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THE BANACH ENVELOPE OF PALEY-WIENER TYPE SPACES

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Abstract. We give an explicit computation of the Banach envelope for the Paley-Wiener type spaces E^p, $0 < p < 1$. This answers a question by Joel Shapiro.

1. Introduction

The Paley-Wiener type space E^p (a precise definition is given below) consists of certain band-limited functions [1]. We will show that for $0 < p < 1$ this space can be identified as a complemented subspace of the direct sum of two classical Hardy spaces. Since the Banach envelopes of the Hardy spaces are known, we are able to establish a necessary and sufficient condition for entire functions to belong to the envelope E^p_c.

For an open subset $\Omega \subseteq \mathbb{C}$ let $\mathbb{A}(\Omega)$ be the space of holomorphic functions on Ω. An entire function f is of exponential type $\tau > 0$ if for all $\epsilon > 0$ there is a $C_\epsilon > 0$ such that for all $z \in \mathbb{C}$ we have $|f(z)| \leq C_\epsilon e^{(\tau+\epsilon)|z|}$. For $0 < p < \infty$ let $E^{p,\tau}$ be the space of entire functions of exponential type τ such that their restrictions to the real axis are in $L^p(\mathbb{R})$:

$$E^{p,\tau} = \{f \in \mathbb{A}(\mathbb{C}) : f \text{ has exponential type } \tau\ , f|_{\mathbb{R}} \in L^p(\mathbb{R})\}\,.$$

We will only consider $\tau = \pi$ and write $E^p = E^{p,\pi}$ from now on. This causes no loss of generality, because we can simply rescale a function $f \in E^{p,\tau}$ to obtain $\operatorname{supp}(f|_{\mathbb{R}})^\wedge \subseteq [-\pi, \pi]$. The quantity

$$\|f\|_{E^p} = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{1/p} = \|f|_{\mathbb{R}}\|_{L^p}$$

defines a norm on E^p for $1 \leq p < \infty$ and a quasi-norm for $0 < p < 1$.

These spaces E^p are complete and hence can be identified with closed subspaces of $L^p(\mathbb{R})$ (e.g. see [6] or [16]). A classical theorem of Paley and Wiener gives a characterization of E^2 as the image of the inverse Fourier transform of $L^2[-\pi, \pi]$. Hence functions in E^2 have compactly supported Fourier transform, i.e. they are band-limited. These functions are important to signal processing due to their sampling properties (Shannon sampling theorem). Since $\|f\|_q \leq C_{pq}\|f\|_p$ for $f \in E^p, 0 < q \leq p$ [16], we have $E^p \subset E^q$ for $0 < p \leq q$. In particular,

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$E^p \subset E^1 \subset E^2$ for $0 < p < 1$. A characterization of E^p for $0 < p < 1$ as a discrete Hardy space is shown in [6].

Let X be a quasi-normed space with separating dual . Then the Banach envelope X_c of X is the completion of $(X, \|\cdot\|_C)$ where C is the convex hull of the closed unit ball B_X and $\|\cdot\|_C$ is the Minkowski functional of C. X_c is a Banach space. The Banach envelope is characterized up to isomorphism by $(X_c)^* = X$ and $\overline{X} = X_c$. Every operator $T : X \to Y$ extends uniquely to $\tilde{T} : X_c \to Y_c$ ([8], [11], [17] and [20]).

The standard example for finding a Banach envelope is ℓ^p for $0 < p < 1$. In this case we have $\ell^p \subset \ell^1$, ℓ^p is dense in ℓ^1 and $(\ell^p)^* = \ell^\infty = (\ell^1)^*$. Therefore, $l^p_c = l^1$. The spaces E^p are nested as well, $E^p \subset E^1$ for $0 < p < 1$. Furthermore, E^p is dense in E^1 since it contains all Schwartz functions with Fourier transform supported in $[-\pi, \pi]$. This makes E^1 a candidate for the envelope of E^p, but it turns out that E^p_c is a certain weighted Bergman space of entire functions different from E^1. The proof relies heavily on the theory of Hardy spaces H^p. We use a deep result by Duren, Romberg and Shields [5] that the Banach envelope of H^p over the unit disk is a certain weighted L^1-Bergman space (see below).

