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SOLITON SOLUTIONS FOR QUASILINEAR SCHRÖDINGER EQUATIONS, I

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ABSTRACT. For a class of quasilinear Schrödinger equations we establish the existence of ground states of soliton type solutions by a minimization argument.

1. INTRODUCTION

This paper is concerned with the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$(1) \qquad i\partial_t z = -\Delta z + W(x)z - f(|z|^2)z - \kappa\Delta h(|z|^2)h'(|z|^2)z$$

where $W(x)$, $x \in \mathbf{R}^N$, is a given potential, κ is a real constant and f, h are real functions of essentially pure power forms. The semilinear case corresponding to $\kappa = 0$ has been studied extensively in recent years (e.g., [3], [9], [24]). Quasilinear equations of form (1) appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of h, the superfluid film equation in plasma physics by Kurihara in [13] (cf. [14]) for $h(s) = s$. In the case $h(s) = (1+s)^{1/2}$, (1) models the self-channeling of a high-power ultra short laser in matter; see [4], [6], [8], [23] and the references in [5]. Equation (1) also appears in plasma physics and fluid mechanics [13], [14], [17], [19], [21], in the theory of Heisenberg ferromagnets and magnons [2], [12], [15], [22], [25], in dissipative quantum mechanics [10] and in condensed matter theory [18]. In the mathematical literature very few results are known about equations of the form (1).

We consider the existence of standing wave solutions for quasilinear Schrödinger equations of form (1) with h and f as pure power functions of the dependent variable (though our method would apply to a more general type of nonlinearity). We consider the case $h(s) = s^\alpha$, $f(s) = \lambda s^{\frac{p-1}{2}}$ and $\kappa > 0$. Putting $z(t) = \exp(-iFt)u(x)$ we obtain a corresponding equation of elliptic type which has a formal variational structure:

$$(2) \qquad -\Delta u + V(x)u - \alpha\kappa(\Delta(|u|^{2\alpha}))|u|^{2\alpha-2}u = \lambda|u|^{p-1}u, \quad u > 0, \text{ in } \mathbf{R}^N,$$

where $V(x) = W(x) - F$ is the new potential function. By a simple scaling, without loss of generality, we may assume $\alpha\kappa = 1$ (corresponding to a new λ) throughout the paper. In the following we always assume $V \in C(\mathbf{R}^N, \mathbf{R})$ and $\inf_{\mathbf{R}^N} V(x) > 0$. We

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also assume $N \geq 2$ since results for $N = 1$ have been given in [20]. Let $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = \infty$ for $N = 2$. We consider several types of potentials.

(V1) $\lim_{|x|\to\infty} V(x) = +\infty$.
(V2) V is radially symmetric, i.e., $V(x) = V(|x|)$.
(V3) V is periodic in each variable of $x_1, \cdots, x_N$.
(V4) $V_\infty := \lim_{|x|\to\infty} V(x) = ||V||_{L^\infty(\mathbf{R}^N)} < \infty$.

Theorem 1.1 (The compact case). *Let $\alpha > \frac{1}{2}$ and $2 < p+1 < 2\alpha 2^*$. Assume* (V1) ***or*** (V2)*. Then there exist $\lambda_n \to \infty$ such that* (2) *has a solution. If in addition $4\alpha \leq p+1$, there also exist $\lambda_n \to 0$ such that* (2) *has a solution.*

Theorem 1.2 (The locally compact case). *Let $\alpha > \frac{1}{2}$ and $4\alpha \leq p+1 < 2\alpha 2^*$. Assume* (V3) ***or*** (V4)*. Then* (2) *has a solution for a sequence of $\lambda_n \to \infty$ and a sequence of $\lambda_n \to 0$.*

Remark 1.3. The second author has given results on (2) in [20]. We emphasize here that for $\alpha > \frac{1}{2}$, $2\alpha 2^* > 2^*$ and the condition $4\alpha \leq p+1 < 2\alpha 2^*$ can be satisfied for all dimensions N. The result in Theorem 1.1 was proved in [20] for the case: $N = 1$, $\alpha = 1$, $2 < p+1$; and the result in Theorem 1.2 was proved in [20] under the assumptions: $N = 1$, $\alpha = 1$, $4 \leq p+1$.

Remark 1.4. It would be interesting to know whether solutions exist for all $\lambda > 0$ in (2). We shall discuss this in a forthcoming paper.

