PERFECT CLIQUES AND $G_\delta$ COLORINGS OF POLISH SPACES

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Abstract. A coloring of a set $X$ is any subset $C$ of $[X]^N$, where $N > 1$ is a natural number. We give some sufficient conditions for the existence of a perfect $C$-homogeneous set, in the case where $C$ is $G_\delta$ and $X$ is a Polish space. In particular, we show that it is sufficient that there exist $C$-homogeneous sets of arbitrarily large countable Cantor-Bendixson rank. We apply our methods to show that an analytic subset of the plane contains a perfect 3-clique if it contains any uncountable $k$-clique, where $k$ is a natural number or $\aleph_0$ (a set $K$ is a $k$-clique in $X$ if the convex hull of any of its $k$-element subsets is not contained in $X$).

1. Introduction

For a set $X$ and natural number $N$, $[X]^N$ denotes the collection of all $N$-element subsets of $X$. A (two-color) coloring of $X$ is (represented by) a set $C \subset [X]^N$. We identify $[X]^N$ with a suitable subspace of the product $X^N$. We are interested in the following problem: find sufficient conditions for the existence of a perfect $C$-homogeneous set $P \subset X$, where $X$ is a Polish space and $C \subset [X]^N$ is open (or more generally $G_\delta$). A natural example for this problem is the following: let $X \subset \mathbb{R}^N$ be closed and $C = \{s \in [X]^k : \text{conv } s \notin X\}$. Then $C$ is open and a $C$-homogeneous set is called a $k$-clique in $X$. It is known (see [3]) that there exists a closed set $X \subset \mathbb{R}^N$ such that $X$ is not a countable union of convex sets but every $k$-clique in $X$ is countable for every $k < \omega$. On the other hand, it is proved in [3] that if a closed set $X \subset \mathbb{R}^N$ contains an uncountable $k$-clique for some $k$, then it contains a perfect 3-clique.

We prove that if $C$ is a $G_\delta$ coloring of a Polish space and there are no perfect $C$-homogeneous sets, then there is a countable ordinal $\gamma$ such that the Cantor-Bendixson rank of every $C$-homogeneous set is $< \gamma$. In the context of cliques, this strengthens the result of Kojman [2] (see Theorem 3.1(a) below). From our result it follows that if $C$ is a $G_\delta$ coloring of an analytic space, then either there exists a perfect $C$-homogeneous set or all $C$-homogeneous sets are countable. This is not true for $F_\sigma$ colorings: a result of Shelah [4] states that consistently there exist $F_\sigma$ 2-colorings with uncountable but not perfect homogeneous sets. Concerning cliques, we investigate analytic subsets of the plane. We prove that if an analytic set $X \subset \mathbb{R}^2$ contains an uncountable $\aleph_0$-clique, then $X$ contains also a perfect 3-clique.
1.1. Notation. Any subset of \([X]^N\) is called a coloring (or an \(N\)-coloring) of \(X\). We write \(\neg C\) instead of \([X]^N \setminus C\). A set \(S \subset X\) is \(C\)-homogeneous if \([A]^N \subset C\). We identify \([X]^N\) with the subspace of \(X^N\) consisting of all \(N\)-tuples \((x_0, \ldots, x_{N-1})\) with \(x_i \neq x_j\) for \(i \neq j\). Thus we may consider topological properties of colorings. If \(f : X \to Y\) is a function, then we write \(f[S]\) for the image of a set \(S \subset X\) and \(f(s)\) for the value at a point \(s \in X\). By a perfect set we mean a compact, nonempty, topological space with no isolated points.

2. On colorings

First we recall a simple result on open 2-colorings of analytic spaces which can be found in Todorcević-Farah’s book [5, p. 81].

**Proposition 2.1.** Let \(X\) be an analytic space and let \(C \subset [X]^2\) be open. Then either there exists a perfect \(C\)-homogeneous set or else \(X\) is a countable union of \(\neg C\)-homogeneous sets, i.e. \(X = \bigcup_{n \in \omega} A_n\) where \([A_n]^2 \cap C = \emptyset\) for every \(n \in \omega\).

