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A LOWER BOUND FOR SUMS OF EIGENVALUES OF THE LAPLACIAN

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ABSTRACT. Let $\lambda_k(\Omega)$ be the kth Dirichlet eigenvalue of a bounded domain Ω in $\mathbb{R}^n$. According to Weyl's asymptotic formula we have

$$\lambda_k(\Omega) \sim C_n(k/V(\Omega))^{2/n}.$$

The optimal in view of this asymptotic relation lower estimate for the sums $\sum_{j=1}^{k} \lambda_j(\Omega)$ has been proven by P.Li and S.T.Yau (*Comm. Math. Phys.* **88** (1983), 309-318). Here we will improve this estimate by adding to its right-hand side a term of the order of k that depends on the ratio of the volume to the moment of inertia of Ω.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq ...$ denote the eigenvalues (repeated with multiplicity) of the Dirichlet Laplacian on Ω, that is, of the eigenvalue problem

$$\begin{aligned} \Delta u + \lambda u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

The asymptotic behavior of $\lambda_k(\Omega)$ as $k \to \infty$ relates to geometric properties of the open set Ω. In fact Weyl's asymptotic formula asserts that

$$\lambda_k(\Omega) \sim C_n(\frac{k}{V(\Omega)})^{2/n} \text{ as } k \to \infty \tag{1.2}$$

where $V(\Omega)$ is the volume of Ω and $C_n = (2\pi)^2\omega_n^{-2/n}$ with ω_n being the volume of the unit ball in $\mathbb{R}^n$. Pólya proved in [4] that the above asymptotic relation is in fact a one-sided inequality if Ω is a plane domain that tiles $\mathbb{R}^2$ (and his proof also works in $\mathbb{R}^n$) and he conjectured, for any domain Ω in $\mathbb{R}^n$, the inequality

$$\lambda_k(\Omega) \geq C_n(\frac{k}{V(\Omega)})^{2/n} \tag{1.3}$$

for all $k \geq 1$.

In this direction Lieb [3] proved an inequality like (1.3) for any domain Ω in $\mathbb{R}^n$ but with a constant $\tilde{C}_n$ that differs from the constant C_n by a factor.

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Then P.Li and S.T.Yau [2] proved that on the average (1.3) is true for any domain Ω in $\mathbb{R}^n$, that is,

$$\sum_{j=1}^{k} \lambda_j(\Omega) \geq \frac{nC_n}{n+2} k^{\frac{n+2}{n}} V(\Omega)^{-\frac{2}{n}} \tag{1.4}$$

which is sharp in view of Weyl's asymptotic formula. This also gives a lower bound for each individual eigenvalue, better than previously known and tending to be optimal as the dimension $n \to \infty$. This inequality was complemented by P.Kröger in [1] who gave an upper bound for the sums of the eigenvalues depending on geometric properties of Ω that have to do with the behavior of the volume of the ε-neigbourhoods of the boundary $\partial\Omega$. Using this he obtained close to optimal estimates for individual eigenvalues that however depend on these geometric assumptions.

Here we will obtain an improvement of the estimate (1.4). Let $I(\Omega)$ denote the "moment of inertia" of Ω, that is,

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x-a|^2 \, dx. \tag{1.5}$$

Then we have the following.

Theorem 1. *For any bounded open set $\Omega \subseteq \mathbb{R}^n$ and any $k \geq 1$ we have*

$$\sum_{j=1}^{k} \lambda_j(\Omega) \geq \frac{nC_n}{n+2} k^{\frac{n+2}{n}} V(\Omega)^{-\frac{2}{n}} + M_n k \frac{V(\Omega)}{I(\Omega)} \tag{1.6}$$

where the constant M_n depends only on the dimension.

In fact one may take $M_n = \frac{c}{n+2}$, c being independent of n, as the proof will show. The proof will follow in part the argument of Li and Yau in [2].

2. Lower estimate for sums of eigenvalues

In this section we will prove Theorem 1.

By translating the open set Ω we may assume that

$$I(\Omega) = \int_{\Omega} |x|^2 \, dx. \tag{2.1}$$

We now fix a $k \geq 1$ and let $u_1, ..., u_k$ denote an orthonormal set of eigenfunctions of (1.1) corresponding to the set of eigenvalues $\lambda_1(\Omega), \lambda_2(\Omega), ..., \lambda_k(\Omega)$. We consider the Fourier transform of each eigenfunction

$$f_j(\xi) = \hat{u}_j(\xi) = (2\pi)^{-n/2} \int_{\Omega} u_j(x) e^{ix\cdot\xi} dx. \tag{2.2}$$

Plancherel's Theorem implies that $f_1, ..., f_k$ is an orthonormal set in $\mathbb{R}^n$. Since $u_1, ..., u_k$ are also orthonormal in $L^2(\Omega)$, Bessel's inequality implies that for every $\xi \in \mathbb{R}^n$

$$\sum_{j=1}^{k} |f_j(\xi)|^2 \leq (2\pi)^{-n} \int_{\Omega} \left|e^{ix\cdot\xi}\right|^2 dx = (2\pi)^{-n} V(\Omega) \tag{2.3}$$

and since

$$\nabla f_j(\xi) = (2\pi)^{-n/2} \int_\Omega ixu_j(x)e^{ix.\xi}dx \tag{2.4}$$

that

$$\sum_{j=1}^{k} |\nabla f_j(\xi)|^2 \le (2\pi)^{-n} \int_\Omega \left|ixe^{ix.\xi}\right|^2 dx = (2\pi)^{-n} I(\Omega). \tag{2.5}$$