Theorems 1.1 and 1.2 will be proved in Sections 2 and 3, respectively.

2. Ground state solutions – The compact case

We consider a family of minimization problems, for $a > 0$

$$m_a = \inf_{M_a} E(u) \tag{3}$$

where

$$M_a = \{u \in X \mid ||u||_{p+1} = a\},$$

$$E(u) = \int_{\mathbf{R}^N} (|\nabla u|^2 + Vu^2)dx + 2\alpha \int_{\mathbf{R}^N} |u|^{2(2\alpha-1)}|\nabla u|^2 dx$$

and in case (V1)

$$X = \{u \in H^{1,2}(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} V(x)u^2 dx < \infty\}$$

with norm given by $||u||^2 = \int_{\mathbf{R}^N}(|\nabla u|^2 + Vu^2)dx$, and in case (V2)

$$X = H_r^{1,2}(\mathbf{R}^N) = \{u \in H^{1,2}(\mathbf{R}^N) \mid u \text{ is radial}\}.$$

In both cases, X is a subspace of $H^{1,2}(\mathbf{R}^N)$. We also need another Sobolev space $D^{1,2}(\mathbf{R}^N)$ for $N \geq 3$ which is the completion of $C_0^\infty(\mathbf{R}^N)$ under the norm $||u||^2 = \int_{\mathbf{R}^N} |\nabla u|^2 dx$. By the Sobolev inequality, $D^{1,2}(\mathbf{R}^N)$ is continuously embedded into $L^{2^*}(\mathbf{R}^N)$. Solutions of (2) will be shown to exist as minimizers of the above minimization problems which are called ground state solutions of (2). Under (V1) or (V2) we have that the embedding from X into $L^{p+1}(\mathbf{R}^N)$ is compact (e.g., [1], [11], [24], [26]).

Lemma 2.1. *For all $a > 0$, m_a is achieved at some $u_a \in M_a$ which is a weak solution of equation* (2) *with $\lambda = \lambda_a$ satisfying $\lambda_a \in (\frac{m_a}{a^{p+1}}, \frac{2\alpha m_a}{a^{p+1}})$.*

Proof. We fix $a > 0$. Let $(u_n) \in M_a$ be a minimizing sequence for m_a. Then by the compact embedding result from X into $L^q(\mathbf{R}^N)$ for $2 \le q < 2^*$, we first have $u_n \rightharpoonup u$ in X (weak convergence) and $u_n \to u$ in $L^q(\mathbf{R}^N)$ for $2 \le q < 2^*$. Since $\nabla(u_n^{2\alpha})$ is uniformly bounded in $L^2(\mathbf{R}^N)$, by the Sobolev inequality we have $||u_n^{2\alpha}||_{2^*} \le C$, which gives $||u_n||_{2\alpha 2^*} \le C$. By Hölder inequality we have $u_n \to u$ in $L^q(\mathbf{R}^N)$ for $2 \le q < 2\alpha 2^*$. Then we claim

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} (|\nabla u_n|^2 + V u_n^2)dx + 2\alpha \int_{\mathbf{R}^N} |u_n|^{2(2\alpha-1)}|\nabla u_n|^2 dx \ge E(u). \tag{4}$$

To see this let us observe that the first integral is the norm of u_n in X which makes it weakly lower semi-continuous in X. The second integral in $E(u_n)$ can be regarded as the $D^{1,2}(\mathbf{R}^N)$ norm of $v_n = (u_n)^{2\alpha}$ when $N \ge 3$. Since v_n is bounded in $D^{1,2}(\mathbf{R}^N)$ there is $v \in D^{1,2}(\mathbf{R}^N)$ such that $v_n \rightharpoonup v$ in $D^{1,2}(\mathbf{R}^N)$ (weak convergence). Also we may assert that $u_n \to u$ a.e. in $\mathbf{R}^N$ and $v_n \to v$ a.e. in $\mathbf{R}^N$. From this we have $v = u^{2\alpha}$. Therefore the second integral is bounded from below by $\int_{\mathbf{R}^N} |u|^{2(2\alpha-1)}|\nabla u|^2 dx$. Thus we have proved the claim when $N \ge 3$. For $N = 2$, we use the continuous embedding X into $L^q(\mathbf{R}^N)$ for any $2 \le q < \infty$ (instead of using $D^{1,2}(\mathbf{R}^N)$) and a similar argument works. Hence we obtain that m_a is achieved at some $u \in M_a$. Since we may assume $u_n \ge 0$ we have $u \ge 0$. By the Lagrange multiplier theorem ([7]) we conclude that u is a weak solution of (2)

