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GENERALIZED SCHWARZ-PICK ESTIMATES

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Abstract. We obtain higher derivative generalizations of the Schwarz-Pick inequality for analytic self-maps of the unit disk as a consequence of recent characterizations of boundedness and compactness of weighted composition operators between Bloch-type spaces.

1. Introduction

Part of the Schwarz-Pick inequality, sometimes called the invariant Schwarz inequality, says that whenever φ is an analytic self-map of the unit disk $\mathbb{D}$, then

$$\frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2} \leq 1$$

for all z in $\mathbb{D}$. If C_φ is the composition operator defined by $C_\varphi(f) = f \circ \varphi$ for f analytic in $\mathbb{D}$, the Schwarz-Pick inequality directly yields the boundedness of all composition operators on the classical Bloch space. We will prove the following generalized Schwarz-Pick estimates.

Theorem 1. *For $n \geq 1$ and φ an analytic self-map of $\mathbb{D}$,*

$$\sup_{z\in\mathbb{D}} \frac{|\varphi^{(n)}(z)|(1-|z|^2)^n}{1-|\varphi(z)|^2} < \infty.$$

Our proof of this theorem will be an application of boundedness criteria for weighted composition operators between various Bloch-type spaces recently obtained in [3]. These Bloch-type spaces and boundedness criteria for weighted composition operators will be discussed in the next section, which also contains the proof of the above theorem. A natural generalization of the above result is given in Theorem 3, when φ satisfies an additional condition. In Section 3 we give "little-oh" versions of Theorems 1 and 3, and in Section 4 we briefly discuss converses to our main results.

2. Proof of the main theorem

The Bloch-type spaces we consider here are defined by

$$\mathcal{B}^\alpha = \{f \text{ analytic in } \mathbb{D} : \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty\}.$$

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These become Banach spaces with norms $|f(0)| + \sup\{(1-|z|^2)^\alpha |f'(z)| : z \in \mathbb{D}\}$. The range of the parameter α can be taken to be $0 < \alpha < \infty$, although our interest here is restricted to the case $\alpha \geq 1$. Note that $\alpha = 1$ gives the classical Bloch space $\mathcal{B}$. A weighted composition operator uC_φ is defined for analytic u on $\mathbb{D}$ and analytic self-map φ of $\mathbb{D}$ by $uC_\varphi(f) = u(f \circ \varphi)$. A characterization of boundedness of uC_φ from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ is given in Theorem 2.1 of [3]; this characterization depends on whether $0 < \alpha < 1$, $\alpha = 1$, or $\alpha > 1$. Here we will only make use of the $\alpha > 1$ case:

Theorem 2 ([3]). *When $\alpha > 1$ and $\beta > 0$ the weighted composition operator uC_φ maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\beta$ if and only if*

(a) $\displaystyle \sup_{z \in \mathbb{D}} |u(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} |\varphi'(z)| < \infty$ *and*

(b) $\displaystyle \sup_{z \in \mathbb{D}} |u'(z)| \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} < \infty.$

Theorem 2 is the key ingredient in our derivation of the generalized Schwarz-Pick estimates. The other ingredient is the observation that since $f \in \mathcal{B}^\alpha$ if and only if $f' \in \mathcal{B}^{\alpha+1}$, and all composition operators are bounded from $\mathcal{B}^1$ to $\mathcal{B}^1$, it follows that the operators $D^n C_\varphi$ are bounded from $\mathcal{B}^1$ to $\mathcal{B}^{n+1}$ for all $n \geq 1$ and all φ, where D^n denotes the n^{th} derivative operator.

Proof of Theorem 1. For $n = 1$, the result is the classical Schwarz-Pick inequality. The rest of the argument proceeds by induction, however it is instructive to look explicitly at the $n = 2$ case. For this, note that DC_φ is bounded from $\mathcal{B}^1$ to $\mathcal{B}^2$ for all φ, as noted above. We have $DC_\varphi(f) = (f' \circ \varphi)\varphi'$. Thus the weighted composition operator $\varphi' C_\varphi$ is bounded from $\mathcal{B}^2$ to $\mathcal{B}^2$, since $f \in \mathcal{B}^1$ if and only if $f' \in \mathcal{B}^2$. In particular by (b) of the boundedness criteria above we have the desired statement for $n = 2$.

