

## PERFECT CLIQUES AND $G_\delta$ COLORINGS OF POLISH SPACES

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(Communicated by Carl G. Jockusch, Jr.)

ABSTRACT. A *coloring* of a set  $X$  is any subset  $C$  of  $[X]^N$ , where  $N > 1$  is a natural number. We give some sufficient conditions for the existence of a perfect  $C$ -homogeneous set, in the case where  $C$  is  $G_\delta$  and  $X$  is a Polish space. In particular, we show that it is sufficient that there exist  $C$ -homogeneous sets of arbitrarily large countable Cantor-Bendixson rank. We apply our methods to show that an analytic subset of the plane contains a perfect 3-clique if it contains any uncountable  $k$ -clique, where  $k$  is a natural number or  $\aleph_0$  (a set  $K$  is a  $k$ -clique in  $X$  if the convex hull of any of its  $k$ -element subsets is not contained in  $X$ ).

### 1. INTRODUCTION

For a set  $X$  and natural number  $N$ ,  $[X]^N$  denotes the collection of all  $N$ -element subsets of  $X$ . A (two-color) *coloring* of  $X$  is (represented by) a set  $C \subset [X]^N$ . We identify  $[X]^N$  with a suitable subspace of the product  $X^N$ . We are interested in the following problem: find sufficient conditions for the existence of a perfect  $C$ -homogeneous set  $P \subset X$ , where  $X$  is a Polish space and  $C \subset [X]^N$  is open (or more generally  $G_\delta$ ). A natural example of this problem is the following: let  $X \subset \mathbb{R}^N$  be closed and  $C = \{s \in [X]^k : \text{conv } s \not\subset X\}$ . Then  $C$  is open and a  $C$ -homogeneous set is called a  $k$ -clique in  $X$ . It is known (see [3]) that there exists a closed set  $X \subset \mathbb{R}^N$  such that  $X$  is not a countable union of convex sets but every  $k$ -clique in  $X$  is countable for every  $k < \omega$ . On the other hand, it is proved in [3] that if a closed set  $X \subset \mathbb{R}^N$  contains an uncountable  $k$ -clique for some  $k$ , then it contains a perfect 3-clique.

We prove that if  $C$  is a  $G_\delta$  coloring of a Polish space and there are no perfect  $C$ -homogeneous sets, then there is a countable ordinal  $\gamma$  such that the Cantor-Bendixson rank of every  $C$ -homogeneous set is  $< \gamma$ . In the context of cliques, this strengthens the result of Kojman [2] (see Theorem 3.1(a) below). From our result it follows that if  $C$  is a  $G_\delta$  coloring of an analytic space, then either there exists a perfect  $C$ -homogeneous set or all  $C$ -homogeneous sets are countable. This is not true for  $F_\sigma$  colorings: a result of Shelah [4] states that consistently there exist  $F_\sigma$  2-colorings with uncountable but not perfect homogeneous sets. Concerning cliques, we investigate analytic subsets of the plane. We prove that if an analytic set  $X \subset \mathbb{R}^2$  contains an uncountable  $\aleph_0$ -clique, then  $X$  contains also a perfect 3-clique.

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Received by the editors August 20, 2001 and, in revised form, October 1, 2001.

2000 *Mathematics Subject Classification*. Primary 52A37, 54H05; Secondary 03E02, 52A10.

*Key words and phrases*. Open ( $G_\delta$ ) coloring, perfect homogeneous set, clique.

1.1. **Notation.** Any subset of  $[X]^N$  is called a *coloring* (or an  $N$ -coloring) of  $X$ . We write  $\neg C$  instead of  $[X]^N \setminus C$ . A set  $S \subset X$  is  $C$ -homogeneous if  $[A]^N \subset C$ . We identify  $[X]^N$  with the subspace of  $X^N$  consisting of all  $N$ -tuples  $(x_0, \dots, x_{N-1})$  with  $x_i \neq x_j$  for  $i \neq j$ . Thus we may consider topological properties of colorings. If  $f: X \rightarrow Y$  is a function, then we write  $f[S]$  for the image of a set  $S \subset X$  and  $f(s)$  for the value at a point  $s \in X$ . By a *perfect set* we mean a compact, nonempty, topological space with no isolated points.

## 2. ON COLORINGS

First we recall a simple result on open 2-colorings of analytic spaces which can be found in Todorćević-Farah's book [5, p. 81].