$$-\Delta u + V(x)u - (\Delta(|u|^{2\alpha}))|u|^{2\alpha-2}u = \lambda_a |u|^{p-1}u, \text{ in } \mathbf{R}^N, \tag{5}$$

where λ_a is the Lagrange multiplier. Multiplying the equation by u and integrating over $\mathbf{R}^N$ we get

$$\lambda_a \in (\frac{m_a}{a^{p+1}}, \frac{2\alpha m_a}{a^{p+1}}). \tag{6}$$

In order to show there exist $\lambda_n \to \infty$ and $\lambda_n \to 0$ such that (2) has a solution, we need the following lemma which does not depend on (V1) and (V2) and will also be used again later.

Lemma 2.2. *Assume m_a is achieved for all $a > 0$. Let u_a be a minimizer and λ_a the corresponding Lagrange multiplier. Then $\lambda_a \to \infty$ as $a \to 0$ when $2 < p + 1$, and $\lambda_a \to 0$ as $a \to \infty$ when $4\alpha \le p + 1$.*

Proof. To show $\lambda_a \to \infty$ as $a \to 0$ we assume to the contrary that there exist $a_n \to 0$ such that $\lambda_n = \lambda_{a_n} \le C_1$. By (2) we have u_n uniformly bounded in X and $v_n = u_n^{2\alpha}$ uniformly bounded in $D^{1,2}(\mathbf{R}^N)$, so by embedding theorems we have u_n bounded in $L^q(\mathbf{R}^N)$ for any $2 \le q \le 2\alpha 2^*$. Especially, u_n is bounded in $L^{p+1}(\mathbf{R}^N)$ and we have with s satisfying $\frac{1}{p+1} = \frac{s}{2} + \frac{1-s}{2\alpha 2^*}$

$$\begin{aligned} & ||u_n||_{p+1}^{p+1} \\ \le\ & ||u_n||_2^{s(p+1)} ||u_n||_{2\alpha 2^*}^{(1-s)(p+1)} \\ \le\ & C(||u_n||_X^{p+1} + ||u_n^{2\alpha}||_{2^*}^{2\alpha(p+1)}) \\ \le\ & C(||u_n||_X^{p+1} + ||\nabla(u_n^{2\alpha})||_2^{2\alpha(p+1)}). \end{aligned} \tag{7}$$

Here C is a constant independent of n. On the other hand, using $||u_n||_{p+1} = a_n \to 0$ and the equation, we have

$$||u_n||_X^2 + (2\alpha)^2 \int_{\mathbf{R}^N} |\nabla u_n^{2\alpha}|^2 dx = \lambda_n \int_{\mathbf{R}^N} |u_n|^{p+1} dx \leq C_1 a_n^{p+1} \to 0.$$

Then using $\alpha(p+1) > \frac{p+1}{2}$ and the above two inequalities we have

$$\begin{aligned} &||u_n||_X^2 + (2\alpha)^2 \int_{\mathbf{R}^N} |\nabla u_n^{2\alpha}|^2 dx \\ &= \lambda_n \int_{\mathbf{R}^N} |u_n|^{p+1} dx \\ &\leq C_1 C(||u_n||_X^2 + \int_{\mathbf{R}^N} |\nabla u_n^{2\alpha}|^2 dx)^{\frac{p+1}{2}} \end{aligned} \tag{8}$$

where C is independent of n. This implies $||u_n||_X^2 + \int_{\mathbf{R}^N} |\nabla u_n^{2\alpha}|^2 dx \geq C_2$ for some $C_2 > 0$, for $p+1 > 2$. This is a contradiction with $||u_n||_X^2 + \int_{\mathbf{R}^N} |\nabla u_n^{2\alpha}|^2 dx \to 0$.