The problem of identifying the Banach envelope of E^p as a space of entire functions was originally posed by Joel Shapiro. I would like to thank Professor Nigel Kalton for communicating this problem and his helpful suggestions. I would also like to thank the referee for some very constructive comments.

2. Preliminaries

We recall the classical Hardy spaces of the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and the upper half planes $\mathbb{C}_\pm = \{z = x + iy \in \mathbb{C} : y \in \mathbb{R}_\pm\}$,

$$H^p(\mathbb{D}) = \left\{ f \in \mathbb{A}(\mathbb{D}) : \|f\|^p_{H^p(\mathbb{D})} = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty \right\},$$

$$H^p(\mathbb{C}_+) = \left\{ f \in \mathbb{A}(\mathbb{C}_+) : \|f\|^p_{H^p(\mathbb{C}_+)} = \sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^p dx < \infty \right\}.$$

Analogously we define for the lower half plane

$$H^p(\mathbb{C}_-) = \left\{ f \in \mathbb{A}(\mathbb{C}_-) : \|f\|^p_{H^p(\mathbb{C}_-)} = \sup_{y<0} \int_{-\infty}^{\infty} |f(x+iy)|^p dx < \infty \right\}.$$

An isometric isomorphism between $H^p(\mathbb{C}_+)$ and $H^p(\mathbb{C}_-)$ is given by $f(z) \mapsto \overline{f(\bar{z})}$. Let $\mathcal{S}$ be the space of Schwartz functions and $\mathcal{S}'$ the space of tempered distributions. Every $f \in H^p(\mathbb{C}_+)$ is uniquely determined by its boundary value distribution $f^b = \lim_{y \to 0} f(x+iy) \in \mathcal{S}'$. Denote the space of these boundary distributions by $H^p_+(\mathbb{R})$, and let $\|f^b\|_{H^p_+(\mathbb{R})} = \|f\|_{H^p(\mathbb{C}_+)}$. In the same way define $H^p_-(\mathbb{R})$. The real and imaginary parts of f have a boundary value distribution in the real Hardy space as defined in [2], [18]. Hence the Fourier transform of $f \in H^p_+(\mathbb{R}), 0 < p < 1$, is a continuous functions and satisfies $\operatorname{supp} \widehat{f} \subseteq [0, \infty)$. More precisely, we have $|\widehat{f}(\xi)| \leq C|\xi|^{1/p-1}\|f\|_{H^p_+(\mathbb{R})}$. Transferring to the lower half plane shows that $f \in H^p_-(\mathbb{R})$ has $\operatorname{supp} \hat{f} \subseteq (-\infty, 0]$. All these results can be found e.g. in [2], [7], [10], [18].

The Bergman spaces over the disc and the upper half plane are defined for $0 < p < \infty$, $\alpha > -1$ as

$$A^{p,\alpha}(\mathbb{D}) = \left\{ f \in \mathbb{A}(\mathbb{D}) : \|f\|_{p,\alpha}^p = \int_{\mathbb{D}} |f(x+iy)|^p (1-|z|^2)^\alpha dx\,dy < \infty \right\},$$

$$A^{p,\alpha}(\mathbb{C}_+) = \left\{ f \in \mathbb{A}(\mathbb{C}_+) : \|f\|_{p,\alpha}^p = \int_{\mathbb{C}_+} |f(x+iy)|^p y^\alpha dx\,dy < \infty \right\}.$$

Analogously we define $A^{p,\alpha}(\mathbb{C}_-)$. The Banach envelope of $H^p(\mathbb{D})$ was identified by Duren, Romberg and Shields [5].

Proposition 2.1. $H_c^p(\mathbb{D}) = A^{1,1/p-2}(\mathbb{D})$.

For a different approach in the setting of Besov and Triebel-Lizorkin spaces see [11] and [12]. In particular, it is shown that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and $1/2 < p < 1$ we have

Proposition 2.2. $H_c^p(\Omega) = A^{1,1/p-2}(\Omega)$.