**Proposition 2.1.** *Let  $X$  be an analytic space and let  $C \subset [X]^2$  be open. Then either there exists a perfect  $C$ -homogeneous set or else  $X$  is a countable union of  $\neg C$ -homogeneous sets, i.e.  $X = \bigcup_{n \in \omega} A_n$  where  $[A_n]^2 \cap C = \emptyset$  for every  $n \in \omega$ .*

The above result is no longer valid when we replace the word "open" with "closed"; see [5, p. 83]. Also, the above proposition cannot be strengthened for colorings of triples: there exists a clopen 3-coloring of  $2^\omega$  such that there are no uncountable homogeneous sets either of this color or of its complement; see Blass' example [1]. In Blass' example, the Cantor-Bendixson rank of any homogeneous set is at most 1. Below we show that in this situation there always exists a countable ordinal which bounds the Cantor-Bendixson ranks of all homogeneous sets. In fact this is true for  $G_\delta$  colorings.

For a topological space  $Y$  and an ordinal  $\alpha$  we denote by  $Y^{(\alpha)}$  the  $\alpha$ -derivative of  $Y$ ; the *Cantor-Bendixson rank* of  $Y$  is the minimal ordinal  $\gamma$  such that  $Y^{(\gamma+1)}$  is empty.

**Theorem 2.2.** *Let  $C$  be a  $G_\delta$   $N$ -coloring of a Polish space  $X$ . If for every countable ordinal  $\gamma$  there exists a  $C$ -homogeneous set of the Cantor-Bendixson rank  $\geq \gamma$ , then  $X$  contains a perfect  $C$ -homogeneous set.*

*Proof.* Fix a countable base  $\mathcal{B}$  in  $X$  and fix a complete metric on  $X$ . Let  $C = \bigcap_{n \in \omega} C_n$ , where each  $C_n$  is open and  $C_{n+1} \subset C_n$ . We will construct a tree of open sets

$$T = \{u_s : s \in 2^{<\omega}\}$$

with the following properties:

- (i)  $\text{cl } u_{s \smallfrown i} \subset u_s$ ,  $\text{cl } u_s \cap \text{cl } u_t = \emptyset$  if  $s, t$  are incompatible and  $\text{diam}(u_s) < 2^{-\text{length}(s)}$ ;
- (ii) if  $k < \omega$  and  $s_0, \dots, s_{N-1} \in 2^k$  are pairwise distinct, then  $\{x_0, \dots, x_{N-1}\} \in C_k$  whenever  $x_i \in u_{s_i}$ ,  $i < N$ ;
- (iii) if  $k < \omega$ , then for each  $\gamma < \omega_1$  there exists a  $C$ -homogeneous set  $P = P_{k,\gamma}$  such that  $P^{(\gamma)} \cap u_s \neq \emptyset$  for each  $s \in 2^k$ .

We start with  $u_\emptyset = X$ . Suppose that  $u_s$  has been defined for all  $s \in 2^{\leq k}$ . Fix  $\gamma < \omega_1$  and consider  $P = P_{k,\gamma+1}$ , as in (iii). Then for each  $s \in 2^k$  the set  $P^{(\gamma)} \cap u_s$  is infinite. Fix  $S \subset P^{(\gamma)}$  such that  $|S \cap u_s| = 2$  for each  $s \in 2^k$ . Next, enlarge each  $x \in S \cap u_s$  to a small open set  $v_x \in \mathcal{B}$ , contained in  $u_s$ , such that  $\{y_0, \dots, y_{N-1}\} \in C_{k+1}$  whenever  $y_i$  are taken from pairwise distinct  $v_x$ 's. This is possible, because  $C_{k+1}$  is open. Let  $\varphi(\gamma) = \{v_x : x \in S\}$ . This defines a mapping  $\varphi: \omega_1 \rightarrow [\mathcal{B}]^{<\omega}$ . As  $\mathcal{B}$  is countable,

there is unbounded  $F \subset \omega_1$  such that  $\varphi \upharpoonright F$  is constant, say  $\{v_{s \smallfrown i} : s \in 2^k, i < 2\}$ , where  $v_{s \smallfrown i} \subset u_s$ . Set  $u_{s \smallfrown i} = v_{s \smallfrown i}$ . Observe that (i) holds if we let  $v_x$ 's be small enough. Also (ii) holds, by the definition of  $v_x$ 's. Finally, (iii) holds, because  $P_{k, \gamma+1}^{(\gamma)} \cap u_t \neq \emptyset$  for  $t \in 2^{k+1}$  whenever  $\gamma \in F$ . By (ii) the perfect set obtained from this construction is  $C$ -homogeneous.  $\square$