Next, we show that if $p+1 \geq 4\alpha$, $\lambda_a \to 0$ as $a \to \infty$. To see this, let us observe that if $u \in M_1, au \in M_a$. Then by a simple scaling argument, for $a \geq 1$ we have

$$m_a \leq a^{4\alpha} m_1$$

and for $a \leq 1$

$$m_a \geq a^{4\alpha} m_1.$$

Then using (6), for $4\alpha < p+1$ we have as $a \to \infty$

$$\lambda_a \leq \frac{2\alpha m_a}{a^{p+1}} \leq 2\alpha m_1 a^{4\alpha-(p+1)} \to 0.$$

The case of $p+1 = 4\alpha$ is treated next. For any $\epsilon > 0$, we first choose $u_\epsilon \in C_0^\infty(\mathbf{R}^N)$ such that $2\alpha \int_{\mathbf{R}^N} |u_\epsilon|^{2(2\alpha-1)} |\nabla u_\epsilon|^2 dx < \epsilon \int_{\mathbf{R}^N} |u_\epsilon|^{p+1} dx$. This can be obtained by choosing $u_\epsilon(x) = u_0(\epsilon x)$ for some $u_0 \in C_0^\infty(\mathbf{R}^N)$ with ϵ small. We may assume $||u_\epsilon||_{p+1} = 1$ (since $p+1 = 4\alpha$) so that $au_\epsilon \in M_a$. Then since $p+1 = 4\alpha > 2$, for a large we have

$$\begin{aligned} &m_a \leq E(au_\epsilon) \\ &\leq a^2 \int_{\mathbf{R}^N} (|\nabla u_\epsilon|^2 + V u_\epsilon^2) dx + 2\alpha a^{4\alpha} \int_{\mathbf{R}^N} |u_\epsilon|^{2(2\alpha-1)} |\nabla u_\epsilon|^2 dx - \epsilon a^{4\alpha} + \epsilon a^{4\alpha} \\ &\leq \epsilon a^{4\alpha}. \end{aligned} \tag{9}$$

This shows $\lim_{a\to\infty} \frac{m_a}{a^{4\alpha}} = 0$. Then using (6) we get $\lambda_a \to 0$ as $a \to \infty$.

Proof of Theorem 1.1. Theorem 1.1 follows from the last two lemmas.

Remark 2.3. (V1) can be replaced by any condition that guarantees the compact embedding from X into $L^{p+1}(\mathbf{R}^N)$; see for example [1], [11].

3. Ground state solutions – The locally compact case

In this section, we consider the cases where the potential V satisfies (V3) or (V4). The space X is taken as $H^{1,2}(\mathbf{R}^N)$. M_a and m_a can be defined as in the last section. Due to Lemma 2.2 we only need to prove that m_a is achieved at some $u_a \in M_a$. Since the proof is the same we just treat the case $a = 1$ and we write $m_1 = m$ and $M_1 = M$ for simplicity.

Lemma 3.1. *Let $(u_n) \subset M$ be a minimizing sequence for m. Then there is $\beta \in (0,1]$ and $x_n \in \mathbf{R}^N$ such that for any $\epsilon > 0$ there exists $R > 0$, for any $R' \geq R$*

$$\underline{\lim}_{n\to\infty} \int_{B_R(x_n)} |u_n|^{p+1} dx \geq \beta - \epsilon \tag{10}$$

and

$$\underline{\lim}_{n\to\infty} \int_{\mathbf{R}^N \setminus B_{R'}(x_n)} |u_n|^{p+1} dx \geq (1-\beta) - \epsilon. \tag{11}$$

Proof. For a minimizing sequence u_n we have that $\int_{\mathbf{R}^N} |u_n|^{2(2\alpha-1)} |\nabla u_n|^2$ is bounded which implies that $\int_{\mathbf{R}^N} |\nabla u_n^{2\alpha}|^2$ is bounded. Thus $u_n^{2\alpha}$ is uniformly bounded in $D^{1,2}(\mathbf{R}^N)$, and we have that u_n is uniformly bounded in $L^{2\alpha 2^*}(\mathbf{R}^N)$ by the Sobolev embedding. Also because u_n is bounded in $L^2(\mathbf{R}^N)$ we get that u_n is bounded in $L^q(\mathbf{R}^N)$ for all $q \in [2, 2\alpha 2^*]$ by the Hölder inequality. Especially, $v_n = u_n^{2\alpha}$ is bounded in $L^2(\mathbf{R}^N)$. This implies that $u_n^{2\alpha}$ is bounded in $H^1(\mathbf{R}^N)$. Note that $||v_n||_{\frac{p+1}{2\alpha}} = 1$. By P.L. Lions' Lemma ([16]), there is $\beta \in (0,1]$ and $x_n \in \mathbf{R}^N$ such that for any $\epsilon > 0$ there exists $R > 0$ such that for any $R' \geq R$, as $n \to \infty$,