The above result is no longer valid when we replace the word “open” with “closed”; see [3] p. 83. Also, the above proposition cannot be strengthened for colorings of triples: there exists a clopen 3-coloring of \(2^\omega\) such that there are no uncountable homogeneous sets either of this color or of its complement; see Blass’ example [4]. In Blass’ example, the Cantor-Bendixson rank of any homogeneous set is at most 1. Below we show that in this situation there always exists a countable ordinal which bounds the Cantor-Bendixson ranks of all homogeneous sets. In fact this is true for \(G_\delta\) colorings.

For a topological space \(Y\) and an ordinal \(\alpha\) we denote by \(Y^{(\alpha)}\) the \(\alpha\)-derivative of \(Y\); the Cantor-Bendixson rank of \(Y\) is the minimal ordinal \(\gamma\) such that \(Y^{(\gamma+1)}\) is empty.

**Theorem 2.2.** Let \(C\) be a \(G_\delta\) \(N\)-coloring of a Polish space \(X\). If for every countable ordinal \(\gamma\) there exists a \(C\)-homogeneous set of the Cantor-Bendixson rank \(\geq \gamma\), then \(X\) contains a perfect \(C\)-homogeneous set.

**Proof.** Fix a countable base \(\mathcal{B}\) in \(X\) and fix a complete metric on \(X\). Let \(C = \bigcap_{n \in \omega} C_n\), where each \(C_n\) is open and \(C_{n+1} \subset C_n\). We will construct a tree of open sets

\[ T = \{u_s : s \in 2^{<\omega}\} \]

with the following properties:

(i) \(\text{cl } u_s^{-1} \subset u_s, \text{cl } u_s \cap \text{cl } u_t = \emptyset\) if \(s, t\) are incompatible and \(\text{diam}(u_s) < 2^{-\text{length}(s)}\);

(ii) if \(k < \omega\) and \(s_0, \ldots, s_{N-1} \in 2^k\) are pairwise distinct, then \(\{x_0, \ldots, x_{N-1}\} \in C_k\) whenever \(x_i \in u_{s_i}, i < N\);

(iii) if \(k < \omega\), then for each \(\gamma < \omega_1\) there exists a \(C\)-homogeneous set \(P = P_{k,\gamma}\) such that \(P^{(\gamma)} \cap u_s \neq \emptyset\) for each \(s \in 2^k\).

We start with \(u_0 = X\). Suppose that \(u_s\) has been defined for all \(s \in 2^{<k}\). Fix \(\gamma < \omega_1\) and consider \(P = P_{k,\gamma+1}\), as in (iii). Then for each \(s \in 2^k\) the set \(P^{(\gamma)} \cap u_s\) is infinite. Fix \(S \subset P^{(\gamma)}\) such that \(|S \cap u_s| = 2\) for each \(s \in 2^k\). Next, enlarge each \(x \in S \cap u_s\) to a small open set \(v_x \in \mathcal{B}\), contained in \(u_s\), such that \(\{y_0, \ldots, y_{N-1}\} \in C_{k+1}\) whenever \(y_i\) are taken from pairwise distinct \(v_x\)’s. This is possible, because \(C_{k+1}\) is open. Let \(\varphi(\gamma) = \{v_x : x \in S\}\). This defines a mapping \(\varphi : \omega_1 \to |\mathcal{B}|^{<\omega}\). As \(\mathcal{B}\) is countable,
there is unbounded $F \subseteq \omega_1$ such that $\varphi \upharpoonright F$ is constant, say \( \{ v_{s,i} : s \in 2^k, i < 2 \} \), where $v_{s,i} \subseteq u_s$. Set $v_{s,i} = v_{s,i}$. Observe that (i) holds if we let $v_s$’s be small enough. Also (ii) holds, by the definition of $v_s$’s. Finally, (iii) holds, because $P^{(\gamma)}_{k,\gamma+1} \cap u_\gamma \neq \emptyset$ for $t \in 2^{k+1}$ whenever $\gamma \in F$. By (ii) the perfect set obtained from this construction is $C$-homogeneous. 

Using the above theorem we obtain the following corollary which, for the case of 2-colorings of Polish spaces, was mentioned by Shelah [4] Remark 1.14:

**Corollary 2.3.** Let $1 \leq N < \omega$ and let $C$ be a $G_\delta$ $N$-coloring of an analytic space $X$. If there exists an uncountable $C$-homogeneous set, then there exists also a perfect one.

