PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 1, Pages 265–274
S 0002-9939(02)06701-1

WEYL FORMULA FOR HYPOELLIPTIC OPERATORS OF SCHRÖDINGER TYPE

ERNESTO BUZANO AND ANDREA ZIGGIOTO

(Communicated by David S. Tartakoff)

Abstract. In this work we consider a general class of hypoelliptic operators, for which we give an estimate of the remainder of the so-called Weyl asymptotic formula for the eigenvalues.

1. Introduction

In this paper we deal with the problem of the asymptotic estimate of the counting function $\mathcal{N}(\tau)$ of a hypoelliptic operator in $\mathbb{R}^n$, with symbol $h(x,\xi)$ diverging at infinity, in terms of the volume $\mathcal{W}(\tau)$ of the set

$$\big\{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n \mid h(x,\xi)\leqslant\tau\big\}.$$

Our starting point is the very general estimate of the relative remainder

$$\frac{\mathcal{N}(\tau)-\mathcal{W}(\tau)}{\mathcal{W}(\tau)}$$

obtained by Dencker in [5], who generalized Hörmander's papers [6] and [7] to locally temperate metrics. As already remarked in [2], Dencker's estimate is not always infinitesimal, as $\tau\to+\infty$.

In this work we identify a wide class of hypoelliptic operators, we call of Schrödinger type, for which we manage to give an estimate of the kind

$$\mathcal{N}(\tau)=\mathcal{W}(\tau)\big(1+O(\tau^{-\epsilon})\big),\qquad \text{as } \tau\to\infty,$$

for a suitable $\epsilon>0$.

We employ the following notation: given two functions $f,g : X\to\mathbb{R}$, and a subset $A\subset X$, we write

$$f(x)\prec g(x),\qquad \forall\, x\in A,$$

if there exists a constant C such that

$$f(x)\leq Cg(x),\qquad \forall\, x\in A.$$

Received by the editors September 4, 2001.
2000 *Mathematics Subject Classification.* Primary 35P20, 47B06.

2. ASYMPTOTIC BEHAVIOR OF THE SPECTRUM OF FORMALLY HYPOELLIPTIC OPERATORS ON $\mathbb{R}^n$

We recall some results from [5] and [6].

Let

$$\phi(x,\xi;y,\eta)=\sum_{j=1}^{n}\xi_j y_j - x_j\eta_j$$

be the standard symplectic form on $\mathbb{R}^n\times\mathbb{R}^n$. A Riemannian metric $g_{x,\xi}(y,\eta)$ on $\mathbb{R}^n\times\mathbb{R}^n$ is *locally ϕ temperate* if it is slowly varying and there exist two positive real numbers r and N and a slowly varying metric $G_x(y)$ on $\mathbb{R}^n$ such that

$$G_x(y)\leqslant g_{x,\xi}(y,\eta),$$

for all (x,ξ), $(y,\eta)\in\mathbb{R}^n\times\mathbb{R}^n$, and

$$g_{x,\xi}(z,\zeta)\prec g_{y,\eta}(z,\zeta)\bigl(1+g^{\phi}_{x,\xi}(x-y,\xi-\eta)\bigr)^N, \tag{1}$$

for all (x,ξ), (y,η), $(z,\zeta)\in\mathbb{R}^n\times\mathbb{R}^n$ such that

$$G_x(x-y)\leqslant r.$$

The quadratic form $g^{\phi}_{x,\xi}(y,\eta)$ appearing in (1) is the *dual metric*

$$g^{\phi}_{x,\xi}(y,\eta)=\sup\bigl\{\bigl(\phi(y,\eta;z,\zeta)\bigr)^2 \mid g_{x,\xi}(z,\zeta)=1\bigr\}.$$

Similarly we say that a positive function $m(x,\xi)$ is *locally ϕ, g temperate* if it is g continuous (see [6]) and

$$m(x,\xi)\prec m(y,\eta)\bigl(1+g^{\phi}_{x,\xi}(x-y,\xi-\eta)\bigr)^N,$$

for all (x,ξ), $(y,\eta)\in\mathbb{R}^n\times\mathbb{R}^n$ such that

$$G_x(x-y)\leqslant r.$$

Definition 1. A differential operator h^{w} is *formally hypoelliptic*[1] if its Weyl symbol $h(x,\xi)$ satisfies the following hypotheses:

1) h is a smooth function such that

$$h(x,\xi)\neq 0,\qquad \forall\,(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n; \tag{2}$$

2) there exists a locally ϕ temperate metric $g_{x,\xi}(y,\eta)$ on $\mathbb{R}^n\times\mathbb{R}^n$ such that $|h|$ is locally ϕ, g temperate and

$$h\in S(|h|\,,g),$$

where $S(|h|\,,g)$ denotes the class of symbols of Weyl-Hörmander (see [6] and [5]).

