

## EXISTENCE OF ALGEBRAIC MINIMAL SURFACES FOR AN ARBITRARY PUNCTURE SET

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ABSTRACT. We will show that any punctured Riemann surface can be conformally immersed into a Euclidean 3-space as a branched complete minimal surface of finite total curvature called an algebraic minimal surface.

### 1. INTRODUCTION

The purpose of this paper is to discuss the puncture set problem for algebraic minimal surfaces. The account of the problem is given in the book [3] in the case of unbranched algebraic minimal surfaces.

An *algebraic minimal surface* in  $\mathbb{R}^3$  is a branched complete conformal minimal immersion  $X: M \rightarrow \mathbb{R}^3$  of finite total curvature. In this case, we can consider  $M$  as a compact Riemann surface  $\bar{M}$  with finitely many points, called *puncture points*, removed. We will call the set of distinct points  $\Sigma$  on  $\bar{M}$  such that there exists an algebraic minimal surface  $X: \bar{M} \setminus \Sigma \rightarrow \mathbb{R}^3$  the *puncture set* of  $\bar{M}$ .

We will state the Puncture Set Problem as follows:

**The Puncture Set Problem.** *Given a compact Riemann surface  $\bar{M}$ , determine the set of puncture sets on  $\bar{M}$ .*

We have known some results on the Puncture Set Problem. Let  $\bar{M}_\ell$  be a compact Riemann surface of genus  $\ell$  and  $\text{PS}_0^r(\bar{M}_\ell)$  the set of  $r$  distinct points  $\Sigma_r$  on  $\bar{M}_\ell$  such that there exists an *unbranched* algebraic minimal surface  $X: \bar{M}_\ell \setminus \Sigma_r \rightarrow \mathbb{R}^3$ . Let  $\mathcal{S}(\bar{M}_\ell; n, m, m') = \{(p_1, \dots, p_n, q_1, \dots, q_{m'})\}$  be the set of  $n + m'$  distinct points on  $\bar{M}_\ell$  such that  $\{p_1, \dots, p_n\} = \text{supp}(F)_-$ ,  $\{q_1, \dots, q_m\} = \text{supp}(dF)_+$  and  $\{q_1, \dots, q_m, \dots, q_{m'}\} = \text{supp}(G)_+$ , where  $F$  and  $G$  are meromorphic functions on  $\bar{M}_\ell$  such that  $(G)_+ \geq (dF)_+$ . Then

- The set  $\text{PS}_0^r(\mathbb{C}P^1)$  consists of all  $r$  distinct points on  $\mathbb{C}P^1$  ([3, pp. 95–100]).
- The set  $\mathcal{S}(\bar{M}_\ell; n, m, m')$  is contained in  $\text{PS}_0^{n+m'}(\bar{M}_\ell)$  for any  $\ell \geq 1$  ([3, p. 127, Proposition 25 and Proposition 26]).

We will consider *branched* algebraic minimal surfaces, too. Let  $\text{PS}^r(\bar{M}_\ell)$  be the set of puncture sets consisting of  $r$  distinct points on  $\bar{M}_\ell$ . Then, we will prove the following theorem:

**Theorem 1.1.** *The set  $\text{PS}^r(\bar{M}_\ell)$  consists of all  $r$  distinct points on  $\bar{M}_\ell$ .*

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In other words, any punctured Riemann surface can be conformally immersed into  $\mathbb{R}^3$  as an algebraic minimal surface.

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## 2. PRELIMINARIES

In this section, we will recall the Weierstrass representation formula for minimal surfaces. For more details, see [3].

Let  $\bar{M}$  be a compact Riemann surface. For a divisor  $D$  on  $\bar{M}$ , we will denote by  $\text{mult}_p D$  the multiplicity of  $D$  at  $p$ ,  $\deg D$  the degree of  $D$ ,  $\text{supp } D$  the support of  $D$  and  $D_+$  and  $D_-$  the effective divisors such that  $D = D_+ - D_-$ :

$$\begin{aligned} D &= \sum_{p \in \bar{M}} (\text{mult}_p D) \cdot p, \quad \deg D = \sum_{p \in \bar{M}} \text{mult}_p D, \\ \text{supp } D &= \{p \in \bar{M} \mid \text{mult}_p D \neq 0\}, \\ D_+ &= \sum_{p \in \bar{M}} \max\{\text{mult}_p D, 0\} \cdot p, \quad D_- = \sum_{p \in \bar{M}} \max\{-\text{mult}_p D, 0\} \cdot p. \end{aligned}$$

For a meromorphic function  $g$  on  $\bar{M}$  and a meromorphic one-form  $\omega$  on  $\bar{M}$ , let  $\text{ord}_p g$  and  $\text{ord}_p \omega$  be the order of  $g$  at  $p$  and the order of  $\omega$  at  $p$ , respectively. Then, the divisor  $(g)$  of  $g$  and the divisor  $(\omega)$  of  $\omega$  are defined by

$$(g) := \sum_{p \in \bar{M}} (\text{ord}_p g) \cdot p, \quad (\omega) := \sum_{p \in \bar{M}} (\text{ord}_p \omega) \cdot p,$$

respectively.