Using the above theorem we obtain the following corollary which, for the case of 2-colorings of Polish spaces, was mentioned by Shelah [4, Remark 1.14]:

**Corollary 2.3.** *Let  $1 \leq N < \omega$  and let  $C$  be a  $G_\delta$   $N$ -coloring of an analytic space  $X$ . If there exists an uncountable  $C$ -homogeneous set, then there exists also a perfect one.*

*Proof.* Let  $f: \omega^\omega \rightarrow X$  be a continuous surjection and define  $C' = \{s \in [\omega^\omega]^2 : f[s] \in C\}$ . If  $K \subset X$  is  $C$ -homogeneous and  $K = f[K']$  where  $f \upharpoonright K'$  is one-to-one, then  $K'$  is  $C'$ -homogeneous. If  $K$  is uncountable then so is  $K'$  and by Theorem 2.2 we get a perfect set  $P \subset \omega^\omega$  which is  $C'$ -homogeneous. Then  $f \upharpoonright P$  is one-to-one and hence  $f[P]$  is a perfect  $C$ -homogeneous set.  $\square$

### 3. APPLICATIONS TO CONVEXITY

Let  $X \subset E$ , where  $E$  is a real vector space. A subset  $K$  of  $X$  is a  $k$ -clique ( $k$  can be a cardinal or just a natural number; we will use this notion for  $k < \omega$  and  $k = \aleph_0$ ) if  $\text{conv } S \not\subset X$  whenever  $S \in [K]^k$ . If  $E$  is finite-dimensional and  $k > \dim E$ , then we can define the notion of a *strong  $k$ -clique* replacing  $\text{conv } S$  by  $\text{int conv } S$  in the definition. A finite set  $S \subset X$  is (*strongly*) *defected* in  $X$  if  $\text{conv } S \not\subset X$  ( $\text{int conv } S \not\subset X$ ). It is clear that the relation of strong defectedness is open and defectedness is open provided that  $X$  is closed.

Applying the results of the previous section we get the following:

**Theorem 3.1.** (a) *Let  $X$  be a closed set in a Polish linear space and let  $N < \omega$ . If  $X$  does not contain a perfect  $N$ -clique, then all  $N$ -cliques in  $X$  are countable. Moreover, there exists an ordinal  $\gamma < \omega_1$  which bounds the Cantor-Bendixson ranks of all  $N$ -cliques in  $X$ .*

(b) *Let  $X$  be an analytic subset of  $\mathbb{R}^m$ . If  $m < N < \omega$  and  $X$  contains an uncountable strong  $N$ -clique, then  $X$  contains also a perfect one.*

Theorem 3.1(a) was proved, under the stronger assumption that  $X$  is a countable union of convex sets, by Kojman in [2].

In [3] we proved, in particular, that in a closed planar set either all cliques are countable or there exists a perfect 3-clique. Here we prove the same for analytic sets, namely:

**Theorem 3.2.** *Let  $X \subset \mathbb{R}^2$  be analytic and assume that  $X$  contains an uncountable  $\aleph_0$ -clique. Then either  $X$  contains a perfect strong 3-clique or else, for some line  $L$ ,  $X \cap L$  contains a perfect 2-clique. In particular  $X$  contains a perfect 3-clique.*

*Proof.* Fix a continuous function  $f: \omega^\omega \rightarrow X$  onto  $X$  and fix an uncountable  $\aleph_0$ -clique  $K \subset X$ . We may assume that every line contains only countably many points of  $L$ : otherwise, for some line  $L$ ,  $X \cap L$  contains an uncountable  $\aleph_0$ -clique, so it contains a perfect 2-clique (Proposition 2.1). Fix uncountable  $K' \subset \omega^\omega$  such that  $f \upharpoonright K'$  is a bijection onto  $K$ .

A finite collection  $\{u_0, \dots, u_{k-1}\}$  of open subsets of  $\omega^\omega$  will be called *relevant* if each  $u_i$  contains uncountably many points of  $K'$ ,  $\text{cl } u_i \cap \text{cl } u_j = \emptyset$  whenever  $i < j < k$  and  $\text{int conv}\{f(x_0), f(x_1), f(x_2)\} \not\subset X$  whenever  $x_0, x_1, x_2$  are taken from pairwise distinct  $u_i$ 's. To find a perfect strong 3-clique in  $X$ , it suffices to construct a perfect tree of open sets in  $\omega^\omega$  with relevant levels. If  $P$  is a perfect set obtained from such a tree, then  $f \upharpoonright P$  is one-to-one and  $f[P]$  is a perfect strong 3-clique.

