \left(\sum_{\mathrm{disc}(g)=0} R_g(z,\ t) \right) d\mu = \mu(A) + k^{-1/2+2\epsilon}.$$

Case 3. $D = t^2 - 4 > 0$.

Here again the class number is finite as in the case $D < 0$, but the isotropy groups are infinite cyclic. Let $g(u,\ v) = \alpha u^2 + \beta uv + \gamma v^2$ be a quadratic form with the discriminant D. The conjugate of Γ_g by some $\gamma \in SL_2(\mathbf{R})$ acts on $\mathbf{H}$ as the infinite cyclic group generated by $z \mapsto \epsilon^2 z$, where $\epsilon > 1$ is the fundamental unit of the order in $\mathbf{Q}(\sqrt{D})$ associated with g. We have

$$(\gamma^T g \gamma)(u,\ v) = \sqrt{D} uv,$$

and we can choose the fundamental domain X_g such that $\gamma^{-1} X_g$ is an annulus defined by

$$\Delta : \quad y > 0; \quad 0 < r_0 \le |z| \le \epsilon^2 r_0.$$

We infer that

$$\begin{aligned} \frac{1}{J_k C_k} \int_{X_g} \chi_A(z) R_g(z,\ t) d\mu &= \frac{1}{J_k C_k} \int_{X_g} \chi_A(z) R_{\gamma^T g \gamma}(\gamma^{-1} z,\ t) d\mu \\ &= \frac{1}{J_k C_k} \int\int_{\Delta} \chi_A(\gamma z)(\sqrt{D} x - ity)^{-2k} y^{2k-2} dx dy \\ &\ll \int_0^{\pi} \int_{r_0}^{\epsilon^2 r_0} \left(\frac{4}{t^2 - 4\cos^2\theta} \right)^k \frac{dr}{r} d\theta \\ &\ll (\log|t|)(4/5)^k \frac{1}{(t^2-4)^2}. \end{aligned}$$

Since $h(D) = h(t^2 - 4) \ll |t|^{1+\epsilon}$, we conclude that

$$\sum_{D>0} \sum_{\mathrm{disc}(g)=D,\, \mathrm{mod}\,\Gamma} \frac{1}{J_k C_k} \int_{X_g} \chi_A(z) R_g(z,\ t) d\mu \ll (4/5)^k.$$

This completes the proof of the theorem.

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Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210

E-mail address: wluo@math.ohio-state.edu