HARMONIC BERGMAN FUNCTIONS
AS RADIAL DERIVATIVES OF BERGMAN FUNCTIONS

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Abstract. In the setting of the half-space of the euclidean $n$-space, we show that every harmonic Bergman function is the radial derivative of a Bergman function with an appropriate norm bound.

1. Introduction

For a positive integer $n \geq 2$, let $H = \mathbb{R}^{n-1} \times \mathbb{R}_+$ denote the upper half $n$-space where $\mathbb{R}_+$ is the set of all positive real numbers. We will often write a point $z \in H$ as $z = (z', z_n)$ where $z' = (z_1, \ldots, z_{n-1}) \in \mathbb{R}^{n-1}$ and $z_n \in \mathbb{R}_+$.

For $1 \leq p < \infty$, a harmonic function $u$ on $H$ is called an $L^p$-Bergman function if

$$||u||_p = \left\{ \int_H |u(z)|^p \, dz \right\}^{1/p} < \infty.$$ 

We let $b^p$ denote the space of all $L^p$-Bergman functions on $H$. The space $b^p$ is a closed subspace of $L^p = L^p(H)$ and hence a Banach space.

A harmonic function $u$ on $H$ is called a Bloch function if

$$||u||_B = \sup z_n |\nabla u(z)| < \infty,$$

where the supremum is taken over all $z \in H$ and $\nabla$ denotes the gradient operator. We let $B$ denote the space of all Bloch functions on $H$. In [5], it is shown that the dual space of $b^1$ can be identified with $B/C$. So we can consider the harmonic Bloch space $B$ as a limiting space of the harmonic Bergman space $b^p$.

It is elementary to see that, given $a \in H$ and a harmonic function $u$ on $H$, there corresponds a unique harmonic function $f$ on $H$ such that $f(a) = 0$ and

$$u(z) - u(a) = \sum_{j=1}^n (z_j - a_j) D_j f(z), \quad z \in H.$$ 

Here and elsewhere, $D_j$ denotes the differentiation with respect to the $j$-th variable for each $j$. In [3] the authors have shown that the maps $u \mapsto D_j f$ are bounded on $b^p$ if and only if $p > n$, while the map $u \mapsto f$ is not bounded on $b^p$ for every $p \geq 1$. Also, these properties are shown to extend to the space $B$. Note that $b^p$-functions
must vanish at $\infty$ along any ray (emanating from the origin). Motivated by these observations, we asked ourselves what would happen if the reference point $a \in \mathbb{H}$ would be replaced by the boundary point $\infty$ and found situations quite different, which is the main result of this paper.

In the following, we let $\mathcal{R}$ denote the radial differentiation of $h \in C^1(\mathbb{H})$ defined by

$$\mathcal{R}h(z) = \sum_{j=1}^{n} z_{j} D_{j} h(z), \quad z \in \mathbb{H}.$$  

Note that $\mathcal{R}h \equiv 0$ if and only if $h$ is radially constant, i.e., $h(z) = h(tz)$ for all $t > 0$ and $z \in \mathbb{H}$. The following is our main result.

**Theorem 1.1.** Let $1 \leq p < \infty$. Then, given $u \in b^p$, there corresponds a unique $\tilde{u} \in b^p$ such that $u = \mathcal{R}\tilde{u}$. In addition, we have the following:

1. The map $u \mapsto \tilde{u}$ is bounded on $b^p$.
2. For $1 \leq p < n$, there exists a positive constant $C_p$ such that

$$\sum_{j=1}^{n} \|D_{j} \tilde{u}\|_p \leq C_p \|D_{n} u\|_p$$

for all $u \in b^p$.

**Remarks.**

1. For the representation $u = \mathcal{R}\tilde{u}$, we will define $\tilde{u}$ as an integral involving $u$ as follows:

$$\tilde{u}(z) = -\int_{1}^{\infty} \frac{u(tz)}{t} \, dt.$$  

The key of this integral representation is that $b^p$-functions vanish with suitable order at $\infty$ along the ray passing through $z$. However, this vanishing property is not shared by Bloch functions, which might have relatively wild (actually logarithmic $[1]$) behavior near $\infty$. Thus, one may not expect such a representation for Bloch functions, as turns out to be the case. See Proposition 3.1.

2. Note that the partial derivatives $D_{j} \tilde{u}$ are obtained by differentiating an integral of $u$. Thus, one may roughly expect that the functions $D_{j} \tilde{u}$ and $u$ are approximately of the same growth. We show that this is not the case in the sense that one cannot replace $D_{n} u$ by $u$ in the right side of (1.1). See Proposition 3.2.