Remark. Under suitable hypotheses on h (see [10], [1] and [4]) one can show that h^{w} is globally hypoelliptic in the following sense:

$$u\in\mathcal{S}'\ \&\ h^{w}(u)\in\mathcal{S}\implies u\in\mathcal{S},$$

where $\mathcal{S}$ is the Schwartz class and $\mathcal{S}'$ are the tempered distributions. This justifies the term formally hypoelliptic in the definition above.

---

[1] This definition is not to be confused with Definition 2.3 of Chapter III of [11].

Theorem 1. *Consider a real positive and formally hypoelliptic symbol* $h \in S(h,g)$ *and assume that there exists a positive real number* γ *such that*

$$g_{x,\xi}(y,\eta) \prec h(x,\xi)^{-\gamma} g^{\phi}_{x,\xi}(y,\eta), \tag{3}$$

for all (x,ξ), $(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$. *Then* h^w *is semi-bounded from below and essentially self-adjoint in* $L^2(\mathbb{R}^n)$.

Moreover, if

$$h(x,\xi) \to +\infty, \qquad \text{as } |x| + |\xi| \to +\infty, \tag{4}$$

then the closure H *of* h^w *in* $L^2(\mathbb{R}^n)$ *has discrete spectrum diverging to* $+\infty$.

Proof. This is Proposition 6.1 of [5]. □

Thanks to this theorem, we can define the *counting function* of the operator H:

$$\mathcal{N}(\tau) = \text{number of eigenfunctions of } H, \text{ corresponding to eigenvalues less than or equal to } \tau. \tag{5}$$

Theorem 2. *Under the hypotheses of Theorem 1 assume there exists* $\kappa > 0$ *such that*

$$h^{-\kappa} \in L^1. \tag{6}$$

Then for all $0 < \delta < \gamma/3$ *we have*

$$\mathcal{N}(\tau) = \mathcal{W}(\tau)\left\{1 + O\big(\mathcal{R}(\tau)\big)\right\}, \qquad \text{as } \tau \to +\infty, \tag{7}$$

where

$$\mathcal{W}(\tau) = (2\pi)^{-n} \int\limits_{h \leqslant \tau} dx\, d\xi \tag{8}$$

and

$$\mathcal{R}(\tau) = \frac{\mathcal{W}\left(\tau + \tau^{1-\delta}\right) - \mathcal{W}\left(\tau - \tau^{1-\delta}\right)}{\mathcal{W}(\tau)}. \tag{9}$$

Remark. Estimate (7) is known as the *Weyl formula.*

Proof. If we change the hypothesis (6) with

$$1 + |x| + |\xi| \prec h(x,\xi)^{\kappa}, \qquad \forall\, (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

this theorem becomes Theorem 6.2 of [5].

A careful inspection of Dencker's argument shows that its proof continues to hold also under our more general hypothesis. □

It is not always true that the remainder $\mathcal{R}(\tau)$ vanishes as $\tau \to +\infty$. In [2] there is an example of a Schrödinger operator with potential with algebraic growth at infinity, thus meeting in particular hypothesis (6), for which $\mathcal{R}(\tau)$ is *unbounded* as $\tau \to +\infty$. In the section which follows we estimate the remainder $\mathcal{R}(\tau)$ under rather general conditions.