$$\int_{B_R(x_n)} |u_n|^{p+1} dx = \int_{B_R(x_n)} |v_n|^{\frac{p+1}{2\alpha}} dx \geq \beta - \epsilon$$

and

$$\int_{\mathbf{R}^N \setminus B_{R'}(x_n)} |u_n|^{p+1} dx = \int_{\mathbf{R}^N \setminus B_{R'}(x_n)} |v_n|^{\frac{p+1}{2\alpha}} \geq (1-\beta) - \epsilon.$$

□

Proof of Theorem 1.2. We first consider the case (V3). Let u_n be a minimizing sequence. From Lemma 3.1, we get $\beta \in (0,1]$, and a sequence x_n such that (10) and (11) hold. We may assume the components of x_n are integer multiples of the periods of $V(x)$. Thus $u_n(\cdot + x_n)$ is still a minimizing sequence. If $\beta = 1$ we get a strong convergence of $u_n(\cdot + x_n) \to u$ in $L^{p+1}(\mathbf{R}^N)$, and a similar argument to that in the proof of Theorem 1.1 finishes the proof. If $\beta < 1$ we derive a contradiction as follows. For $\epsilon > 0$ and $R > 0$ given in Lemma 3.1, let $\eta_R(t)$ be a smooth function defined on $[0, \infty)$ satisfying $\eta_R(t) = 1$ for $t \leq R$ and $\eta_R(t) = 0$ for $t \geq 2R$ and $\eta_R'(t) \leq \frac{2}{R}$. Let $\eta_R^c(t) = 1 - \eta_R(t)$. Define

$$v_n(x) = \eta_R(|x - x_n|) u_n(x) \quad \text{and} \quad w_n(x) = \eta_R^c(|x - x_n|) u_n(x).$$

Then it is easy to see for n large

$$|\int |v_n|^{p+1} dx - \beta| \leq \epsilon$$

and

$$|\int |w_n|^{p+1} dx - (1-\beta)| \leq \epsilon.$$

Also a direct computation shows that

(12)
$$\begin{aligned}
&\int_{\mathbf{R}^N}(|\nabla u_n|^2+Vu_n^2)dx+2\alpha\int_{\mathbf{R}^N}|u_n|^{2(2\alpha-1)}|\nabla u_n|^2dx\\
&\geq\int_{\mathbf{R}^N}(|\nabla v_n|^2+Vv_n^2)dx+2\alpha\int_{\mathbf{R}^N}|v_n|^{2(2\alpha-1)}|\nabla v_n|^2dx\\
&\qquad+\int_{\mathbf{R}^N}(|\nabla w_n|^2+Vw_n^2)dx+2\alpha\int_{\mathbf{R}^N}|w_n|^{2(2\alpha-1)}|\nabla w_n|^2dx-\frac{C}{R},
\end{aligned}$$

where $C>0$ is independent of n,ϵ,R. Assume $u_n(\cdot+x_n)$ converges weakly to u in X and let $u_R=u|_{B_R(0)}$. Then $u_R\neq 0$ for R large because $\beta>0$. Then by the fact $p+1\geq 4\alpha>2$, we have

(13)
$$\begin{aligned}
&m+o(1)\\
&=E(u_n)\\
&\geq ||v_n||_{p+1}^{4\alpha}E(v_n/||v_n||_{p+1})+||w_n||_{p+1}^{4\alpha}E(w_n/||w_n||_{p+1})\\
&\quad+(||v_n||_{p+1}^2-||v_n||_{p+1}^{4\alpha})\int_{\mathbf{R}^N}(|\nabla v_n|^2+Vv_n^2)\\
&\quad+(||w_n||_{p+1}^2-||w_n||_{p+1}^{4\alpha})\int_{\mathbf{R}^N}(|\nabla w_n|^2+Vw_n^2)-\frac{C}{R}\\
&\geq m(||v_n||_{p+1}^{4\alpha}+||w_n||_{p+1}^{4\alpha})+(||u_R||_{p+1}^2-||u_R||_{p+1}^{4\alpha})\int_{\mathbf{R}^N}(|\nabla u_R|^2+Vu_R^2)-\frac{C}{R}\\
&\geq m[(\beta-\epsilon)^{\frac{4\alpha}{p+1}}+(1-\beta-\epsilon)^{\frac{4\alpha}{p+1}}]\\
&\qquad+(||u_R||_{p+1}^2-||u_R||_{p+1}^{4\alpha})\int_{\mathbf{R}^N}(|\nabla u_R|^2+Vu_R^2)-\frac{C}{R}.
\end{aligned}$$

Letting $n\to\infty$ and then $\epsilon\to 0$ (which implies $R\to\infty$), we get a contradiction for $p+1\geq 4\alpha$. This shows that $\beta=1$. This completes the proof for the case of (V3).