Since each u_j vanishes on the boundary of Ω it is easy to see that (see [2])

$$\int_{\mathbb{R}^n} |\xi|^2 |f_j(\xi)|^2 d\xi = \int_\Omega |\nabla u_j(x)|^2 dx = \lambda_j(\Omega) \tag{2.6}$$

for each j. Hence setting

$$F(\xi) = \sum_{j=1}^{k} |f_j(\xi)|^2 \tag{2.7}$$

we have $0 \le F(\xi) \le (2\pi)^{-n}V(\Omega)$,

$$|\nabla F(\xi)| \le 2(\sum_{j=1}^{k} |f_j(\xi)|^2)^{1/2}(\sum_{j=1}^{k} |\nabla f_j(\xi)|^2)^{1/2} \le 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)} \tag{2.8}$$

for every $\xi \in \mathbb{R}^n$ and moreover

$$\int_{\mathbb{R}^n} F(\xi)d\xi = k \text{ and } \int_{\mathbb{R}^n} |\xi|^2 F(\xi)d\xi = \sum_{j=1}^{k} \lambda_j(\Omega). \tag{2.9}$$

Now let $F^*(\xi) = \phi(|\xi|)$ denote the decreasing radial rearrangement of F where we may assume (by approximating F) that the decreasing function $\phi : [0, +\infty) \to [0, (2\pi)^{-n}V(\Omega)]$ is absolutely continuous. Setting $\mu(t) = |\{F^* > t\}| = |\{F > t\}|$ the coarea formula implies that

$$\mu(t) = \int_t^{(2\pi)^{-n}V(\Omega)} \int_{\{F=s\}} |\nabla F|^{-1} d\sigma_s ds. \tag{2.10}$$

Since F^* is radial we have $\mu(\phi(s)) = |\{F^* > \phi(s)\}| = \omega_n s^n$ and so differentiating we get $n\omega_n s^{n-1} = \mu'(\phi(s))\phi'(s)$ for almost every s. But (2.8), (2.10) and the isoperimetric inequality imply that, with $\rho = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}$,

$$\begin{aligned} -\mu'(\phi(s)) &= \int_{\{F=\phi(s)\}} |\nabla F|^{-1} d\sigma_{\phi(s)} \ge \rho^{-1}\mathrm{Vol}_{n-1}(\{F = \phi(s)\}) \\ &\ge \rho^{-1} n\omega_n s^{n-1} \end{aligned} \tag{2.11}$$

and so

$$-\rho \le \phi'(s) \le 0 \tag{2.12}$$

for almost every s.

Now (2.9) implies that

$$k = \int_{\mathbb{R}^n} F(\xi)d\xi = \int_{\mathbb{R}^n} F^*(\xi)d\xi = n\omega_n \int_0^\infty s^{n-1}\phi(s)ds \tag{2.13}$$

and

$$\sum_{j=1}^{k} \lambda_j(\Omega) = \int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi \geq \int_{\mathbb{R}^n} |\xi|^2 F^*(\xi) d\xi = n\omega_n \int_0^\infty s^{n+1}\phi(s)ds \tag{2.14}$$

since $\xi \to |\xi|^2$ is radial and increasing.

To prove Theorem 1 we will also need the following lemma.

Lemma 1. *Let* $n \geq 1, \rho, A > 0$ *and* $\psi : [0, +\infty) \to [0, +\infty)$ *be decreasing (and absolutely continuous) such that*

$$-\rho \leq \psi'(s) \leq 0 \tag{2.15}$$

and

$$\int_0^\infty s^{n-1}\psi(s)ds = A. \tag{2.16}$$

Then

$$\int_0^\infty s^{n+1}\psi(s)ds \geq \frac{1}{n+2}(nA)^{\frac{n+2}{n}}\psi(0)^{-\frac{2}{n}} + \frac{A\psi(0)^2}{6(n+2)\rho^2}. \tag{2.17}$$

Proof. By considering the function $\alpha\psi(\beta t)$ for appropriate $\alpha, \beta > 0$ we may assume that $\rho = 1$ and $\psi(0) = 1$. We also assume that $B = \int_0^\infty s^{n+1}\psi(s)ds < \infty$, otherwise there is nothing to prove, and so $\lim_{j\to\infty} T_j^{n+1}\psi(T_j) = 0$ for some sequence (T_j) with $T_j \to \infty$ as $j \to \infty$.

