EXPOSED 2-HOMOGENEOUS POLYNOMIALS
ON HILBERT SPACES

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Abstract. We show that every extreme point of the unit ball of 2-homogeneous polynomials on a separable real Hilbert space is its exposed point and that the unit ball of 2-homogeneous polynomials on a non-separable real Hilbert space contains no exposed points. We also show that the unit ball of 2-homogeneous polynomials on any infinite dimensional real Hilbert space contains no strongly exposed points.

We recall that a unit vector \( x \) in a real Banach space \( E \) is exposed if there is a unit vector \( f \in E^* \) so that \( f(x) = 1 \) and \( f(y) < 1 \) for each \( y \in B_E \) with \( y \neq x \), where \( B_E \) is the closed unit ball of \( E \). It is easy to see that every exposed point of \( B_E \) is an extreme point. A unit vector \( x \) in a real Banach space \( E \) is strongly exposed if there is a unit vector \( f \in E^* \) so that \( f(x) = 1 \) and given any sequence \((x_k)\) in \( B_E \) with \( f(x_k) \to 1 \) we can conclude that \( x_k \to x \) in norm. It is easy to see that every strongly exposed point of \( B_E \) is its exposed point. We denote by \( \text{ext} B_E, \text{exp} B_E \) and \( \text{sexp} B_E \) the sets of extreme points, exposed points and strongly exposed points of \( B_E \), respectively.

Let \( \mathcal{P}^2(H) \) be the Banach space of continuous 2-homogeneous polynomials on a real Hilbert space \( H \). Recently many authors (see [1]-[7]) studied extremal problems for polynomials on a Banach space. The extreme points of the unit ball of this space has been characterized by Grecu [6]. The object of this note is to determine the exposed and strongly exposed points of the unit ball of \( \mathcal{P}^2(H) \). Grecu showed that for a real Hilbert space \( H \), \( P \in \text{ext} \mathcal{P}^2(H) \) if and only if there exists an orthogonal decomposition of \( H = H_1 \oplus H_2 \) such that \( P(x) = \|\pi_1(x)\|^2 - \|\pi_2(x)\|^2 \), where \( \pi_j : H \to H_j \) are the orthogonal projections of \( H \) onto \( H_j \) \( (j = 1, 2) \). Using this result we show the following results:

1. If \( H \) is a separable real Hilbert space, then every extreme point of the unit ball of \( \mathcal{P}^2(H) \) is exposed.

2. If \( H \) is a non-separable real Hilbert space, then the unit ball of \( \mathcal{P}^2(H) \) contains no exposed points.

3. If \( H \) is an infinite dimensional real Hilbert space, then the unit ball of \( \mathcal{P}^2(H) \) contains no strongly exposed points.

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Theorem 1. Let $H$ be a separable real Hilbert space. Then every extreme point of the unit ball of $P(2H)$ is exposed.

Proof. Since $\exp B_{P(2H)} \subset \text{ext} B_{P(2H)}$, it suffices to show that if $P \in \text{ext} B_{P(2H)}$, then $P \in \exp B_{P(2H)}$.

Let $P \in \text{ext} B_{P(2H)}$. By Theorem 1.6 of [6], $P(x) = \|\pi_1(x)\|^2 - \|\pi_2(x)\|^2$ where $H = H_1 \oplus H_2$ and $\pi_j : H \to H_j$ are the orthogonal projections of $H$ onto $H_j$ $(j = 1, 2)$. Clearly $\|\pi_j\| = 1$. Let $\{e_\alpha\}_{\alpha \in A}$ and $\{t_\beta\}_{\beta \in B}$ be orthonormal bases of $H_1$ and $H_2$, respectively. It is clear that $\{e_\alpha, t_\beta\}_{\alpha \in A, \beta \in B}$ is an orthonormal basis of $H$.