We will need the analogous statement for the upper and lower half plane. The following proposition is certainly well known; for completeness we give a proof using a conformal map from $\mathbb{C}_+$ onto $\mathbb{D}$.

Proposition 2.3. $H_c^p(\mathbb{C}_\pm) = A^{1,1/p-2}(\mathbb{C}_\pm)$.

Proof. It is enough to consider the upper half plane. We use the conformal map $w = \phi(z) = \frac{i-z}{i+z}$ from $\mathbb{C}_+$ onto $\mathbb{D}$. Then with $F(z) = f(\phi(z)), z = x + iy$, we have $f(w) \in H^p(\mathbb{D})$ if and only if $F(z)/(z+i)^{2/p} \in H^p(\mathbb{C}_+)$ (see [4], [10]). For $\alpha > -1$ we get

$$\int_{\mathbb{D}} |f(w)|^p (1-|w|^2)^\alpha dx\,dy = \int_{\mathbb{C}_+} |F(z)|^p (1-|\phi(z)|^2)^\alpha |\phi'(z)|^2 dx\,dy.$$

A short computation shows $1 - |\phi(z)|^2 = \frac{4y}{|z+i|^2}$ and $|\phi'(z)|^2 = \frac{4}{|z+i|^4}$. Therefore,

$$\int_{\mathbb{D}} |f(w)|^p (1-|w|^2)^\alpha dx\,dy = 4^{\alpha+1} \int_{\mathbb{C}_+} \left| \frac{F(z)}{(z+i)^{(2\alpha+4)/p}} \right|^p y^\alpha dx\,dy.$$

Hence $F(z) \in H_c^p(\mathbb{C}_+)$ if and only if $f(w)(\phi^{-1}(w)+i)^{2/p} \in A^{1,\alpha}(\mathbb{D})$ where $\alpha = 1/p - 2$. Then $2\alpha + 4 = 2/p$, and mapping back to the upper half plane shows $F(z) \in H_c^p(\mathbb{C}_+)$ if and only if $F(z) \in A^{1,\alpha}(\mathbb{C}_+)$. □

3. The Banach envelope of E^p

The next proposition is due to Plancherel and Pólya [14].

Proposition 3.1. *Let $0 < p < \infty$ and $f \in E^p$. Then for every $y \in \mathbb{R}$ we have*

$$\int_{-\infty}^{\infty} |f(x+iy)|^p dx \le e^{p\pi|y|} \int_{-\infty}^{\infty} |f(x)|^p dx.$$

For an entire function f let $f_{\pm\pi}(z) = e^{\pm i\pi z} f|_{\mathbb{C}_\pm}(z)$ and $j(f) = (f_\pi, f_{-\pi})$. Then from Proposition 3.1 it follows that $f \mapsto f_{\pm\pi}$ is an isometric isomorphism of E^p into $H^p(\mathbb{C}_\pm)$. Hence j embeds E^p into $H^p(\mathbb{C}_+) \oplus H^p(\mathbb{C}_-)$.

Crucial to compute the envelope of E^p is the following.

Lemma 3.2. *$j(E^p)$ is complemented in $H^p(\mathbb{C}_+) \oplus H^p(\mathbb{C}_-)$.*

Proof. We construct a bounded projection onto $j(E^p)$. Choose $\phi, \psi \in \mathcal{S}$ such that $\operatorname{supp}\hat{\phi} \subset [-2\pi, \pi], \operatorname{supp}\hat{\psi} \subset [-\pi, 2\pi]$ and $\hat{\phi}(x) + \hat{\psi}(x) = 1$ for all $x \in [-\pi, \pi]$. This can be done by a suitable partition of unity on the Fourier transform side. Then let $T : \mathcal{S}' \times \mathcal{S}' \mapsto \mathcal{S}'$ be defined by