Let  $M$  be an open Riemann surface,  $X: M \rightarrow \mathbb{R}^3$  a branched complete conformal minimal immersion and  $\Phi_i$  the holomorphic one-form on  $M$  defined by  $\Phi_i = (\partial X_i / \partial z) dz$ , where  $z$  is a local holomorphic coordinate of  $M$  ( $i = 1, 2, 3$ ). The triplet  $\Phi = (\Phi_1, \Phi_2, \Phi_3)$  satisfies the following relation:

$$(2.1) \quad (\Phi_1 \otimes \Phi_1) + (\Phi_2 \otimes \Phi_2) + (\Phi_3 \otimes \Phi_3) = 0.$$

We will call the holomorphic map  $[\Phi]: M \rightarrow \mathbb{C}P^2$  defined by  $[\Phi] := [\Phi_1, \Phi_2, \Phi_3]$  in homogenous coordinates on  $\mathbb{C}P^2$  the *generalized Gauss map* of  $X$ . The generalized Gauss map is called an *algebraic Gauss map* if  $M$  is biholomorphic to a compact Riemann surface with a finite set of points removed and  $[\Phi]$  extends holomorphically on  $M$ . The removed points are called the *puncture points* of  $X$ .

**Definition 2.1.** We will call a branched complete conformal minimal immersion  $X: M \rightarrow \mathbb{R}^3$  an *algebraic minimal surface* if the generalized Gauss map is algebraic.

We can see that the following lemma holds ([1]).

**Lemma 2.2.** *Let  $X: M \rightarrow \mathbb{R}^3$  be a branched complete conformal minimal immersion. Then, the total curvature of  $X$  is finite if and only if the generalized Gauss map is algebraic.*

From the above lemma, we can see the following:

**Corollary 2.3.** *Let  $X: M \rightarrow \mathbb{R}^3$  be a branched complete conformal minimal immersion. Then,  $X$  is an algebraic minimal surface if and only if the holomorphic one-form  $\Phi_i$  on  $M$  extends meromorphically on a compact Riemann surface  $\bar{M}$  ( $i = 1, 2, 3$ ).*

In the following, we will assume that  $X$  is an algebraic minimal surface and that  $M$  is biholomorphic to a compact Riemann surface  $\bar{M}_\ell$  of genus  $\ell$  with puncture points  $\{p_1, \dots, p_r\}$  removed.

**Definition 2.4.** For  $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ , we will define the divisor  $(\Phi)$  of  $\Phi$  by

$$(\Phi) := \sum_{p \in \bar{M}_\ell} \left( \min_{i=1,2,3} \text{ord}_p \Phi_i \right) \cdot p.$$

Then, we can see that  $\text{supp}(\Phi)_+$  is the set of all the branch points of  $X$  and  $\text{supp}(\Phi)_-$  is the set of all the puncture points of  $X$ . From the completeness of  $X$ , the multiplicity of  $(\Phi)_-$  at each puncture point is equal to or greater than 2:

$$(2.2) \quad \text{supp}(\Phi)_- = \{p_1, \dots, p_r\}, \quad \text{mult}_{p_i}(\Phi)_- \geq 2 \quad (i = 1, \dots, r).$$

Conversely, from a triplet  $\Phi$  of meromorphic one-forms on  $\bar{M}_\ell$  satisfying the condition (2.1) and (2.2), we can obtain a multi-valued algebraic minimal surface  $X: M \rightarrow \mathbb{R}^3$  by integration

$$(2.3) \quad X(p) := \text{Re} \int_{p_0}^p \Phi,$$

where  $p$  and  $p_0$  are non-puncture points.

**Definition 2.5.** We will call the condition (2.2) the *divisor condition*.

If we choose another base point in the integral, then the image of the map shifts by a translation in  $\mathbb{R}^3$ .