Then for each $x \in H$, we have

$$x = \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha + \sum_{\beta \in B} \langle x, t_\beta \rangle t_\beta$$

and

$$P(x) = \sum_{\alpha \in A} \langle x, e_\alpha \rangle^2 - \sum_{\beta \in B} \langle x, t_\beta \rangle^2.$$ 

Note that $P(e_\alpha) = 1$ for all $\alpha \in A$ and $P(t_\beta) = -1$ for all $\beta \in B$. Let $\{a_\alpha\}_{\alpha \in A}$ and $\{b_\beta\}_{\beta \in B}$ be collections of reals such that $a_\alpha > 0, b_\beta < 0$ and

$$\sum_{\alpha \in A} a_\alpha - \sum_{\beta \in B} b_\beta = 1.$$ 

Define $f \in P(2H)^*$ such that for each $Q \in P(2H)$,

$$f(Q) = \sum_{\alpha \in A} Q(e_\alpha)a_\alpha + \sum_{\beta \in B} Q(t_\beta)b_\beta.$$ 

Then $\|f\| = 1$. Indeed, for each $Q \in P(2H)$ with $\|Q\| = 1$, we have $|Q(e_\alpha)| \leq 1$ for all $\alpha \in A, |Q(t_\beta)| \leq 1$ for all $\beta \in B$ and

$$|f(Q)| \leq \sum_{\alpha \in A} |Q(e_\alpha)|a_\alpha + \sum_{\beta \in B} |Q(t_\beta)|(-b_\beta) \leq \sum_{\alpha \in A} 1 \cdot a_\alpha - \sum_{1 \cdot \beta \in B} b_\beta = 1$$

and

$$f(P) = \sum_{\alpha \in A} P(e_\alpha)a_\alpha + \sum_{\beta \in B} P(t_\beta)b_\beta = \sum_{\alpha \in A} 1 \cdot a_\alpha - \sum_{\beta \in B} 1 \cdot b_\beta = 1.$$ 

We will show that this functional $f$ exposes the polynomial $P$.

Let $Q \in P(2H)$ be such that $f(Q) = \|Q\|$. We claim that $Q(e_\alpha) = 1$ for all $\alpha \in A$ and $Q(t_\beta) = -1$ for all $\beta \in B$. Indeed,

$$1 = f(Q) = \sum_{\alpha \in A} Q(e_\alpha)a_\alpha + \sum_{\beta \in B} Q(t_\beta)b_\beta = \sum_{\alpha \in A} 1 \cdot a_\alpha - \sum_{\beta \in B} 1 \cdot b_\beta = 1,$$

so $Q(e_\alpha) = 1$ for all $\alpha \in A$ and $Q(t_\beta) = -1$ for all $\beta \in B$ because of $a_\alpha > 0, b_\beta < 0$.

Let $Q$ be the corresponding continuous symmetric bilinear form to the polynomial $Q$.

We claim:

1. $Q(e_\alpha, e_{\alpha'}) = 0$ $(\alpha \neq \alpha' \in A)$;
2. $Q(t_\beta, t_{\beta'}) = 0$ $(\beta \neq \beta' \in B)$;
3. $Q(e_\alpha, t_\beta) = 0$ $(\alpha \in A, \beta \in B)$.
Proof of (1). It follows that
\[ 1 = \|Q\| \geq \sup_{x^2 + x_0^2 = 1} |Q(x_\alpha e_\alpha + x_{\alpha'} e_{\alpha'})| \]
\[ = \sup_{x^2 + x_0^2 = 1} |\hat{Q}(x_\alpha e_\alpha + x_{\alpha'} e_{\alpha''), x_\alpha e_\alpha + x_{\alpha'} e_{\alpha''})| \]
\[ = \sup_{x^2 + x_0^2 = 1} |Q(e_\alpha) x^2_\alpha + Q(e_{\alpha'}) x^2_{\alpha'} + 2\hat{Q}(e_\alpha, e_{\alpha'}) x_\alpha x_{\alpha'}| \]
\[ = \sup_{x^2 + x_0^2 = 1} |x^2_\alpha + x^2_{\alpha'} + 2\hat{Q}(e_\alpha, e_{\alpha'}) x_\alpha x_{\alpha'}| \]
\[ = \sup_{x^2 + x_0^2 = 1} |1 + 2\hat{Q}(e_\alpha, e_{\alpha'}) x_\alpha x_{\alpha'}| \]
which implies \( \hat{Q}(e_\alpha, e_{\alpha'}) = 0 \).