3. The inequality (1.1) does not mean $\|D_{n} u\|_p \leq \|u\|_p$ at all. In fact, if $\|D_{n} u\|_p < \infty$ would happen for functions $u \in b^p$, then all the partial derivatives of $b^p$-functions would also be $b^p$-functions by Lemma 2.2 below, which cannot be expected in general. An explicit proof showing the existence of $u \in b^p$ with $D_{n} u \notin b^p$, $1 \leq p < \infty$, is included for completeness. See Proposition 3.3. Also, for $1 \leq p < \infty$, we have

$$\|z_{n} D_{n} u\|_p \approx \|u\|_p$$

as $u$ ranges over all $b^p$-functions. See [5].

4. The range of $p$ for the inequality (1.1) is sharp. That is, (1.1) fails to hold for $n \leq p < \infty$. See Proposition 3.3.

Section 2 is devoted to the proof of Theorem 1.1. In Section 3 we provide examples concerning the above remarks.

**Constants.** Throughout the paper we will use the same letter $C$ to denote various constants, often with subscripts indicating dependency, which may change at each occurrence. We will often write $A \lesssim B$ or $B \gtrsim A$ for nonnegative quantities $A, B$. 

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if $A$ is dominated by $B$ times some *inessential* positive constant. Also, we write $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

2. **Proof of Theorem 1.1**

Before proceeding to the proof, we review some preliminary results on reproducing kernels for the spaces $b^p$. The reproducing kernel $R(z, w)$ for $b^p$ is given by

\begin{equation}
R(z, w) = \frac{4}{n\sigma_n} \frac{n(z_n + w_n)^2 - |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}}, \quad z, w \in \mathbb{H},
\end{equation}

where $\sigma_n$ denotes the volume of the unit ball in $\mathbb{R}^n$ and $\overline{w} = (w', -w_n)$. That is, we have

\[ u(z) = \int_{\mathbb{H}} u(w)R(z, w) \, dw, \quad z \in \mathbb{H}, \]

for all $u \in b^p$, $1 \leq p < \infty$. See [2], [5] for details and related topics. A generalized reproducing property of the kernel $R(z, w)$ is also available [5]:

\begin{equation}
\tag{2.2}
\[ u(z) = -2 \int_{\mathbb{H}} w_n [D_n u(w)]R(z, w) \, dw, \quad z \in \mathbb{H}, \]
\end{equation}

for all $u \in b^p$, $1 \leq p < \infty$. This generalized reproducing formula shows that $b^p$-functions are completely determined by their normal derivatives. In particular, it is not too surprising to see that the $L^p$-size of derivatives of $b^p$-functions is controlled by that of normal derivatives. To see this, we first need a lemma, which is a special case of Lemma 4.4 of [4].

**Lemma 2.1.** Let $1 \leq p < \infty$. For $\psi \in L^p$, define

\[ T \psi(z) = \int_{\mathbb{H}} \psi(w) \frac{w_n}{|z - \overline{w}|^{n+1}} \, dw, \quad z \in \mathbb{H}. \]

Then, $T : L^p \to L^p$ is bounded.

**Lemma 2.2.** Let $1 \leq p < \infty$. Then we have

\[ \sum_{j=1}^n ||D_j u||_p \leq C_p ||D_n u||_p \]

for functions $u \in b^p$.

**Proof.** By a straightforward calculation using (2.1), it is easily checked that

\begin{equation}
\tag{2.3}
|D_j R(z, w)| \lesssim \frac{1}{|z - \overline{w}|^{n+1}}, \quad z, w \in \mathbb{H},
\end{equation}

for all $j$. Here, the ambiguous notation $D_j R(z, w)$ means $D_j[R(\cdot, w)]$ evaluated at $z$ for each fixed $w$. Thus, by (2.2) and (2.3), we obtain

\[ |D_j u(z)| \lesssim \int_{\mathbb{H}} |D_n u(w)| \frac{w_n}{|z - \overline{w}|^{n+1}} \, dw, \quad z \in \mathbb{H}, \]

for each $j$. Now, the lemma follows from Lemma 2.1. The proof is complete. □
We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Since any $b^p$-function vanishes at $\infty$ along any ray, the uniqueness follows from the fact that radially constant functions vanishing at $\infty$ along any ray must be identically 0. Let $1 \leq p < \infty$, fix $u \in b^p$ and define

$$\tilde{u}(z) = -\int_1^\infty \frac{u(tz)}{t} \, dt, \quad z \in \mathbb{H}. \quad (2.4)$$