$$T(u, v) = u * \phi + v * \psi.$$

We have $\operatorname{supp} T(u, v)^\wedge \subseteq [-2\pi, 2\pi]$, and if u has $\operatorname{supp}\hat{u} \subseteq [-\pi, \pi]$, then $T(u, u) = u$. Write $u_{\pm\pi} = e^{\pm i\pi x}u$ for $u \in \mathcal{S}'$. Suppose $(u, v) \in H^p_+(\mathbb{R}) \oplus H^p_-(\mathbb{R})$. Then $T(u_{-\pi}, v_\pi)^\wedge$ is continuous and supported in $[-\pi, \pi]$. This shows that $T(u_{-\pi}, v_\pi)$ has an extension to a function in E^2. Non-tangential (distributional) boundary values of functions in $H^p(\mathbb{C}_+)$ are in $L^p(\mathbb{R})$, and we have $\|u * \Phi\|_{H^p_+(\mathbb{R})} \leq C\|u\|_{H^p_+(\mathbb{R})}$ for $u \in H^p_+(\mathbb{R}), \Phi \in \mathcal{S}'$ [18]. Hence $T(u_{-\pi}, v_\pi)|_\mathbb{R} \in L^p(\mathbb{R})$, and $T(u_{-\pi}, v_\pi)$ extends to a function in E^p. This extension has the explicit form $Q : H^p(\mathbb{C}_+) \oplus H^p(\mathbb{C}_-) \mapsto E^p$,

$$Q(f, g)(z) = \left\langle T(f^b_{-\pi}, g^b_\pi), e^{itz} \right\rangle = \int_{-\pi}^{\pi} T(f^b_{-\pi}, g^b_\pi)e^{itz}dt.$$

By choice of ϕ, ψ we have $Q(j(f)) = f$, and hence $P = jQ$ is the desired projection. $\square$

Now we arrive at our characterization of $E^p_{c,q}$.

Theorem 3.3. *An entire function f belongs to E^p_c if and only if*

$$\|f\| = \int_\mathbb{C} e^{-\pi|y|}|y|^{1/p-2}|f(x + iy)|dx\, dy < \infty.$$

Moreover, $\|\cdot\|$ is equivalent to the norm of E^p_c.

Proof. Let $\alpha = 1/p - 2$. Define

$$Z = \{f \in \mathbb{A}(\mathbb{C}) : f_\pi \in A^{1,\alpha}(\mathbb{C}_+), f_{-\pi} \in A^{1,\alpha}(\mathbb{C}_-)\} \subset A^{1,\alpha}(\mathbb{C}_+) \oplus A^{1,\alpha}(\mathbb{C}_-).$$

We will show $E^p_c = Z$ with equivalence of norms. We have $H^p(\mathbb{C}_\pm) \subset A^{1,\alpha}(\mathbb{C}_\pm)$, and hence $E^p \subset Z$. It is crucial to observe that the operator Q from the previous proof extends to $\tilde{Q} : A^{1,\alpha}(\mathbb{C}_+) \oplus A^{1,\alpha}(\mathbb{C}_-) \mapsto E^p_c$ while preserving the defining equation, i.e. $\tilde{Q}(f, g)(z) = \langle T(f^b_{-\pi}, g^b_\pi), e^{itz}\rangle$. This follows from the characterization of boundary value distributions for functions in $A^{1,\alpha}(\mathbb{C}_+)$ [15]. These distributions are uniquely determined by their values on $\mathcal{S}$, and passing from $f \in A^{q,\alpha}(\mathbb{C}_+)$ to $f^b \in A^{1,\alpha}_+(\mathbb{R})$ is a continuous operation. Therefore, we have $E^p_c \subset E^2$ and $\tilde{Q}(j(f)) = f$ for $f \in Z$. We conclude that $j(E^p_c)$ is a complemented subspace of $H^p_c(\mathbb{C}_+) \oplus H^p_c(\mathbb{C}_-) = A^{1,\alpha}(\mathbb{C}_+) \oplus A^{1,\alpha}(\mathbb{C}_-)$. Hence $E^p_c \subseteq Z$, and on $j(E^p_c)$ the envelope-norm is equivalent to the norm of $A^{1,\alpha}(\mathbb{C}_+) \oplus A^{1,\alpha}(\mathbb{C}_-)$. It remains to establish density of $j(E^p)$ in $j(Z)$. Pick $f \in Z$. Then there are $(h_n, g_n) \in H^p(\mathbb{C}_+) \oplus H^p(\mathbb{C}_-)$ such that $(h_n, g_n) \to (f_\pi, f_{-\pi}) = j(f) \in A^{1,\alpha}(\mathbb{C}_+) \oplus A^{1,\alpha}(\mathbb{C}_-)$. Then $Q(h_n, g_n) \in E^p$ and $Q(h_n, g_n) \to \tilde{Q}(j(f)) = f$. $\square$