Proof of (2). By the similar proof of (1), we have
\[ 1 = \|Q\| \geq \sup_{x^2 + x_0^2 = 1} |Q(x_\beta t_\beta + x_{\beta'} t_{\beta'})| = \sup_{x^2 + x_0^2 = 1} |1 + 2\hat{Q}(t_\beta, t_{\beta'}) x_\beta x_{\beta'}| \]
which implies \( \hat{Q}(t_\beta, t_{\beta'}) = 0 \).

Proof of (3). By the similar proof of (1), we have
\[ 1 = \|Q\| \geq \sup_{x^2 + x_0^2 = 1} |Q(x_\alpha e_\alpha + x_\beta t_\beta)| \]
\[ = \sup_{x^2 + x_0^2 = 1} |Q(e_\alpha) x^2_\alpha + Q(t_\beta) x^2_\beta + 2\hat{Q}(e_\alpha, t_\beta) x_\alpha x_\beta| \]
\[ = \sup_{x^2 + x_0^2 = 1} |x^2_\alpha - x^2_\beta + 2\hat{Q}(e_\alpha, t_\beta) x_\alpha x_\beta| .\]

By Lemma 2.1 of [3], we have \( \hat{Q}(e_\alpha, t_\beta) = 0 \). For \( x \in H \), it follows that
\[ Q(x) = \hat{Q}(x, x) \]
\[ = \hat{Q}\left(\sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha + \sum_{\beta \in B} \langle x, t_\beta \rangle t_\beta, \sum_{\alpha \in A} \langle x, e_\alpha \rangle e_\alpha + \sum_{\beta \in B} \langle x, t_\beta \rangle t_\beta\right) \]
\[ = \sum_{\alpha, \alpha' \in A} \langle x, e_\alpha \rangle \langle x, e_{\alpha'} \rangle \hat{Q}(e_\alpha, e_{\alpha'}) + \sum_{\beta, \beta' \in B} \langle x, t_\beta \rangle \langle x, t_{\beta'} \rangle \hat{Q}(t_\beta, t_{\beta'}) \]
\[ + 2 \sum_{\alpha \in A, \beta \in B} \langle x, e_\alpha \rangle \langle x, t_\beta \rangle \hat{Q}(e_\alpha, t_\beta) \]
\[ = \sum_{\alpha \in A} Q(e_\alpha) \langle x, e_\alpha \rangle^2 + \sum_{\beta \in B} Q(t_\beta) \langle x, t_\beta \rangle^2 \quad \text{(by claims (1)-(3))} \]
\[ = \sum_{\alpha \in A} \langle x, e_\alpha \rangle^2 - \sum_{\beta \in B} \langle x, t_\beta \rangle^2 = P(x), \]
showing that \( f \) exposes \( P \). Therefore \( P \in \exp B_{\mathcal{P}(2^H)} \).

Theorem 2. Let \( H \) be an infinite dimensional real Hilbert space. Then the unit ball of \( \mathcal{P}(2^H) \) contains no strongly exposed points.

Proof. Since \( \sexp B_{\mathcal{P}(2^H)} \subset \ext B_{\mathcal{P}(2^H)} \), it suffices to show that
\[ \sexp B_{\mathcal{P}(2^H)} \cap \ext B_{\mathcal{P}(2^H)} = \emptyset. \]
Let $P \in \operatorname{ext}B_{\mathcal{P}(2H)}$. By Theorem 1.6 of [6] $P(x) = \|\pi_1(x)\|^2 - \|\pi_2(x)\|^2$ where $H = H_1 \oplus H_2$ and $\pi_j : H \to H_j$ are the orthogonal projections of $H$ onto $H_j$ \((j = 1, 2)\). Clearly $\|\pi_j\| = 1$. Without loss of generality, assume that $\dim(H_1) = \infty$.