Now fix an integer $n \geq 2$ and assume by induction that the generalized Schwarz-Pick estimates hold for all positive integers less than or equal to n. We will show that the estimate holds for $n+1$. Consider the bounded operator $D^n C_\varphi : \mathcal{B}^1 \to \mathcal{B}^{n+1}$. If we can show that $\varphi^{(n)} C_\varphi$ is bounded from $\mathcal{B}^2$ to $\mathcal{B}^{n+1}$, then again part (b) of the boundedness criteria above will yield the generalized Schwarz-Pick estimate for $n+1$. To see why the boundedness of $\varphi^{(n)} C_\varphi : \mathcal{B}^2 \to \mathcal{B}^{n+1}$ follows from the boundedness of $D^n C_\varphi : \mathcal{B}^1 \to \mathcal{B}^{n+1}$ we consider the expansion of $D^n(f \circ \varphi) = (f \circ \varphi)^{(n)}$ by Faà di Bruno's formula (see, for example, [4]):

$$(f \circ \varphi)^{(n)}(z) = \sum \frac{n!}{k_1! k_2! \cdots k_n!} f^{(k)}(\varphi(z)) \prod_{j=1}^{n} \left(\frac{\varphi^{(j)}(z)}{j!} \right)^{k_j},$$

where $k = k_1 + k_2 + \cdots + k_n$ and this sum is over all non-negative integers $k_1, k_2, \cdots, k_n$ satisfying $k_1 + 2k_2 + \cdots + nk_n = n$. In particular, one of the terms of this sum is $f'(\varphi(z))\varphi^{(n)}(z)$ and the remaining terms involve products of $f^{(k)} \circ \varphi(z)$ $(1 < k \leq n)$ with products of derivatives of φ. Writing Faà di Bruno's formula in operator notation we have

$$D^n C_\varphi = \sum \frac{n!}{k_1! k_2! \cdots k_n!} \prod_{j=1}^{n} \left(\frac{D^j \varphi}{j!} \right)^{k_j} C_\varphi D^k. \tag{1}$$

With $k_n = 1$ (and therefore also $k_1 = k_2 = \cdots = k_{n-1} = 0$) we obtain on the right the term $\varphi^{(n)} C_\varphi D$. If $k_n = 0$ we obtain (constant multiples of) the terms

$$\prod_{j=1}^{n-1} \left(\varphi^{(j)}\right)^{k_j} C_\varphi D^k,$$

where $k = k_1 + \cdots + k_{n-1}$, and $k_1 + 2k_2 + \cdots + (n-1)k_{n-1} = n$. Set

$$u(z) = \prod_{j=1}^{n-1} \left(\varphi^{(j)}(z)\right)^{k_j}, \tag{2}$$

where the non-negative integers $k_1, \ldots, k_{n-1}$ are as just described. Our goal is to show that each weighted composition operator uC_φ is bounded from $\mathcal{B}^{k+1}$ to $\mathcal{B}^{n+1}$; this together with the boundedness of $D^n C_\varphi : \mathcal{B}^1 \to \mathcal{B}^{n+1}$ will imply the boundedness of $\varphi^{(n)} C_\varphi$ from $\mathcal{B}^2$ to $\mathcal{B}^{n+1}$. To show that uC_φ is bounded from $\mathcal{B}^{k+1}$ to $\mathcal{B}^{n+1}$ we must verify conditions (a) and (b) of Theorem 2.

For condition (a) we observe that the product

$$|u(z)| \frac{(1-|z|^2)^{n+1}}{(1-|\varphi(z)|^2)^{k+1}} |\varphi'(z)|$$

can be written as

$$\left(\frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\right)^{k_1+1} \prod_{j=2}^{n-1} \left(\frac{(1-|z|^2)^j |\varphi^{(j)}(z)|}{1-|\varphi(z)|^2}\right)^{k_j}, \tag{3}$$

since $n+1 = (k_1+1) + 2k_2 + \cdots + (n-1)k_{n-1}$. Using the induction hypothesis we see that

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{n+1}}{(1-|\varphi(z)|^2)^{k+1}} |u(z)|\, |\varphi'(z)| < \infty.$$

For condition (b) of Theorem 2, notice that

$$u'(z) = \sum_{i=1}^{n-1} k_i \left(\varphi^{(i)}(z)\right)^{k_i-1} \varphi^{(i+1)}(z) \prod_{j=1, j\neq i}^{n-1} \left(\varphi^{(j)}(z)\right)^{k_j}. \tag{4}$$

We claim that

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{n+1}}{(1-|\varphi(z)|^2)^{k}} |u'(z)| < \infty.$$

To see this, note that when $k_i \neq 0$ we see by the induction hypothesis that

$$\left|\varphi^{(i)}(z)\right|^{k_i-1} |\varphi^{(i+1)}(z)| \prod_{j=1, j\neq i}^{n-1} \left|\varphi^{(j)}(z)\right|^{k_j}$$

is bounded above by a constant multiple of

$$\frac{(1-|\varphi(z)|^2)^{k_i-1}}{(1-|z|^2)^{i(k_i-1)}} \frac{1-|\varphi(z)|^2}{(1-|z|^2)^{i+1}} \prod_{j=1, j\neq i}^{n-1} \left(\frac{1-|\varphi(z)|^2}{(1-|z|^2)^j}\right)^{k_j} = \frac{(1-|\varphi(z)|^2)^k}{(1-|z|^2)^{n+1}}, \tag{5}$$

and our claim follows.