Let $h(s) = -\psi'(s)$ for $s \geq 0$. Then $0 \leq h(s) \leq 1$ and $\int_0^\infty h(s)ds = \psi(0) = 1$. Moreover integration by parts shows that

$$\int_0^\infty s^n h(s)ds = n\int_0^\infty s^{n-1}\psi(s)ds = nA \tag{2.18}$$

since $\lim_{j\to\infty} T_j^{n+1}\psi(T_j) = 0$, and

$$\int_0^\infty s^{n+2}h(s)ds = \lim_{T\to\infty}(-T^{n+2}\psi(T) + (n+2)\int_0^T s^{n+1}\psi(s)ds) \leq (n+2)B. \tag{2.19}$$

Next let $a \geq 0$ be such that

$$\int_a^{a+1} s^n ds = \int_0^\infty s^n h(s)ds = nA. \tag{2.20}$$

Such an a exists since by the same argument as in Lemma 1 of [2] one can easily show that the assumptions on h imply $\int_0^\infty s^n h(s)ds \geq \int_0^1 s^n ds$. Indeed this follows by integrating the inequality $(s^n - 1)(h(s) - \xi(s)) \geq 0$ over $[0, +\infty)$ where ξ is the characteristic function of the interval $[0, 1]$. We also choose $\lambda, \mu \in \mathbb{R}$ such that the function

$$q(s) = s^{n+2} - \lambda s^n + \mu \tag{2.21}$$

satisfies $q(a) = q(a+1) = 0$. Since the derivative $q'(s)$ has at most one zero in $[0, +\infty)$ we conclude that $q(s) < 0$ in $(a, a+1)$ and $q(s) > 0$ in $[0, +\infty)\backslash(a, a+1)$ (and also $\lambda, \mu \geq 0$). Thus letting $\chi(s)$ denote the characteristic function of the interval $[a, a+1]$, the assumptions on h imply that

$$q(s)(\chi(s) - h(s)) \leq 0 \text{ on } [0, +\infty). \tag{2.22}$$

Integrating the inequality (2.22), taking into account the choice of a and using (2.19) we have

$$(n+2)B \geq \int_0^\infty s^{n+2}h(s)ds \geq \int_a^{a+1} s^{n+2}ds. \tag{2.23}$$

To estimate the last integral we take $\tau > 0$ to be chosen later and integrate the inequality

$$ns^{n+2} - (n+2)\tau^2 s^n + 2\tau^{n+2} \geq 2\tau^n(s-\tau)^2 \tag{2.24}$$

(that can be proved by dividing the left-hand side by $(s-\tau)^2$) over $[a, a+1]$ to get, also using (2.23), that

$$\begin{aligned} n(n+2)B - (n+2)\tau^2 nA + 2\tau^{n+2} &\geq 2\tau^n \int_a^{a+1} (s-\tau)^2 ds \geq \\ &\geq 2\tau^n \int_{-1/2}^{1/2} t^2 dt = \frac{\tau^n}{6}. \end{aligned} \tag{2.25}$$

Now choosing $\tau = (nA)^{1/n}$ we get

$$B \geq \frac{1}{n+2}(nA)^{\frac{n+2}{n}} + \frac{A}{6(n+2)} \tag{2.26}$$

and this completes the proof of the Lemma. □

To complete the proof of Theorem 1 we apply Lemma 1 to the function ϕ with $A = (n\omega_n)^{-1}k$, $\rho = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}$ and get in view of (2.14) that

$$\sum_{j=1}^{k} \lambda_j(\Omega) \geq \frac{n}{n+2}\omega_n^{-\frac{2}{n}} k^{\frac{n+2}{n}} \phi(0)^{-\frac{2}{n}} + \frac{ck\phi(0)^2}{(n+2)\rho^2} \tag{2.27}$$

where c is any constant such that $0 < c < \dfrac{1}{6}$.

Now observe that $0 < \phi(0) \leq (2\pi)^{-n}V(\Omega)$ and that if R is such that $\omega_n R^n = V(\Omega)$, then $I(\Omega) \geq \int_{B(R)} |x|^2\, dx = \dfrac{n\omega_n R^{n+2}}{n+2}$ and so

$$\rho \geq 2(2\pi)^{-n}\sqrt{\frac{n}{n+2}\omega_n^{-\frac{2}{n}} V(\Omega)^{\frac{n+2}{n}+1}} \geq (2\pi)^{-n}\omega_n^{-\frac{1}{n}} V(\Omega)^{\frac{n+1}{n}}. \tag{2.28}$$

On the other hand the function $g(t) = \dfrac{n}{n+2}\omega_n^{-\frac{2}{n}} k^{\frac{n+2}{n}} t^{-\frac{2}{n}} + \dfrac{ckt^2}{(n+2)\rho^2}$ would be decreasing on $(0, (2\pi)^{-n}V(\Omega)]$ if $g'((2\pi)^{-n}V(\Omega)) \leq 0$ which in view of (2.28) and since $k \geq 1$ will be satisfied if

$$c < (2\pi)^2 \omega_n^{-\frac{4}{n}}. \tag{2.29}$$

It is easy to see that we can choose c independent of n that satisfies (2.29). Then we can replace $\phi(0)$ by $(2\pi)^{-n}V(\Omega)$ in (2.27) which gives inequality (1.6) and so completes the proof of Theorem 1.

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