3. Hypoelliptic differential operators of Schrödinger type and estimate of the remainder in the Weyl formula

Definition 2. A formally hypoelliptic differential operator h^w is called *of Schrödinger type* if

$$h(x,\xi) = p(\xi) + q(x)$$

where p is a real polynomial such that

$$p(0) = 0, \tag{10}$$

and q is a real-valued function, called a *potential.*

Theorem 3. *Under the hypotheses of Theorem 1, consider a formally hypoelliptic operator of Schrödinger type $h^w = p^w + q$, and set*

$$\mathcal{V}(\tau) = \int\limits_{q \leqslant \tau} dx.$$

Assume that there exists $\tau_0 > 0$, such that

$$\mathcal{V}(2\tau) \prec \mathcal{V}(\tau), \qquad \forall\, \tau \geqslant \tau_0. \tag{11}$$

Then there exists $a > 0$ such that for all $0 < \epsilon < \frac{a}{1+a} \min\left\{1, \frac{\gamma}{3}\right\}$ we have

$$\mathcal{N}(\tau) = \mathcal{W}(\tau)\left\{1 + O\left(\tau^{-\epsilon}\right)\right\}, \qquad \text{as } \tau \to +\infty. \tag{12}$$

Remark. When p is a multi-quasi-elliptic polynomial, one can compute the constant a; see [3].

Proof. First we show that hypothesis (6) is satisfied, so that Weyl formula (7) holds; then we estimate the remainder $\mathcal{R}(\tau)$.

Lemma 1. *There exists $\kappa > 0$ such that (6) is satisfied.*

Proof. Since $h > 0$, we have

$$p(\xi) + q(x) > 0, \qquad \forall\, x\,, \xi \in \mathbb{R}^n.$$

Because $p(0) = 0$ and $q(x) \to +\infty$, as $|x| \to +\infty$, by (4), it follows that

$$L = \inf_{x \in \mathbb{R}^n} q(x) > 0. \tag{13}$$

Hence there exists $R > 0$ such that

$$p(\xi) + q(x) \geqslant \sqrt{p(\xi) q(x)}, \qquad \forall\, \xi \notin B(R) \;\&\; \forall\, x \in \mathbb{R}^n,$$

where $B(R)$ is the open ball with center at the origin and radius R. Therefore it suffices to show that there exists $\kappa_1 > 0$ and $\kappa_2 > 0$ such that $p^{-\kappa_1} \in L^1\big(\mathbb{R}^n \backslash B(R)\big)$ and $q^{-\kappa_2} \in L^1(\mathbb{R}^n)$.

From (2) and (4) we have that $p \to +\infty$, as $|\xi| \to +\infty$. Then, because p is a polynomial, it follows from the Tarski-Seidenberg Theorem that there exists $k_0 > 0$ such that

$$p(\xi) \succ |\xi|^{k_0}\,, \qquad \forall\; |\xi| \geqslant R. \tag{14}$$

In fact,

$$E = \left\{(s,t,\xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \mid s = |\xi|^2 \;\&\; p(\xi) = t\right\}$$

is algebraic in $\mathbb{R}^{2+n}$, and therefore

$$f(s) = \inf_{|\xi|^2 = s} p(\xi) = \inf\bigl\{t \mid \exists s\, \exists \xi\, \bigl((s,t,\xi) \in E\bigr)\bigr\} < +\infty$$

is semi-algebraic. Moreover $f(s) \to +\infty$, as $s \to +\infty$; hence (14) follows from Corollary A.2.6 of [8].

It is clear that estimate (14) implies the existence of κ_1.

Now we prove the existence of κ_2. First we show that there exists $c_0 > 0$ such that

$$\mathcal{V}(\tau) \prec \tau^{c_0}, \qquad \forall\, \tau > 0. \tag{15}$$

From (13) we have that $\mathcal{V}(\tau) = 0$ for $\tau \leqslant L$, hence it suffices to prove (15) for $\tau \geqslant \tau_0 > 0$. From (11) we have that there exists $c > 0$ such that

$$\mathcal{V}(2\tau) \leqslant c\mathcal{V}(\tau), \qquad \forall\, \tau \geqslant \tau_0. \tag{16}$$