By the mean-value property, Jensen’s inequality and Cauchy’s estimates, we have (see Corollary 8.2 of [2])

$$|D^\alpha u(tz)| \leq C_\alpha \frac{\|u\|_p}{(tz^a)^{a+n/p}}, \quad t > 0, \quad z \in \mathbb{H},$$

for every multi-index $\alpha$. It follows that

$$\int_1^\infty \frac{|D^\alpha u(tz)|}{t} \, dt < \infty$$

for each $z \in \mathbb{H}$ and multi-index $\alpha$. Taking $\alpha = (0, 0, \ldots, 0)$, we see that $\tilde{u}$ is well defined. Also, via the dominated convergence theorem, we see that differentiation, as many times as we want, under the integral sign of (2.4), is justified. In particular, $\tilde{u}$ is harmonic on $\mathbb{H}$ and for each $j$ we have

$$D_j \tilde{u}(z) = -\int_1^\infty D_j u(tz) \, dt. \quad (2.5)$$

Now, since $u$ vanishes at $\infty$ along any ray, we have by (2.5)

$$u(z) = -\int_1^\infty \frac{du}{dt}(tz) \, dt
= -\int_1^\infty \sum_{j=1}^n z_j D_j u(tz) \, dt
= -\sum_{j=1}^n z_j \int_1^\infty D_j u(tz) \, dt
= -\sum_{j=1}^n z_j D_j \tilde{u}(z)
= \mathcal{R} \tilde{u}(z)$$

for all $z \in \mathbb{H}$. This proves the existence.

We now prove (1) and (2). First, by Minkowski’s integral inequality we have

$$||\tilde{u}||_p \leq \int_1^\infty \left\{ \int_\mathbb{H} |u(tz)|^p \, dz \right\}^{1/p} \frac{dt}{t^{1+n/p}}
= \int_1^\infty \left\{ \int_\mathbb{H} |u(z)|^p \, dz \right\}^{1/p} \frac{dt}{t^{1+n/p}}
= ||u||_p \int_1^\infty \frac{dt}{t^{1+n/p}}.$$
Since the above integral is finite for all $1 \leq p < \infty$, we have (1). Next, by Minkowski’s integral inequality again we have

$$||D_j \bar{u}||_p \leq \int_1^\infty \left\{ \int_{\mathbb{H}} |D_j u(tz)|^p \, dz \right\}^{1/p} \, dt$$

$$= \int_1^\infty \left\{ \int_{\mathbb{H}} |D_j u(z)|^p \, dz \right\}^{1/p} \frac{dt}{t^{n/p}}$$

for each $j$. Note that the above integral is finite for $1 \leq p < n$. This, together with Lemma 2.2 yields (2). The proof is complete. □

3. Examples

In this section we provide examples concerning the remarks in the Introduction. First, we show that the radial derivative representation fails to hold in general for Bloch functions.

**Proposition 3.1.** There is a function $u \in \mathcal{B}$ such that $u \neq \mathcal{R} v$ for any $v \in \mathcal{B}$.

**Proof.** Let $u(z) = \log(z_1^2 + z_n^2)$ for $z \in \mathbb{H}$. Then we can easily check that $u \in \mathcal{B}$. Suppose $u = \mathcal{R} v$ for some $v \in \mathcal{B}$. Then we have

$$\log(z_1^2 + z_n^2) = \sum_{j=1}^n z_j D_j v(z)$$

for all $z \in \mathbb{H}$. By plugging $(0, \ldots, 0, m)$ into $z$ and then letting $m \to \infty$ on both sides of (3.1), we see that the left side of (3.1) is unbounded but the right side of (3.1) is bounded by $\|v\|_B$. Therefore we get a contradiction and the proof is complete. □

Next, we show that (1.1) is no longer true if $||D_n \bar{u}||_p$ is replaced by $||u||_p$. For that purpose, it suffices to show that the map $u \mapsto D_j \bar{u}$ cannot be bounded on any $b^p$, $1 \leq p < \infty$.

**Proposition 3.2.** Given $1 \leq p < \infty$, there is a function $u \in b^p$ satisfying $D_n \bar{u} \notin b^p$.

**Proof.** In order to derive a contradiction, suppose that there is some $1 \leq p < \infty$ for which $D_n \bar{u} \in b^p$ for all $u \in b^p$. Recall that the map $u \mapsto \bar{u}$ is bounded on $b^p$ by Theorem 1.1. Thus, since the convergence in $b^p$ implies uniform convergence on compact subsets, the closed graph theorem implies that the map $u \mapsto D_n \bar{u}$ is bounded on $b^p$.