Let \(\{e\alpha\}_{\alpha \in A}\) be an orthonormal basis of $H_1$. Let \(\{\alpha_j\}_{j = 1}^\infty \subset A\). It is clear that for each $x \in H$, we have $\pi_1(x) = \sum_{\alpha \in A} \langle \pi_1(x), e\alpha \rangle e\alpha$. Then

$$P(x) = \sum_{\alpha \in A} \langle \pi_1(x), e\alpha \rangle^2 - \|\pi_2(x)\|^2.$$  

Suppose that $P \in \text{sexp}B_{\mathcal{P}(2H)}$. Then there is an $f \in \mathcal{P}(2H)^*$ such that $\|f\| = 1 = f(P)$ and given any sequence \(\{P_j\}\) in $B_{\mathcal{P}(2H)}$ with $f(P_j) \to 1$, we have $\|P_j - P\| \to 0$.

For each $\alpha_j$, we have

$$f\left(\sum_{\alpha \neq \alpha_j} \langle \pi_1(x), e\alpha \rangle^2 - \langle \pi_1(x), e\alpha_j \rangle^2 - \|\pi_2(x)\|^2\right) < 1$$

so $f(\langle \pi_1(x), e\alpha_j \rangle^2) > 0$. We will show $f(\langle \pi_1(x), e\alpha_j \rangle^2) \to 0$ as $j \to \infty$. For each $n$,

$$\sum_{1 \leq j \leq n} f(\langle \pi_1(x), e\alpha_j \rangle^2) = f\left(\sum_{1 \leq j \leq n} \langle \pi_1(x), e\alpha_j \rangle^2\right)$$

$$\leq \|f\| \sum_{1 \leq j \leq n} \|\langle \pi_1(x), e\alpha_j \rangle\|^2 = \|f\| \sum_{1 \leq j \leq n} \|\langle \pi_1(x), e\alpha_j \rangle\|^2$$

$$= \sup_{\|x\| = 1} \sum_{1 \leq j \leq n} \|\langle \pi_1(x), e\alpha_j \rangle\|^2 = \sup_{\|x\| = 1} \|\pi_1(x)\|^2 = \|\pi_1\|^2 = 1.$$  

Thus $\sum_{1 \leq j < \infty} f(\langle \pi_1(x), e\alpha_j \rangle^2) \leq 1$, so we have $f(\langle \pi_1(x), e\alpha_j \rangle^2) \to 0$.

Define $P_j(x) = P(x) - \langle \pi_1(x), e\alpha_j \rangle^2 \in \mathcal{P}(2H)$. Then $\|P_j\| = 1$ and $|f(P_j) - 1| = |f(P_j - P)| = f(\langle \pi_1(x), e\alpha_j \rangle^2) \to 0$. But

$$\|P_j - P\| = \|\langle \pi_1(x), e\alpha_j \rangle^2\| = \sup_{\|x\| = 1} \|\pi_1(x)\|^2 = \|\pi_1\|^2 = 1,$$  

so we have a contradiction. Thus $\text{sexp}B_{\mathcal{P}(2H)} = \emptyset$.

**Theorem 3.** Let $H$ be non-separable real Hilbert space. Then the unit ball of $\mathcal{P}(2H)$ contains no exposed points.

**Proof.** Let $P$ be an extreme point, so that

$$P(x) = \sum_{\alpha \in A} \langle x, t\alpha \rangle^2 - \sum_{\beta \in B} \langle x, t\beta \rangle^2$$  

relative to a suitably chosen orthonormal basis whose indexing set is the disjoint union $A \cup B$. Suppose that the functional $f$ exposes $P$. As in the proof of Theorem 2 it follows that $f(\langle x, e\alpha \rangle^2) > 0$ for every $\alpha \in A$ and similarly, $f(\langle x, e\alpha \rangle^2) < 0$ for every $\alpha \in B$. But

$$\sum_{\alpha \in A} f(\langle x, t\alpha \rangle^2) = f\left(\sum_{\alpha \in A} \langle x, t\alpha \rangle^2\right) \leq 1,$$  

and hence $A$ must be countable. Similarly, $B$ is countable. Thus $H$ is separable.
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