

## A REMARK ON REAL COBOUNDARY COCYCLES IN $L^\infty$ -SPACE

RYOTARO SATO

(Communicated by Michael Handel)

ABSTRACT. Let  $T$  be an ergodic automorphism of a probability measure space  $(\Omega, \mathcal{A}, m)$  and let  $f$  be a real-valued measurable function on  $\Omega$ . We deduce a necessary and sufficient condition for the existence of  $L^\infty$ -solutions of the cohomology equation  $f = h \circ T - h$ , by using the recent result of Alonso, Hong and Obaya.

Let  $(\Omega, \mathcal{A}, m)$  be a probability measure space and let  $T$  be an ergodic automorphism of  $(\Omega, \mathcal{A}, m)$ . Every real-valued measurable function  $f$  on  $\Omega$  defines an additive real  $T$ -cocycle

$$(1) \quad S_n f = \sum_{j=0}^{n-1} f \circ T^j, \quad n > 0.$$

Recently Alonso, Hong and Obaya proved in [1] that a necessary and sufficient condition for the existence of  $L^p$ -solutions of the cohomology equation  $f = h \circ T - h$ , with  $0 < p < \infty$ , is that  $f \in L^p(\Omega, m)$  and there exists a measurable subset  $A \subset \Omega$  with  $m(A) > 0$  and an increasing sequence  $\{n_k\}$  of positive integers with

$$(2) \quad \sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \int_A |S_j f|^p dm < \infty.$$

For related results we refer the reader to [2] and [3]. In this note we would like to consider the case  $p = \infty$  and prove the following theorem.

**Theorem.** *Let  $f \in L^p(\Omega, m)$  for some  $p$  with  $0 < p \leq \infty$ . Then there exists a function  $h$  in  $L^\infty(\Omega, m)$  with  $f = h \circ T - h$  if and only if there exists a measurable subset  $A \subset \Omega$  with  $m(A) > 0$  and an increasing sequence  $\{n_k\}$  of positive integers with*

$$(3) \quad \sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \|S_j f \cdot \chi_A\|_\infty < \infty.$$

*Proof.* Suppose  $f = h \circ T - h$  for some  $h \in L^\infty(\Omega, m)$ . Then, since  $S_j f = h \circ T^j - h$  for  $j \geq 1$ , we have  $\|S_j f\|_\infty \leq 2\|h\|_\infty$ , whence (3) holds for  $A = \Omega$  and  $n_k = k$ .

---

Received by the editors August 31, 2001.

2000 *Mathematics Subject Classification.* Primary 37A20, 28D05, 47A35.

*Key words and phrases.* Ergodic automorphism, additive real cocycle, cohomology equation, coboundary.

Conversely, suppose the second condition of the Theorem holds. Since  $0 < r < p$  implies  $L^p(\Omega, m) \subset L^r(\Omega, m)$ , we may assume from the beginning that  $0 < p \leq 1$ . Then we have

$$\int_A |S_j f|^p dm \leq 1 + \|S_j f \cdot \chi_A\|_\infty \quad \text{for } j \geq 1,$$

which implies

$$\sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \int_A |S_j f|^p dm < \infty.$$

Thus, by Theorem 3.2 of [1], there exists a function  $h$  in  $L^p(\Omega, m)$  with  $f = h \circ T - h$ . It suffices to show that  $h$  is a function in  $L^\infty(\Omega, m)$ .

To do so, we notice that there exists a positive number  $\delta$  with

$$m(A \cap \{\omega : |h(\omega)| \leq \delta\}) > 0.$$

Therefore, we may assume from the beginning that

$$(4) \quad |h| \leq \delta \quad \text{on } A.$$

Then, using the relations

$$\|h \circ T^j \cdot \chi_A\|_\infty = \|(S_j f + h) \cdot \chi_A\|_\infty \leq \|S_j f \cdot \chi_A\|_\infty + \delta,$$

we see from (3) that

$$(5) \quad M := \sup_{k \geq 1} \frac{1}{n_k} \sum_{j=1}^{n_k} \|h \circ T^j \cdot \chi_A\|_\infty < \infty.$$

By this inequality, we will prove below that  $h \in L^\infty(\Omega, m)$ .

For this purpose, let  $\alpha$  be a positive number such that there exists a measurable subset  $B \subset \Omega$  with

$$(6) \quad m(B) > 0 \quad \text{and} \quad |h| \geq \alpha \quad \text{on } B.$$

Since  $T$  is an ergodic automorphism by hypothesis, the Birkhoff Ergodic Theorem implies that

$$\lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{j=1}^{n_k} \chi_A(T^{-j}\omega) = m(A) > 0$$

for almost all  $\omega \in \Omega$ . Hence, we can choose a measurable subset  $B_1 \subset B$  with  $m(B_1) > 0$  and a positive integer  $K$  with

$$(7) \quad \frac{1}{n_K} \sum_{j=1}^{n_K} \chi_A(T^{-j}\omega) > 2^{-1} m(A) \quad \text{for all } \omega \in B_1.$$

Thus, taking into account (5) and (6), we get

$$\begin{aligned} \infty > M &\geq \frac{1}{n_K} \sum_{j=1}^{n_K} \|h \circ T^j \cdot \chi_A\|_\infty = \frac{1}{n_K} \sum_{j=1}^{n_K} \|h \cdot \chi_A \circ T^{-j}\|_\infty \\ &\geq \frac{1}{n_K} \sum_{j=1}^{n_K} \|\alpha \chi_B \cdot (\chi_A \circ T^{-j})\|_\infty \geq \alpha \cdot \left\| \chi_{B_1} \cdot \left( \frac{1}{n_K} \sum_{j=1}^{n_K} \chi_A \circ T^{-j} \right) \right\|_\infty \\ &\geq \alpha \cdot 2^{-1} m(A). \end{aligned}$$

This shows that  $\alpha$  cannot be so large, and thus  $h \in L^\infty(\Omega, m)$  must follow. This completes the proof.  $\square$

## REFERENCES

- [1] A. I. Alonso, J. Hong and R. Obaya, Absolutely continuous dynamics and real coboundary cocycles in  $L^p$ -spaces,  $0 < p < \infty$ , *Studia Math.* 138 (2000), 121–134. MR **2001i**:37010
- [2] I. Assani, A note on the equation  $Y = (I - T)X$  in  $L^1$ , *Illinois J. Math.* 43 (1999), 540–541. MR **2000j**:47016
- [3] M. Lin and R. Sine, Ergodic theory and the functional equation  $(I - T)x = y$ , *J. Operator Theory* 10 (1983), 153–166. MR **84m**:47015

DEPARTMENT OF MATHEMATICS, OKAYAMA UNIVERSITY, OKAYAMA, 700-8530 JAPAN  
*E-mail address*: [satoryot@math.okayama-u.ac.jp](mailto:satoryot@math.okayama-u.ac.jp)