Now, let

$$\varphi(z) = \begin{cases} 
\log|z| & \text{for } n = 2, \\
|z|^{2-n} & \text{for } n \geq 3,
\end{cases}$$

and let $v(z) = D_n^3 \varphi(z)$ for $z \in \mathbb{H}$. Then it is easy to see that

$$v(z) = \frac{f(z)}{|z|^{n+4}}, \quad D_n v(z) = \frac{g(z)}{|z|^{n+6}}.$$
for some homogeneous harmonic polynomials $f$ and $g$ of degree 3 and 4, respectively. For $\delta > 0$, put $\nu_\delta(z) = v(z', z_n + \delta)$, $z \in \mathbb{H}$. Then, by homogeneity of $f$, we have

$$
\|\nu_\delta\|_p^p = \int_{\mathbb{H}} \frac{|f(z', z_n + \delta)|^{p}}{|(z', z_n + \delta)^{p(n+1)}} \, dz
$$

$$
= \frac{\delta^{n+3p}}{\delta^{p(n+4)}} \int_{\mathbb{H}} \frac{|f(z', z_n + 1)|^{p}}{|(z', z_n + 1)^{p(n+4)}} \, dz
$$

$$
\approx \delta^{n-pn-p} \int_{\mathbb{H}} \frac{|f(z)|^{p}}{1 + |z|^{p(n+4)}} \, dz
$$

$$
\lesssim \delta^{n-pn-p} \left( \sup_{|\zeta| = 1} |f(\zeta)|^p \right) \int_{\mathbb{H}} \frac{dz}{1 + |z|^{p(n+1)}}.
$$

Since the last integral of the above is finite, we see $v_\delta \in b^p$ and

(3.2) \quad \|v_\delta\|_p \approx \delta^{n/p-n-1}.

On the other hand, by (2.5) and homogeneity of $g$, we have

$$
\|D_n \nu_\delta\|_p^p = \int_{\mathbb{H}} \left\| \int_1^{\infty} \frac{g(t z', t z_n + \delta)}{|(t z', t z_n + \delta)^{n+6}} \, dt \right\|_p^p \, dz
$$

$$
= \frac{\delta^{n+4p}}{\delta^{p(n+6)}} \int_{\mathbb{H}} \left\| \int_1^{\infty} \frac{g(t z', t z_n + 1)}{|(t z', t z_n + 1)^{n+6}} \, dt \right\|_p^p \, dz.
$$

Note that the last integral above is independent of $\delta$. If it is not finite (actually this is the case for $n \leq p < \infty$), then we already have a contradiction. If it is finite (actually this is the case for $p < n$), then we have

$$
\|D_n \nu_\delta\|_p \approx \delta^{n/p-n-2}
$$

and thus by (3.2)

$$
\frac{\|D_n \nu_\delta\|_p}{\|v_\delta\|_p} \approx \delta^{-1} \rightarrow \infty \quad \text{as} \quad \delta \rightarrow 0.
$$

This shows that the map $u \mapsto D_n u$ is not bounded on $b^p$, which is again a contradiction. The proof is complete.

We also show that the right side of (1.1) is possibly infinite.

**Proposition 3.3.** Given $1 \leq p < \infty$, there is a function $u \in b^p$ satisfying $D_n u \notin b^p$.

**Proof.** In order to derive a contradiction, suppose that there is some $1 \leq p < \infty$ for which $D_n u \in b^p$ for all $u \in b^p$. Then, we again see via the closed graph theorem that the map $u \mapsto D_n u$ is bounded on $b^p$.

We continue using the notations defined in the proof of Proposition 3.2. Using a similar argument as in the estimate of $\|v_\delta\|_p$, one can verify that

$$
\|D_n \nu_\delta\|_p \approx \delta^{n/p-n-2}.
$$

This, together with (3.2), yields

$$
\frac{\|D_n \nu_\delta\|_p}{\|v_\delta\|_p} \approx \delta^{-1} \rightarrow \infty \quad \text{as} \quad \delta \rightarrow 0.
$$

This shows that the map $u \mapsto D_n u$ is not bounded on $b^p$, which is a contradiction as desired. The proof is complete.
Finally, we show that (1.1) is sharp for the range of $p$. For that purpose, it suffices to show the following.

**Proposition 3.4.** There is a function $u \in \bigcap_{p \geq n} b^p$ satisfying $D_n u \in \bigcap_{p \geq n} b^p$, but $D_n \tilde{u} \notin \bigcup_{p \geq n} b^p$.