Let $k_1 \in \mathbb{N}$ be such that

$$\tau_0 \leqslant 2^{-k_1}\tau \leqslant 2\tau_0.$$

By applying k_1 times inequality (16), we have

$$\begin{aligned}\mathcal{V}(\tau) &\leqslant c^{k_1}\mathcal{V}(2^{-k_1}\tau)\\ &\leqslant c^{\log_2(\tau/\tau_0)}\mathcal{V}(2\tau_0)\\ &= \left(\frac{\tau}{\tau_0}\right)^{\log_2 c}\mathcal{V}(2\tau_0),\end{aligned}$$

which is (15). Now we prove the integrability of $q^{-\kappa_2}$ for $\kappa_2 > c_0 + 1$. In fact, from (15) we have

$$\begin{aligned}\int q(x)^{-\kappa_2}\,dx &= \sum_{k=0}^{+\infty} \int\limits_{k \leqslant q \leqslant k+1} q(x)^{-\kappa_2}\,dx\\ &\leqslant \int\limits_{q \leqslant 1} q(x)^{-\kappa_2}\,dx + \sum_{k=1}^{+\infty} k^{-\kappa_2}\mathcal{V}(k+1)\\ &\prec \sum_{k=1}^{+\infty} k^{-\kappa_2}(k+1)^{c_0} < +\infty,\end{aligned}$$

for $\kappa_2 > c_0 + 1$. □

Thanks to Lemma 1 and Theorem 2 we have that (9) holds.

Now we recall the following result of Nilsson:

Theorem 4. *The function*

$$\Phi(\lambda) = \int\limits_{p(\xi) \leqslant \lambda} d\xi$$

is a real analytic for λ real and large and there exist a positive rational number a and a non-negative integer b, such that for all $k \in \mathbb{N}$ there exists $A_k \in \mathbb{R}$ such that the following estimate holds:

$$\Phi^{(k)}(\lambda) = A_k \lambda^{a-k}(\log \lambda)^b\bigl(1 + o(1)\bigr), \qquad \text{as } \lambda \to +\infty. \tag{17}$$

Proof. This is a special case of Theorem 3 of [9], where the author only assumes that p is a polynomial such that $p(\xi) \to +\infty$, as $|\xi| \to +\infty$. $\square$

Let us draw some consequences from estimate (17):

Corollary 1. *There exists* $\lambda_0 > 0$ *and* $A > 0$ *such that*

$$\Phi(\lambda_2) \leqslant \Phi(\lambda_1) \left(\frac{\lambda_2}{\lambda_1}\right)^A, \qquad \forall\, \lambda_2 \geqslant \lambda_1 \geqslant \lambda_0.$$

Proof. Because $\Phi(\lambda)$ is positive and increasing, it is clear that in the estimate (17) we must have A_0, $A_1 > 0$. In particular, from (17) there exists $\lambda_0 > 0$ such that

$$\Phi(\lambda) \geqslant \frac{A_0}{2} \lambda^a (\log \lambda)^b$$

and

$$\Phi'(\lambda) \leqslant \frac{3A_1}{2} \lambda^{a-1} (\log \lambda)^b,$$

for all $\lambda \geqslant \lambda_0$. It follows that

$$\frac{\Phi'(\lambda)}{\Phi(\lambda)} \leqslant \frac{3A_1}{A_0} \frac{1}{\lambda}, \qquad \forall\, \lambda \geqslant \lambda_0. \tag{18}$$

Let

$$\phi(\lambda) = \frac{\Phi'(\lambda)}{\Phi(\lambda)} = \log\bigl(\Phi(\lambda)\bigr)';$$

then we have

$$\log \frac{\Phi(\lambda_2)}{\Phi(\lambda_1)} = \int_{\lambda_1}^{\lambda_2} \phi(t)\, dt.$$

From (18) it follows that

$$\begin{aligned} \frac{\Phi(\lambda_2)}{\Phi(\lambda_1)} &\leqslant \exp\left(\int_{\lambda_1}^{\lambda_2} \frac{3A_1}{A_0} \frac{1}{t}\, dt\right) \\ &= \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{3A_1}{A_0}}, \qquad \forall\, \lambda_2 \geqslant \lambda_1 \geqslant \lambda_0. \end{aligned}$$

$\square$

Now we estimate the remainder $\mathcal{R}(\tau)$ with $0 < \delta < 1$. Let $0 < s < 1$ be a number to be chosen later and write

$$\mathcal{W}\left(\tau + \tau^{1-\delta}\right) - \mathcal{W}\left(\tau - \tau^{1-\delta}\right) = \mathcal{W}_1(\tau) + \mathcal{W}_2(\tau),$$

where

$$\mathcal{W}_1(\tau) = (2\pi)^{-n} \int\limits_{\substack{\tau - \tau^{1-\delta} < p+q \leqslant \tau + \tau^{1-\delta} \\ q \leqslant \tau - \tau^{1-s\delta}}} dx\, d\xi \tag{19}$$

and

$$\mathcal{W}_2(\tau) = (2\pi)^{-n} \int\limits_{\substack{\tau - \tau^{1-\delta} < p+q \leqslant \tau + \tau^{1-\delta} \\ q > \tau - \tau^{1-s\delta}}} dx\, d\xi. \tag{20}$$