For the case of (V4) we again consider a minimizing sequence $u_n\subset M$. Applying Lemma 3.1, we get $\beta>0$ and $x_n\in\mathbf{R}^N$ such that for any $\epsilon>0$ there is $R>0$ such that (10) and (11) hold. We show here that $\beta=1$ and x_n is bounded in $\mathbf{R}^N$, which together imply that u_n converge strongly in $L^{p+1}(\mathbf{R}^N)$. We show $\beta=1$ first. Assume for contradiction $\beta<1$. Similar to the proofs above, we define v_n and w_n with ϵ and $R>0$. Let

$$m_\infty=\inf_{u\in M}(\int_{\mathbf{R}^N}(|\nabla u|^2+V_\infty u^2)dx+2\alpha\int_{\mathbf{R}^N}|u|^{2(2\alpha-1)}|\nabla u|^2dx),$$

i.e., the infimum of $E(u)$ over M with $V(x)$ replaced by V_∞ in $E(u)$. Then m_∞ is achieved by the proof in the first half above since (V3) is satisfied by a constant, say at $u\in M$ which we may assume to be positive in $\mathbf{R}^N\setminus\{0\}$. Using this u as a test function we can show that if $V(x)$ is not identically equal to V_∞, then $m<m_\infty$.

Now if $\beta \in (0,1)$, we can follow a similar argument as above to get a contradiction:

(14)

$$\begin{aligned}
& m + o(1) \\
&= E(u_n) \\
&\geq ||v_n||_{p+1}^{4\alpha} E(v_n/||v_n||_{p+1}) + ||w_n||_{p+1}^{4\alpha} E(w_n/||w_n||_{p+1}) \\
&\quad + (||v_n||_{p+1}^2 - ||v_n||_{p+1}^{4\alpha}) \int_{\mathbf{R}^N} (|\nabla v_n|^2 + V v_n^2) \\
&\quad + (||w_n||_{p+1}^2 - ||w_n||_{p+1}^{4\alpha}) \int_{\mathbf{R}^N} (|\nabla w_n|^2 + V w_n^2) - \frac{C}{R} \\
&\geq m(||v_n||_{p+1}^{4\alpha} + ||w_n||_{p+1}^{4\alpha}) + (||u_R||_{p+1}^2 - ||u_R||_{p+1}^{4\alpha}) \int_{\mathbf{R}^N} (|\nabla u_R|^2 + V u_R^2) - \frac{C}{R} \\
&\geq m[(\beta - \epsilon)^{\frac{4\alpha}{p+1}} + (1 - \beta - \epsilon)^{\frac{4\alpha}{p+1}}] \\
&\quad + (||u_R||_{p+1}^2 - ||u_R||_{p+1}^{4\alpha}) \int_{\mathbf{R}^N} (|\nabla u_R|^2 + V u_R^2) - \frac{C}{R}
\end{aligned}$$

by sending $n \to \infty$, $\epsilon \to 0$ ($R \to \infty$). Thus, $\beta = 1$.

Next, we assume that $\beta = 1$ and $|x_n| \to \infty$ as $n \to \infty$. Then we have $w_n \to 0$ in $L^2(\mathbf{R}^N)$. Then as we send $n \to \infty$, $\epsilon \to 0$,

$$\begin{aligned}
m + o(1) &= E(u_n) \\
&\geq ||v_n||_{p+1}^{4\alpha} E(v_n/||v_n||_{p+1}) - \frac{C}{R} \\
&\geq ||v_n||_{p+1}^{4\alpha} E_\infty(v_n/||v_n||_{p+1}) - \frac{C}{R} \\
&\geq m_\infty,
\end{aligned} \tag{15}$$

a contradiction with $m < m_\infty$. The proof is complete.

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