**Proof.** Let $f : \omega^\omega \to X$ be a continuous surjection and define $C' = \{ s \in [\omega^\omega]^2 : f[s] \in C \}$. If $K \subseteq X$ is $C$-homogeneous and $K = f[K']$ where $f \upharpoonright K'$ is one-to-one, then $K'$ is $C'$-homogeneous. If $K$ is uncountable then so is $K'$ and by Theorem 2.2 we get a perfect set $P \subseteq \omega^\omega$ which is $C'$-homogeneous. Then $f \upharpoonright P$ is one-to-one and hence $f[P]$ is a perfect $C$-homogeneous set. □

### 3. Applications to Convexity

Let $X \subseteq E$, where $E$ is a real vector space. A subset $K$ of $X$ is a $k$-clique ($k$ can be a cardinal or just a natural number; we will use this notion for $k < \omega$ and $k = \aleph_0$) if $\text{conv } S \not\subseteq X$ whenever $S \subseteq [K]^k$. If $E$ is finite-dimensional and $k > \dim E$, then we can define the notion of a strong $k$-clique replacing $\text{conv } S$ by $\text{int } \text{conv } S$ in the definition. A finite set $S \subseteq X$ is (strongly) defected in $X$ if $\text{conv } S \not\subseteq X$ ($\text{int } \text{conv } S \not\subseteq X$). It is clear that the relation of strong defectedness is open and defectedness is open provided that $X$ is closed.

Applying the results of the previous section we get the following:

**Theorem 3.1.** (a) Let $X$ be a closed set in a Polish linear space and let $N < \omega$. If $X$ does not contain a perfect $N$-clique, then all $N$-cliques in $X$ are countable. Moreover, there exists an ordinal $\gamma < \omega_1$ which bounds the Cantor-Bendixson ranks of all $N$-cliques in $X$.

(b) Let $X$ be an analytic subset of $\mathbb{R}^m$. If $m < N < \omega$ and $X$ contains an uncountable strong $N$-clique, then $X$ contains also a perfect one.

Theorem 3.1(a) was proved, under the stronger assumption that $X$ is a countable union of convex sets, by Kojman in [2].

In [3] we proved, in particular, that in a closed planar set either all cliques are countable or there exists a perfect 3-clique. Here we prove the same for analytic sets, namely:

**Theorem 3.2.** Let $X \subseteq \mathbb{R}^2$ be analytic and assume that $X$ contains an uncountable $\aleph_0$-clique. Then either $X$ contains a perfect strong 3-clique or else, for some line $L$, $X \cap L$ contains a perfect 2-clique. In particular $X$ contains a perfect 3-clique.

**Proof.** Fix a continuous function $f : \omega^\omega \to X$ onto $X$ and fix an uncountable $\aleph_0$-clique $K \subseteq X$. We may assume that every line contains only countably many points of $L$: otherwise, for some line $L$, $X \cap L$ contains an uncountable $\aleph_0$-clique, so it contains a perfect 2-clique (Proposition 2.1). Fix uncountable $K' \subseteq \omega^\omega$ such that $f \upharpoonright K'$ is a bijection onto $K$. 

A finite collection \( \{u_0, \ldots, u_{k-1}\} \) of open subsets of \( \omega^\omega \) will be called relevant if each \( u_i \) contains uncountably many points of \( K' \), \( \text{cl} \, u_i \cap \text{cl} \, u_j = \emptyset \) whenever \( i < j < k \) and \( \text{int conv} \{ f(x_0), f(x_1), f(x_2) \} \not\subset X \) whenever \( x_0, x_1, x_2 \) are taken from pairwise distinct \( u_i \)'s. To find a perfect strong 3-clique in \( X \), it suffices to construct a perfect tree of open sets in \( \omega^\omega \) with relevant levels. If \( P \) is a perfect set obtained from such a tree, then \( f \upharpoonright P \) is one-to-one and \( f[P] \) is a perfect strong 3-clique.