Let us estimate $\mathcal{W}_1$. On the domain of integration of $\mathcal{W}_1$ we have

$$\tau + \tau^{1-\delta} - q \geqslant p > \tau - \tau^{1-\delta} - q \geqslant \tau^{1-s\delta} - \tau^{1-\delta}.$$

Because

$$\tau^{1-s\delta} - \tau^{1-\delta} = \tau^{1-s\delta}\left(1 - \tau^{(s-1)\delta}\right) \to +\infty, \qquad \text{as } \tau \to +\infty,$$

there exists $\tau_1 > 0$ such that

$$\tau^{1-s\delta} - \tau^{1-\delta} \geqslant \lambda_0, \qquad \forall\, \tau \geqslant \tau_1.$$

Then from (19) and Corollary 1 we have

$$\begin{aligned}(2\pi)^n \mathcal{W}_1(\tau) &= \int\limits_{q \leqslant \tau - \tau^{1-s\delta}} \left(\int\limits_{\tau - \tau^{1-\delta} - q < p \leqslant \tau + \tau^{1-\delta} - q} d\xi \right) dx \\ &= \int\limits_{q \leqslant \tau - \tau^{1-s\delta}} \left\{ \Phi\left(\tau + \tau^{1-\delta} - q(x)\right) - \Phi\left(\tau - \tau^{1-\delta} - q(x)\right) \right\} dx \\ &\leqslant \int\limits_{q \leqslant \tau - \tau^{1-s\delta}} \Phi(\tau - q) \left\{ \left(1 + \frac{\tau^{1-\delta}}{\tau - q}\right)^A - \left(1 - \frac{\tau^{1-\delta}}{\tau - q}\right)^A \right\} dx,\end{aligned}$$

for all $\tau \geqslant \tau_1$. This in turn implies that there exists $\tau_2 \geqslant \tau_1$ such that

$$\begin{aligned}\mathcal{W}_1(\tau) &\prec \int\limits_{q \leqslant \tau - \tau^{1-s\delta}} \Phi(\tau - q) \frac{\tau^{1-\delta}}{\tau - q}\, dx \\ &\leqslant \tau^{(s-1)\delta} \int\limits_{q \leqslant \tau - \tau^{1-s\delta}} \Phi(\tau - q)\, dx \\ &= \tau^{(s-1)\delta} \mathcal{W}(\tau), \qquad \forall\, \tau \geqslant \tau_2.\end{aligned} \tag{21}$$

Now we estimate $\mathcal{W}_2$. Because p is non-negative as $|\xi| \to +\infty$, there exists $\tau_3 \geqslant 1$ such that, on the domain of integration, we have

$$\begin{aligned}&p \leqslant \tau + \tau^{1-\delta} - q < \tau^{1-\delta} + \tau^{1-s\delta} = \tau^{1-s\delta}\left(\tau^{(s-1)\delta} + 1\right) \leqslant 2\tau^{1-s\delta}, \\ &q \leqslant \tau + \tau^{1-\delta} - p \leqslant 2\tau,\end{aligned}$$

for $\tau \geqslant \tau_3$.

It follows that

$$\begin{aligned}\mathcal{W}_2(\tau) &\leqslant (2\pi)^{-n} \int\limits_{p(\xi) \leqslant 2\tau^{1-s\delta}} d\xi \int\limits_{q(x) \leqslant 2\tau} dx \\ &= (2\pi)^{-n} \Phi\left(2\tau^{1-s\delta}\right) \mathcal{V}(2\tau),\end{aligned}$$

for all $\tau \geqslant \tau_3$. Now, from (11) we have

$$\mathcal{V}(2\tau) \prec \mathcal{V}\left(\frac{\tau}{2}\right), \qquad \forall \tau \geqslant \tau_0.$$