We can see that the multi-valued algebraic minimal surface  $X$  can be reduced to a single-valued algebraic minimal surface if and only if

$$(2.4) \quad \text{Re} \int_{\gamma_i} \Phi = (0, 0, 0) \quad (i = 1, \dots, 2\ell + r - 1),$$

where  $\gamma = \{\gamma_1, \dots, \gamma_{2\ell+r-1}\}$  is a basis of the first homology group  $H_1(M, \mathbb{Z})$  of  $M$ .

**Definition 2.6.** We will call the condition (2.4) the *period condition*.

We can obtain a pair  $(g, \omega)$  consisting of a meromorphic function  $g$  on  $\bar{M}_\ell$  and a meromorphic one-form  $\omega$  on  $\bar{M}_\ell$  which is equivalent to  $\Phi$  by

$$(g, \omega) = \left( \frac{\Phi_3}{\Phi_1 - \sqrt{-1}\Phi_2}, \Phi_1 - \sqrt{-1}\Phi_2 \right),$$

$$(\Phi_1, \Phi_2, \Phi_3) = (1 - g^2, \sqrt{-1}(1 + g^2), 2g) \frac{\omega}{2}.$$

When the triplet  $\Phi$  corresponds to a single-valued minimal surface  $X: M \rightarrow \mathbb{R}^3$  by (2.3), the pair  $(g, \omega)$  is called the *Weierstrass data* of  $X$ . It is well-known that  $g$  is the stereographic projection of the Gauss map of  $X$  from the north pole.

We can see that the following relation holds among  $(g)$ ,  $(\omega)$ , and  $(\Phi)$ :

$$(\Phi) = -2(g)_- + (\omega).$$

Then, the divisor condition (2.2) becomes as follows:

$$(2.5) \quad \begin{aligned} \text{supp}(-2(g)_- + (\omega))_- &= \{p_1, \dots, p_r\}, \\ \text{mult}_{p_i}(-2(g)_- + (\omega))_- &\geq 2 \quad (i = 1, \dots, r). \end{aligned}$$

3. PROOF OF THEOREM 1.1

In this section, we will prove Theorem 1.1.

Let  $\text{Div}(\bar{M}_\ell)$  be the set of divisors on  $\bar{M}_\ell$  and  $\text{Div}_+^d(\bar{M}_\ell) = \{D \in \text{Div}(\bar{M}_\ell) \mid D > 0, \deg D = d\}$ . For a divisor  $D \in \text{Div}(\bar{M}_\ell)$ , let  $\mathcal{I}(D)$  be the complex vector space of meromorphic one-forms on  $\bar{M}_\ell$  defined by  $\mathcal{I}(D) := \{\xi \mid (\xi) \geq D\}$  and  $i(D)$  the complex dimension of  $\mathcal{I}(D)$ .

We will assume the following lemmas:

**Lemma 3.1** ([3, Proposition 3]). *Let  $\omega$  be a meromorphic one-form on  $\bar{M}_\ell$ . Then, we have  $\deg(\omega) = 2\ell - 2$ .*

**Lemma 3.2** ([3, p. 35]). *Let  $-D \in \text{Div}_+^d(\bar{M}_\ell)$ . Then, we have  $i(D) = \ell + d - 1$ .*

**Lemma 3.3** ([3, Proposition 10]). *If  $D \in \text{Div}_+^d(\bar{M}_\ell)$ ,  $d \geq 2\ell$ , then there exists a meromorphic function  $g$  such that  $(g)_- = D$ .*

Then, we can prove the following lemmas:

**Lemma 3.4.** *For any distinct points  $\{p_1, \dots, p_r\} \subset \bar{M}_\ell$ , there exists a meromorphic function  $g$  on  $\bar{M}_\ell$  such that  $\text{supp}(g)_- = \{p_1, \dots, p_r\}$ .*

*Proof.* Let  $d \geq 2\ell$  and  $D \in \text{Div}_+^d(\bar{M}_\ell)$  such that  $\text{supp} D = \{p_1, \dots, p_r\}$ . Then, there exists a meromorphic function  $g$  such that  $(g)_- = D$  by Lemma 3.3. Then,  $\text{supp}(g)_- = \text{supp} D = \{p_1, \dots, p_r\}$ . □

**Lemma 3.5.** *Let  $-D \in \text{Div}_+^d(\bar{M}_\ell)$  with  $\text{supp} D = \{p_1, \dots, p_r\}$ . If  $g$  is a meromorphic function on  $\bar{M}_\ell$  such that  $\text{supp}(g)_- = \{p_1, \dots, p_r\}$  and  $d \leq 2(\text{mult}_{p_i}(g)_- - \ell)$  ( $i = 1, \dots, r$ ), then the pair  $(g, \omega)$  satisfies the condition (2.5) for any  $\omega \in \mathcal{I}(D) \setminus \{0\}$ .*