Let us see that $E^p_c \neq E^1$. For $f \in E^1, y > 0$ let $f_y(x) = f(x + iy)$. Then by Proposition 3.1 we have $\|f_y\|_{L^1} \leq e^{\pi y}\|f_0\|_{L^1} = \|f\|_{E^1}$. For a given $y > 0$ we choose a function that satisfies the converse inequality up to a constant as follows: Take $\phi_\epsilon \in \mathcal{S}$ such that $\operatorname{supp}\phi_\epsilon \subseteq [-\pi, -\pi + \epsilon]$, and let $f^\epsilon(z) = \left\langle \phi_\epsilon, e^{itz} \right\rangle$. Then for fixed

$y_0 > 0$ we choose $\epsilon > 0$ small enough to give $\|f^\epsilon_y\|_{L^1} \geq Ce^{\pi y}\|f^\epsilon\|_{E^1}$ for all $0 \leq y \leq y_0$. We obtain $\|f^\epsilon\|_{E^p_c} = \int_{-\infty}^{\infty} e^{-\pi|y|}|y|^{1/p-2}\|f^\epsilon_y\|_{L^1}dy \geq C\int_{-\infty}^{y_0} y^{1/p-2}\|f^\epsilon\|_{E^1}dy = (1/p-1)^{-1}y_0^{1/p-1}\|f^\epsilon\|$. Since $1/p-1>0$ we conclude that the E^1-norm and the E^p_c-norm are not equivalent.

A result from [9] (see also [20]) is $A^{1,\alpha}(\mathbb{D}) \approx \ell^1$, and therefore we use a result of Pełczyński [13] that every complemented subspace of ℓ^1 is isomorphic to ℓ^1:

Corollary 3.4. $E^p_c \approx \ell^1$.

4. The q-envelope of E^p

The result from the previous section can be generalized to the q-envelopes. Let X be a quasi-normed space with separating dual and $0 < q \leq 1$. Then the Banach q-envelope $X_{c,q}$ of X is the completion of $(X, \|\cdot\|_{C_q})$ where C_q is the q-convex hull of the closed unit ball B_X and $\|\cdot\|_{C_q}$ is the Minkowski functional of C_q. $X_{c,q}$ is a complete quasi-normed space.

The results in [3] and [20] give the q-envelopes of the Hardy spaces.

Proposition 4.1. *For $0 < p < q \leq 1$ we have $H^p_{c,q}(\mathbb{D}) = A^{q,q/p-2}(\mathbb{D})$.*

The proofs of Theorem 3.3 and Proposition 2.3 work analogously for the q-envelope, $0 < q < 1$, if we use $\alpha = q/p-2$ and $A^{q,\alpha}(\mathbb{C}_\pm)$. Hence

Theorem 4.2. *An entire function f belongs to $E^p_{c,q}$ if and only if*

$$\|f\| = \left(\int_{\mathbb{C}} e^{-q\pi|y|}|y|^{q/p-2}|f(x+iy)|^q dx\,dy\right)^{1/q} < \infty.$$

Moreover, $\|\cdot\|$ is equivalent to the quasi-norm of $E^p_{c,q}$.

In [9] (see also [20]) it is shown that $A^{q,\alpha}(\mathbb{D}) \approx \ell^q$, and therefore we use a result of Stiles [19] or [8] for $q < 1$ that every complemented subspace of ℓ^q is isomorphic to ℓ^q:

Corollary 4.3. $E^p_{c,q} \approx \ell^q$.

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