On the other side, from (17), we have that

$$\begin{aligned}\Phi\left(2\tau^{1-s\delta}\right) &= A_0\left(2\tau^{1-s\delta}\right)^a\left(\log\left(2\tau^{1-s\delta}\right)\right)^b\big(1+o(1)\big)\\ &= 2^a(1-s\delta)^b A_0\tau^{(1-s\delta)a}(\log\tau)^b\big(1+o(1)\big)\\ &= 2^{2a}(1-s\delta)^b A_0\tau^{-s\delta a}\left(\frac{\tau}{2}\right)^a\left(\log\left(\frac{\tau}{2}\right)\right)^b\big(1+o(1)\big)\\ &= 2^{2a}(1-s\delta)^b\tau^{-s\delta a}\Phi\left(\frac{\tau}{2}\right)\big(1+o(1)\big), \qquad \text{for } \tau\to+\infty.\end{aligned}$$

But these estimates imply that there exists $\tau_4 \geqslant \tau_0$ such that

$$\begin{aligned}\mathcal{W}_2(\tau) &\prec \tau^{-s\delta a}\Phi\left(\frac{\tau}{2}\right)\mathcal{V}\left(\frac{\tau}{2}\right)\\ &\leqslant \tau^{-s\delta a}\int\limits_{p\leqslant\tau/2,\ q\leqslant\tau/2} dx\,d\xi\\ &\leqslant \tau^{-s\delta a}\mathcal{W}(\tau), \qquad \forall\,\tau\geqslant\tau_4.\end{aligned} \tag{22}$$

Eventually, from (21) and (22), we have that it suffices to choose s such that

$$(s-1)\delta = -s\delta a,$$

that is,

$$s = \frac{1}{1+a},$$

in order to obtain

$$\mathcal{W}\left(\tau+\tau^{1-\delta}\right)-\mathcal{W}\left(\tau-\tau^{1-\delta}\right) = O\left(\tau^{-\epsilon}\right)\mathcal{W}(\tau), \quad \text{as } \tau\to+\infty,$$

with

$$\epsilon = \frac{\delta a}{1+a}.$$

□

4. An example

We end our paper by giving an example of potential verifying our hypotheses. First we prove the following

Proposition 1. *Let $q:\mathbb{R}^n\to\mathbb{R}$ be a smooth function such that there exist $\mu\in\mathbb{R}^n_+$, $C_\mu>1$ and $R_0\geqslant 0$ such that*

$$q(\mu\cdot x)\geqslant C_\mu q(x), \qquad \forall\,|x|\geqslant R_0,$$

where

$$\mu\cdot x = \sum_{i=1}^n \mu_i x_i.$$

Let N be the smallest integer such that

$$C_\mu^N \geqslant 2. \tag{23}$$

Then $\mathcal{V}$ satisfies condition (11) with

$$\tau_0 = \tfrac{C_\mu^N}{2}\max_{|x|\leqslant R_0} q(x).$$

Proof. Letting $x = \mu\cdot y$ and using (23) yields

$$\begin{aligned}
\int\limits_{q(x)\leqslant 2\tau} dx &= \mu_1\cdots\mu_n \int\limits_{q(\mu\cdot y)\leqslant 2\tau} dy \\
&\leqslant \mu_1\cdots\mu_n \left\{ \int\limits_{|y|<R_0} dy + \int\limits_{\substack{q(\mu\cdot y)\leqslant 2\tau \\ |y|\geqslant R_0}} dy \right\} \\
&\leqslant \mu_1\cdots\mu_n \left\{ \int\limits_{|y|<R_0} dy + \int\limits_{\substack{C_\mu q(y)\leqslant 2\tau \\ |y|>R_0}} dy \right\} \\
&\leqslant \mu_1\cdots\mu_n \int\limits_{q(y)\leqslant 2C_\mu^{-1}\tau} dy,
\end{aligned}$$

for $\tau \geqslant \tau_0$. By iterating, we obtain

$$\mathcal{V}(2\tau) = \int\limits_{q(x)\leqslant 2\tau} dx \leqslant (\mu_1\cdots\mu_n)^N \int\limits_{q(x)\leqslant 2C_\mu^{-N}\tau} dx \leqslant (\mu_1\cdots\mu_n)^N\, \mathcal{V}(\tau),$$

for all $\tau \geqslant \tau_0$. ☐

Now we can construct a class of operators of Schrödinger type, with potential verifying condition (11).