Conditions (a) and (b) in Theorem 2 are satisfied and the operator uC_φ maps $\mathcal{B}^{k+1}$ boundedly into $\mathcal{B}^{n+1}$. The operator D^k maps $\mathcal{B}^1$ boundedly onto $\mathcal{B}^{k+1}$. Thus, for each k as above with $k_n = 0$, the operator

$$\prod_{j=1}^{n-1} \left(\varphi^{(j)}\right)^{k_j} C_\varphi D^k$$

maps $\mathcal{B}^1$ boundedly into $\mathcal{B}^{n+1}$. We conclude that $\varphi^{(n)}\, C_\varphi D$ maps $\mathcal{B}^1$ boundedly into $\mathcal{B}^{n+1}$. Since D maps $\mathcal{B}^1$ onto $\mathcal{B}^2$, the weighted composition operator $\varphi^{(n)}\, C_\varphi$ maps $\mathcal{B}^2$ boundedly onto $\mathcal{B}^{n+1}$. By condition (b) of Theorem 2 this implies that

$$|\varphi^{(n+1)}(z)| \frac{(1-|z|^2)^{n+1}}{1-|\varphi(z)|^2} = |(\varphi^{(n)})'(z)| \frac{(1-|z|^2)^{n+1}}{1-|\varphi(z)|^2}$$

is bounded. This completes the induction, and the proof. □

Theorem 1 can easily be generalized as follows.

Theorem 3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map such that*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} < \infty$$

for some $\alpha, \beta > 0$. Then for each integer $n \geq 2$,

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta+n-1} |\varphi^{(n)}(z)|}{(1-|\varphi(z)|^2)^\alpha} < \infty.$$

Proof. The hypothesis insures that C_φ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ ([3], Corollary 2.4) so DC_φ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}^{\beta+1}$. Since $DC_\varphi = \varphi' C_\varphi D$ it follows that $\varphi' C_\varphi$ must be bounded from $\mathcal{B}^{\alpha+1}$ to $\mathcal{B}^{\beta+1}$. Part (b) of Theorem 2 gives the desired conclusion for $n = 2$. We proceed by induction in much the same way as was done in the proof of Theorem 1. Assume the result holds for all positive integers less than or equal to n. To obtain the result for $n+1$ we show that $\varphi^{(n)} C_\varphi$ is bounded from $\mathcal{B}^{\alpha+1}$ to $\mathcal{B}^{\beta+n}$, and then appeal to Theorem 2. As in the proof of Theorem 1, boundedness of $\varphi^{(n)} C_\varphi$ will follow from the boundedness of $D^n C_\varphi$ from $\mathcal{B}^\alpha$ to $\mathcal{B}^{\beta+n}$ and (1) if we can show that uC_φ is bounded from $\mathcal{B}^{\alpha+k}$ to $\mathcal{B}^{\beta+n}$, $1 \leq k < n$, when u is given by (2). Condition (a) of Theorem 2 follows from the observation that

$$\begin{aligned} &|u(z)| \frac{(1-|z|^2)^{\beta+n}}{(1-|\varphi(z)|^2)^{\alpha+k}} |\varphi'(z)| \\ &\quad = \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} \prod_{j=1}^{n-1} \left(\frac{(1-|z|^2)^j |\varphi^{(j)}(z)|}{1-|\varphi(z)|^2} \right)^{k_j}. \end{aligned}$$

The first factor is bounded on $\mathbb{D}$ by hypothesis, and the other factors are bounded on $\mathbb{D}$ by Theorem 1.