Suppose that we have a relevant collection \( \{u_0, \ldots, u_k\} \). We have to show that it is possible to split each \( u_i \) to obtain again a relevant collection. We will split \( u_k \). Let \( L = K' \cap u_k \) and pick \( y_i \in u_i \) for \( i < k \). Define \( c_i : [L]^2 \to 2 \) by letting \( c_i(x_0, x_1) = 1 \) if \( \text{conv} \{ f(x_0), f(x_1), f(y_i) \} \not\subset X \). Observe that there are no infinite \( c_i \)-homogeneous sets of color 0: if \( S \subset L \) is infinite, then, by Carathéodory’s theorem, there is \( s \in [S]^3 \) such that \( f[s] \) is defected in \( X \) (because \( f[S] \) is defected) and hence for some \( x_0, x_1 \in s \) we have \( \text{conv} \{ f(x_0), f(x_1), f(y_i) \} \not\subset X \), because \( \text{conv} \mathcal{T} \subset \bigcup_{x,y \in \mathcal{T}} \text{conv} \{x, y, p\} \) for \( \mathcal{T} \subset \mathbb{R}^2 \), \( p \in \mathbb{R}^2 \). Using \( k \) times the theorem of Dushnik-Miller we obtain uncountable \( L' \subset L \) which is \( c_i \)-homogeneous of color 1 for \( i < k \). Shrinking \( L' \) we may assume that each nonempty open subset of \( L' \) is uncountable. Now choose disjoint open sets \( v_0, v_1 \) with \( \text{cl} \, v_j \subset u_k \) and \( v_j \cap L' \neq \emptyset \) for \( j < 2 \). To finish the proof we need the following geometric property of the plane:

**Claim 3.3.** Let \( A, B \subset X \subset \mathbb{R}^2 \) and \( c \in \mathbb{R}^2 \) be such that \( A, B \) are uncountable, each line contains countably many points of \( A \cup B \) and \( \text{conv} \{a, b, c\} \subset X \) whenever \( a \in A, b \in B \). Then there are \( a_0 \in A, b_0 \in B \) such that \( \text{int conv} \{ a_0, b_0, c \} \subset X \).

**Proof.** Suppose that this is true. Observe that, replacing \( A \) and \( B \) if necessary, we may assume that for some \( b_0 \in B \), \( [a, b_0] \cup [a, c] \not\subset X \) whenever \( a \in A \). Indeed, if \( [b, c] \subset X \) for some \( b \in B \), then we take \( b_0 = b \), otherwise we take any \( a_0 \in A \) and we replace \( A \) and \( B \). Now, without loss of generality, we may assume that \( b_0 = (-1, 0) \), \( c = (1, 0) \) and \( A \) is contained in \( (-1, 1) \times (0, 1) \). Now, if some vertical line contains two elements of \( A \), then we are done: we take \( a_0 \in A \) such that some \( a_1 \in A \) is below \( a_0 \); then the relative interiors of segments \( [b_0, a_1], [a_1, c] \) are contained in the interior of \( \text{conv} \{a_0, b_0, c\} \).

Assume that each vertical line contains at most one element of \( A \). As \( A \) is uncountable, there is \( a_1 \in A \) such that arbitrarily close to \( a_1 \) there are uncountably many points both on the left and the right side of \( a_1 \). Suppose now that e.g. \( \{b_0, a_1\} \) is defected in \( X \). As \( [b_0, a_1] \) contains only countably many points of \( A \), we can find \( a_2 \in A \) which is close enough to \( a_1 \), on the left side of \( a_1 \) and not in \( [b_0, a_1] \). If \( a_2 \) is below \( [b_0, a_1] \), then we can set \( a_0 = a_1 \), otherwise we can set \( a_0 = a_2 \).

Let \( i = 0 \). Using **Claim 3.3** for \( A = f[v_0 \cap L'], B = f[v_1 \cap L'] \) and \( c = f(y_i) \) we get \( x_j \in v_j \) such that \( \text{int conv} \{ f(x_0), f(x_1), f(y_i) \} \not\subset X \). By continuity, shrink \( v_0, v_1 \) and enlarge \( y_i \) to an open set \( u'_i \subset u_i \) such that each triple selected from \( f[v_0] \times f[v_1] \times f[u'_i] \) is strongly defected in \( X \). Repeat the same argument for each \( i < k \), obtaining a relevant collection \( \{u'_0, \ldots, u'_{k-1}, v'_0, v'_1\} \) which realizes the splitting of \( u_k \). This completes the proof.

**References**


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