Proposition 2. *Let us consider*

- *a real polynomial $p(\xi)$ vanishing at the origin and hypoelliptic, i.e. such that*

  $$\lim_{|\xi|\to+\infty} p(\xi) = +\infty$$

  and

  $$\nabla p(\xi) \prec p(\xi)^{1-\rho}, \qquad \text{for } p(\xi) \geqslant 1,$$

  with $0 < \rho \leqslant 1$,
- *a positive smooth function $q(x)$ such that*
  - *$p(\xi) + q(x) > 0$, $\forall\,(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$,*
  - *there exists $0 \leqslant \nu < \rho$ such that for all $\alpha \in \mathbb{N}^n$ we have $D^\alpha q(x) \prec q(x)^{1+\nu|\alpha|}$, $\forall x \in \mathbb{R}^n$,*
  - *it satisfies the hypothesis of Proposition 1.*

Then $h^{\mathrm{w}} = p^{\mathrm{w}} + q$ satisfies the hypotheses of Theorem 3.

Proof. We give a brief sketch of the proof. By Proposition 1 condition (11) is satisfied.

Then it suffices to verify the hypotheses of Theorem 1. Consider the metric

$$g_{x,\xi}(y,\eta) = h(x,\xi)^{2\nu}\,|y|^2 + h(x,\xi)^{-2\rho}\,|\eta|^2 .$$

Then one verifies that g and h are locally ϕ temperate; for example see [3]. Moreover one can show that $h \to +\infty$, as $|x| + |\xi| \to +\infty$, $h \in S(h,g)$ and (3) is verified. We omit the details. ☐

For example we can take (see [1], section 1.1)

$$p(\xi) = \left(\xi_1^{2k} - \xi_2^{2k-1}\right)^2 + \xi_1^{2k}\xi_2^{2k-2},$$
$$q(x) = \exp\left(x_1^{2m_1} + x_2^{2m_2}\right),$$

where k is an integer greater than 1 and m_1 and m_2 are positive integers.

References

[1] Boggiatto, P., Buzano, E. and Rodino, L., *Global hypoellipticity and spectral theory*, Mathematical Research, vol. 92, Akademie Verlag, Berlin, 1996. MR **97m:**35040

[2] Buzano, E., *Some remarks on the Weyl asymptotics by the approximate spectral projection method*, Boll. U.M.I. **3-B** (2000), no. 8, 775–792 MR **2001k:**35232

[3] Buzano, E. and Ziggioto, A., *Weyl formula for multi-quasi-elliptic operators of Schrödinger type*, Ann. Mat. Pura Appl. **180** (2001), 223–243. MR **2002f:**35179

[4] Buzano, E. and Nicola, F., *Hypoelliptic symbols and complex powers of pseudodifferential operators in the Weyl-Hörmander classes*, in preparation.

[5] Dencker, N., *The Weyl calculus with locally temperate metrics and weights*, Arkiv för Mat. **24** (1986), 59–79. MR **87m:**47111

[6] Hörmander, L., *The Weyl calculus of pseudo-differential operators*, Comm. Pure Appl. Math. **32** (1979), 359–443. MR **80j:**47060

[7] ———, *On the asymptotic distribution of the eigenvalues of pseudodifferential operators in* $\mathbb{R}^n$, Arkiv för Mat. **17** (1979), 297–313. MR **82i:**35140

[8] ———, *The analysis of linear partial differential operators II*, Grundleheren der mathematischen Wissenschaften, vol. 257, Springer-Verlag, Berlin, 1983. MR **85g:**35002b

[9] Nilsson, N., *Monodromy and asymptotic properties of certain multiple integrals*, Arkiv för Mat. **18** (1980), 181–198. MR **83i:**32020

[10] Shubin, M. A., *Pseudodifferential operators and spectral theory*, Springer-Verlag, Berlin, 1987. MR **88c:**47105

[11] Taylor M. E., *Pseudodifferential Operators*, Princeton Mathematical Series, vol. 34, Princeton University Press, Princeton, New Jersey, 1981. MR **82i:**35172

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italia
E-mail address: `buzano@dm.unito.it`

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italia
E-mail address: `ziggioto@dm.unito.it`