*Proof.* Let  $\omega \in \mathcal{I}(D) \setminus \{0\}$ . It is clear that if  $p \notin \{p_1, \dots, p_r\}$ , then

$$\text{mult}_p(-2(g)_- + (\omega))_- = 0.$$

If  $p \in \{p_1, \dots, p_r\}$ , then

$$\begin{aligned} \text{mult}_p(-2(g)_- + (\omega))_- &\geq \text{mult}_p(-2(g)_- + (\omega)_+)_- \\ &= \max\{\text{mult}_p(2(g)_- - (\omega)_+), 0\} \\ &\geq \max\{(2\ell + d) - (2\ell + d - 2), 0\} = 2, \end{aligned}$$

since  $\deg(\omega)_+ = 2\ell - 2 + \deg(\omega)_- \leq 2\ell + d - 2$  by Lemma 3.1 and the assumption of Lemma 3.5. Hence, the pair  $(g, \omega)$  satisfies (2.5). □

*Remark 3.6.* From the assumption in Lemma 3.5, we can see that  $r \leq d$ .

Let  $(g, \omega)$  be a pair satisfying (2.5) and let  $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{3(2\ell+r-1)}) : \mathcal{I}(D) \rightarrow \mathbb{R}^{3(2\ell+r-1)}$  be a real linear map defined by

$$\begin{aligned} \mathcal{P}_i(\omega) &= \text{Re} \int_{\gamma_i} (1 - g^2) \omega, \\ \mathcal{P}_{(2\ell+r-1)+i}(\omega) &= \text{Re} \int_{\gamma_i} \sqrt{-1} (1 + g^2) \omega, \\ \mathcal{P}_{2(2\ell+r-1)+i}(\omega) &= \text{Re} \int_{\gamma_i} g \omega \end{aligned}$$

( $i = 1, \dots, 2\ell + r - 1$ ), where  $\{\gamma_1, \dots, \gamma_{2\ell+r-1}\}$  is a basis for  $H_1(M, \mathbb{Z})$ . We will denote by  $\ker \mathcal{P}$  the kernel of  $\mathcal{P}$  and  $\dim_{\mathbb{R}} \ker \mathcal{P}$  the real dimension of  $\ker \mathcal{P}$ .

**Lemma 3.7.** *If  $2d > 4\ell + 3r - 1$ , then  $\dim_{\mathbb{R}} \ker \mathcal{P} > 0$ .*

*Proof.* By Lemma 3.2, the real dimension of  $\mathcal{I}(D)$  is equal to  $2(\ell + d - 1)$ . Hence, if  $2(\ell + d - 1) > 3(2\ell + r - 1)$ , that is,  $2d > 4\ell + 3r - 1$ , then  $\dim_{\mathbb{R}} \ker \mathcal{P} \neq 0$ . Thus,  $\dim_{\mathbb{R}} \ker \mathcal{P} > 0$ .  $\square$

*Proof of Theorem 1.1.* Let  $\{p_1, \dots, p_r\}$  be a set of distinct points on  $\bar{M}_\ell$  and  $d$  a positive integer such that  $d \geq r$  and  $2d > 4\ell + 3r - 1$ . By Lemma 3.4, we can see that there exists a meromorphic function  $g$  such that  $\text{supp}(g)_- = \{p_1, \dots, p_r\}$ . We may assume that  $2 \text{mult}_{p_i}(g)_- \geq 2\ell + d$  ( $i = 1, \dots, r$ ). Let  $-D \in \text{Div}_+^d(\bar{M}_\ell)$  such that  $\text{supp } D = \{p_1, \dots, p_r\}$ . By Lemma 3.5, the pair  $(g, \omega)$  satisfies the condition (2.5) for any  $\omega \in \mathcal{I}(D) \setminus \{0\}$ . We can see that  $\ker \mathcal{P} \setminus \{0\}$  is a set of Weierstrass data for algebraic minimal surfaces from  $\bar{M}_\ell \setminus \{p_1, \dots, p_r\}$  into  $\mathbb{R}^3$ . By Lemma 3.7,  $\dim_{\mathbb{R}} \ker \mathcal{P} > 0$ . Hence,  $\ker \mathcal{P} \setminus \{0\} \neq \emptyset$ .  $\square$

*Remark 3.8.* When we consider that algebraic minimal surfaces are similar to meromorphic functions on a compact Riemann surface, we may consider that Theorem 1.1 is similar to Lemma 3.4. We will discuss deformations of unbranched algebraic minimal surfaces from this point of view in [2].

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