Suppose that we have a relevant collection  $\{u_0, \dots, u_k\}$ . We have to show that it is possible to split each  $u_i$  to obtain again a relevant collection. We will split  $u_k$ . Let  $L = K' \cap u_k$  and pick  $y_i \in u_i$  for  $i < k$ . Define  $c_i: [L]^2 \rightarrow 2$  by letting  $c_i(x_0, x_1) = 1$  iff  $\text{conv}\{f(x_0), f(x_1), f(y_i)\} \not\subset X$ . Observe that there are no infinite  $c_i$ -homogeneous sets of color 0: if  $S \subset L$  is infinite, then, by Carathéodory's theorem, there is  $s \in [S]^3$  such that  $f[s]$  is defected in  $X$  (because  $f[S]$  is defected) and hence for some  $x_0, x_1 \in s$  we have  $\text{conv}\{f(x_0), f(x_1), f(y_i)\} \not\subset X$ , because  $\text{conv } T \subset \bigcup_{x,y \in T} \text{conv}\{x, y, p\}$  for  $T \subset \mathbb{R}^2$ ,  $p \in \mathbb{R}^2$ . Using  $k$  times the theorem of Dushnik-Miller we obtain uncountable  $L' \subset L$  which is  $c_i$ -homogeneous of color 1 for  $i < k$ . Shrinking  $L'$  we may assume that each nonempty open subset of  $L'$  is uncountable. Now choose disjoint open sets  $v_0, v_1$  with  $\text{cl } v_j \subset u_k$  and  $v_j \cap L' \neq \emptyset$  for  $j < 2$ . To finish the proof we need the following geometric property of the plane:

**Claim 3.3.** *Let  $A, B \subset X \subset \mathbb{R}^2$  and  $c \in \mathbb{R}^2$  be such that  $A, B$  are uncountable, each line contains countably many points of  $A \cup B$  and  $\text{conv}\{a, b, c\} \not\subset X$  whenever  $a \in A, b \in B$ . Then there are  $a_0 \in A, b_0 \in B$  such that  $\text{int conv}\{a_0, b_0, c\} \not\subset X$ .*

*Proof.* Suppose this is not true. Observe that, replacing  $A$  and  $B$  if necessary, we may assume that for some  $b_0 \in B$ ,  $[a, b_0] \cup [a, c] \not\subset X$  whenever  $a \in A$ . Indeed, if  $[b, c] \subset X$  for some  $b \in B$ , then we take  $b_0 = b$ , otherwise we take any  $a_0 \in A$  and we replace  $A$  and  $B$ . Now, without loss of generality, we may assume that  $b_0 = (-1, 0)$ ,  $c = (1, 0)$  and  $A$  is contained in  $(-1, 1) \times (0, 1)$ . Now, if some vertical line contains two elements of  $A$ , then we are done: we take  $a_0 \in A$  such that some  $a_1 \in A$  is below  $a_0$ ; then the relative interiors of segments  $[b_0, a_1]$ ,  $[c, a_1]$  are contained in the interior of  $\text{conv}\{a_0, b_0, c\}$ .

Assume that each vertical line contains at most one element of  $A$ . As  $A$  is uncountable, there is  $a_1 \in A$  such that arbitrarily close to  $a_1$  there are uncountably many points both on the left and the right side of  $a_1$ . Suppose now that e.g.  $\{b_0, a_1\}$  is defected in  $X$ . As  $[b_0, a_1]$  contains only countably many points of  $A$ , we can find  $a_2 \in A$  which is close enough to  $a_1$ , on the left side of  $a_1$  and not in  $[b_0, a_1]$ . If  $a_2$  is below  $[b_0, a_1]$ , then we can set  $a_0 = a_1$ , otherwise we can set  $a_0 = a_2$ .  $\square$

Let  $i = 0$ . Using Claim 3.3 for  $A = f[v_0 \cap L']$ ,  $B = f[v_1 \cap L']$  and  $c = f(y_i)$  we get  $x_j \in v_j$  such that  $\text{int conv}\{f(x_0), f(x_1), f(y_i)\} \not\subset X$ . By continuity, shrink  $v_0, v_1$  and enlarge  $y_i$  to an open set  $u'_i \subset u_i$  such that each triple selected from  $f[v_0] \times f[v_1] \times f[u'_i]$  is strongly defected in  $X$ . Repeat the same argument for each  $i < k$ , obtaining a relevant collection  $\{u'_0, \dots, u'_{k-1}, v'_0, v'_1\}$  which realizes the splitting of  $u_k$ . This completes the proof.  $\square$

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