**Proof.** First, consider the case $n > 2$. Fix $z_0 = (0', 1)$. Let $\varphi(z) = |z + z_0|^{2-n}$ and put $u = c_n D_1 D_2 \cdots D_{n-1} \varphi$ where $c_n$ is chosen so that

\begin{equation}
\tag{3.3} u(z) = \frac{z_1 z_2 \cdots z_{n-1}}{|z + z_0|^{3n-4}}, \quad z \in H.
\end{equation}

Harmonicity of $\varphi$ on $H$ implies that of $u$. A straightforward calculation yields

\begin{equation}
\tag{3.4} D_n u(z) = (4 - 3n) \frac{z_1 z_2 \cdots z_{n-1} (z_n + 1)}{|z + z_0|^{3n-2}}, \quad z \in H.
\end{equation}

From (3.3) and (3.4), we have

$$|u(z)| \lesssim (1 + |z|)^{3-2n}, \quad |D_n u(z)| \lesssim (1 + |z|)^{2-2n}$$

and therefore both $u$ and $D_n u$ belong to $b^p$ for any $p \geq n$. Also, we have

$$D_n \tilde{u}(z) = -\int_1^\infty D_n u(tz) dt = (3n - 4) z_1 \cdots z_{n-1} \int_1^\infty \frac{t^{n-1}(tz_n + 1)}{|tz + z_0|^{3n-2}} dt$$

and therefore

\begin{equation}
\tag{3.5} |D_n \tilde{u}(z)| \gtrsim |z_1 z_2 \cdots z_{n-1}| \int_1^{1/|z|} t^{n-1} dt \approx \frac{|z_1 z_2 \cdots z_{n-1}|}{|z|^{n}}, \quad z \in H.
\end{equation}

Let $E$ be the set of all points $z \in H$ such that $0 < z_n < 1$ and $z_n/2 \leq z_j \leq 2z_n$ for each $j$. Note that $z_j \approx z_n \approx |z|$ for $z \in E$. Thus, for any $p \geq n$, we have by (3.5)

$$\int_E |D_n \tilde{u}(z)|^p dz \gtrsim \int_E \left\{ \frac{z_1 z_2 \cdots z_{n-1}}{|z|^n} \right\}^p dz$$

$$\gtrsim \int_{E \setminus z_n} \frac{dz}{z_n}$$

$$= \left(\frac{3}{2}\right)^{n-1} \int_0^{1/2} \frac{z_n^{n-1}}{z_n^p} dz_n$$

$$= \infty$$

as desired.

Now, consider the case $n = 2$. Let $u$ be the imaginary part of the holomorphic function $(z + iy)^{-2}$ on $H$. Here, $z = (x, y) = x + iy$. Then, we have $D_2 u(z) = -2\Re(z + i)^{-3}$. Since

$$|u(z)| \lesssim (1 + |z|)^{-2}, \quad |D_2 u(z)| \lesssim (1 + |z|)^{-3},$$

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we see that both $u$ and $D^2 u$ belong to $b^p$ for any $p \geq 2$. Note that
\[
D^2 \tilde{u}(z) = -\int_1^\infty D^2 u(tz) \, dt
\]
\[
= 2\Re \int_1^\infty (tz + i)^{-3} \, dt
\]
\[
= \Re \left( \frac{1}{z} (z + i)^{-1} \right)
\]
\[
= \frac{x(x^2 - 3y^2 - 4y - 1)}{|z|^2 |z + i|^4}.
\]
Let $E$ be the set of all points $z \in \mathbb{H}$ such that $|z| \leq 1/10$. If $z \in E$, then we have
\[
|x^2 - 3y^2 - 2y - 1| \geq 1 - (x^2 + 3y^2 + 2y) \geq 1 - 5|z| \geq \frac{1}{2}
\]
so that
\[
|D^2 \tilde{u}(z)| \gtrsim \frac{|x|}{|z|^2}.
\]
Therefore, for any $p \geq 2$, we have
\[
\int_\mathbb{H} |D^2 \tilde{u}(z)|^p \, dxdy \geq \int_E |D^2 \tilde{u}(z)|^p \, dxdy
\]
\[
\gtrsim \int_E \frac{|x|^p}{|z|^{2p}} \, dxdy
\]
\[
= \int_0^\pi \int_0^{1/10} \left( \frac{\cos \theta}{r} \right)^p \, drd\theta
\]
\[
= \infty
\]
as desired. This completes the proof. $\square$

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