Similarly, to check condition (b) we must show that

$$\frac{|u'(z)|(1-|z|^2)^{\beta+n}}{(1-|\varphi(z)|^2)^{\alpha+k-1}}$$

is bounded on $\mathbb{D}$. Using the expression for $u'(z)$ given in (4) this follows by observing that for $k_i \geq 1$ the expression

$$\left|\varphi^{(i)}(z)\right|^{k_i-1} |\varphi^{(i+1)}(z)| \prod_{j=1, j\neq i}^{n-1} \left|\varphi^{(j)}(z)\right|^{k_j}$$

is bounded above by a constant multiple of

$$\frac{(1-|\varphi(z)|^2)^\alpha}{(1-|z|^2)^{\beta+i}} \frac{(1-|\varphi(z)|^2)^{k_i-1}}{(1-|z|^2)^{i(k_i-1)}} \prod_{j=1, j\neq i}^{n-1} \left(\frac{1-|\varphi(z)|^2}{(1-|z|^2)^j}\right)^{k_j} = \frac{(1-|\varphi(z)|^2)^{\alpha+k-1}}{(1-|z|^2)^{\beta+n}},$$

which gives the desired result. This completes the verification of the boundedness of uC_φ from $\mathcal{B}^{\alpha+k}$ to $\mathcal{B}^{\beta+n}$, and the theorem follows exactly as in Theorem 1. □

3. The hyperbolic little Bloch class

Recall that an analytic self-map of the disk φ is said to be in the hyperbolic little Bloch class $\mathcal{B}_0^h$ if

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} = 0.$$

Note this implies in particular that φ is in the little Bloch space $\mathcal{B}_0$, the subspace of $\mathcal{B}$ consisting of Bloch functions f satisfying $\lim_{|w|\to 1^-} |f'(w)|(1-|w|^2) = 0$. The hyperbolic little Bloch class appears in the characterization of those composition operators which are compact on the little Bloch space: C_φ is compact from $\mathcal{B}_0$ to itself if and only if $\varphi \in \mathcal{B}_0^h$ ([2], Theorem 1).

A particular case of the next result shows that functions in the hyperbolic little Bloch class satisfy a little-oh version of our generalized Schwarz-Pick estimates.

Theorem 4. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map such that*

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0$$

for some $\alpha, \beta > 0$. Then for each integer $n \geq 2$,

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)^{\beta+n-1}|\varphi^{(n)}(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0.$$

In particular, if $\varphi \in \mathcal{B}_0^h$, then

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)^n|\varphi^{(n)}(z)|}{1-|\varphi(z)|^2} = 0$$

for every positive integer n.

Theorem 4 can be proved by similar techniques to those employed in Theorem 3, using Theorem 3.1 of [3] which characterizes compactness of weighted composition operators from $\mathcal{B}_0^\alpha$ to $\mathcal{B}_0^\beta$ by little oh analogues of (a) and (b) of Theorem 2. We omit the details.

4. Converse results

For certain positive α and β the implications in Theorem 3 and Theorem 4 are actually logical equivalences.

Theorem 5. *Let φ be an analytic self-map of the unit disk and $\beta > \alpha > 0$. Then*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} < \infty$$

if and only if

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\beta+n-1}|\varphi^{(n)}(z)|}{(1-|\varphi(z)|^2)^\alpha} < \infty$$

for each positive integer n.

Furthermore,

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0$$

if and only if

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)^{\beta+n-1}|\varphi^{(n)}(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0$$

for each positive integer n.

We do not give the proof of this result here, but note that the interest in the first part of Theorem 5 in the "if" direction is when $0 < \alpha < \beta < 1$, as the condition

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta|\varphi'(z)|}{1-|\varphi(z)|^2)^\alpha} < \infty$$

holds automatically for all self-maps when $\alpha \leq \beta$ and $\beta \geq 1$.

The "if" directions of the two statements in Theorem 5 need not hold if $\beta < \alpha$. For example, if $n \geq 2$ and $\varphi(z) = \frac{1}{2}z^{n-1} + \frac{1}{2}$, then $\varphi^{(n)}(z) = 0$ so that

$$\sup_{z\in\mathbb{D}} \frac{|\varphi^{(n)}(z)|(1-|z|^2)^{\beta+n-1}}{(1-|\varphi(z)|^2)^\alpha} = \lim_{|z|\to 1^-} \frac{|\varphi^{(n)}(z)|(1-|z|^2)^{\beta+n-1}}{(1-|\varphi(z)|^2)^\alpha} = 0$$

for any $\alpha, \beta > 0$. However if we consider $z = r \in (0,1)$ we have that

$$\frac{(1-r^2)^\beta|\varphi'(r)|}{(1-|\varphi(r)|^2)^\alpha}$$

is unbounded as $r \to 1^-$ if $\beta < \alpha$, and tends to a finite positive constant as $r \to 1^-$ if $\beta = \alpha$. Thus the hypothesis $\alpha < \beta$ is sharp for the second statement in Theorem 5, and close to sharp for the first statement.

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