

SEGAL-BARGMANN TRANSFORMS
OF ONE-MODE INTERACTING FOCK SPACES
ASSOCIATED WITH GAUSSIAN AND POISSON MEASURES

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ABSTRACT. Let μ_g and μ_p denote the Gaussian and Poisson measures on \mathbb{R} , respectively. We show that there exists a unique measure $\tilde{\mu}_g$ on \mathbb{C} such that under the Segal-Bargmann transform S_{μ_g} the space $L^2(\mathbb{R}, \mu_g)$ is isomorphic to the space $\mathcal{H}L^2(\mathbb{C}, \tilde{\mu}_g)$ of analytic L^2 -functions on \mathbb{C} with respect to $\tilde{\mu}_g$. We also introduce the Segal-Bargmann transform S_{μ_p} for the Poisson measure μ_p and prove the corresponding result. As a consequence, when μ_g and μ_p have the same variance, $L^2(\mathbb{R}, \mu_g)$ and $L^2(\mathbb{R}, \mu_p)$ are isomorphic to the same space $\mathcal{H}L^2(\mathbb{C}, \tilde{\mu}_g)$ under the S_{μ_g} - and S_{μ_p} -transforms, respectively. However, we show that the multiplication operators by x on $L^2(\mathbb{R}, \mu_g)$ and on $L^2(\mathbb{R}, \mu_p)$ act quite differently on $\mathcal{H}L^2(\mathbb{C}, \tilde{\mu}_g)$.

1. INTRODUCTION

Let μ be a probability measure on \mathbb{R} having finite moments of all orders. In the paper [1] Accardi and Bożejko discovered a canonical unitary isomorphism between the Hilbert space $L^2(\mu)$ and the one-mode interacting Fock space $\Gamma(\lambda)$ associated with a sequence λ arising from μ . Under this isomorphism the number vectors $\Phi_n, n \geq 0$, correspond to the orthogonal polynomials $P_n(x)$ associated with μ and the modified field operator on $\Gamma(\lambda)$ corresponds to the multiplication operator by x on $L^2(\mu)$.

Being motivated by Accardi-Bożejko's discovery, Asai has recently introduced in [3] coherent vectors which are used to define the S_μ -transform. The S_μ -transform is shown in [3] to be a unitary operator from $L^2(\mu)$ onto a Hilbert space \mathcal{H}_λ of analytic functions on a disk $\Omega_\lambda \subset \mathbb{C}$, where λ is determined by μ . The composition of the Accardi-Bożejko isomorphism and the S_μ -transform gives the interacting Fock space counterpart of the well-known Segal-Bargmann transform (cf. [9, 10, 17, 22, 23]).

The purpose of this paper is to apply the results of Accardi-Bożejko [1] and Asai [3] to the cases of Gaussian measure μ_g and Poisson measure μ_p . In particular, when μ_g and μ_p have the same variance, we will see that by Theorems 4.6 and 4.8

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the Segal-Bargmann transforms S_{μ_g} and S_{μ_p} take $L^2(\mu_g)$ and $L^2(\mu_p)$, respectively, to the same space $\mathcal{HL}^2(\mathbb{C}, \tilde{\mu}_g)$ of analytic L^2 -functions, where $\tilde{\mu}_g$ is the Gaussian measure on \mathbb{C} . However, the Segal-Bargmann representation of multiplication by x on $L^2(\mu_g)$ is quite different from that on $L^2(\mu_p)$.

2. ONE-MODE INTERACTING FOCK SPACE

Let μ be a probability measure on \mathbb{R} having finite moments of all orders. It is well known [25] that there exist (1) a complete orthogonal system $\{P_n(x)\}_{n=0}^\infty$ of polynomials for $L^2(\mu)$ with $P_0 = 1$, (2) a sequence $\{\omega_n\}_{n=1}^\infty$ of nonnegative real numbers, and (3) a sequence $\{\alpha_n\}_{n=0}^\infty$ of real numbers such that the following equalities hold for all $n \geq 0$:

$$(2.1) \quad \begin{aligned} (x - \alpha_n)P_n(x) &= P_{n+1}(x) + \omega_n P_{n-1}(x), \\ \langle P_n, P_m \rangle_{L^2(\mu)} &= \delta_{n,m} \omega_1 \cdots \omega_n, \end{aligned}$$

where $\omega_0 P_{-1} = 0$ by convention. We have the fact that the sequence $\alpha_n = 0$ for all n if and only if μ is symmetric.

For a probability measure μ with the associated sequence $\{\omega_n\}_{n=1}^\infty$, we define a sequence $\{\lambda_n\}_{n=0}^\infty$ by

$$(2.2) \quad \lambda_0 = 1, \quad \lambda_n = \omega_1 \omega_2 \cdots \omega_n, \quad n \geq 1.$$

Assume that the sequence $\{\lambda_n\}_{n=0}^\infty$ satisfies the condition:

$$(2.3) \quad (\star) \quad \inf_{n \geq 0} \lambda_n^{\frac{1}{n}} > 0.$$

With such a sequence $\lambda = \{\lambda_n\}_{n=0}^\infty$, we define $\Gamma(\lambda)$ by

$$\Gamma(\lambda) = \left\{ (a_0, a_1, \dots, a_n, \dots) : a_n \in \mathbb{C}, \sum_{n=0}^{\infty} \lambda_n |a_n|^2 < \infty \right\}$$

and a norm $\|\cdot\|_\lambda$ on $\Gamma(\lambda)$ by

$$\|(a_n)\|_\lambda = \left(\sum_{n=0}^{\infty} \lambda_n |a_n|^2 \right)^{1/2}.$$

Then $\Gamma(\lambda)$ is a Hilbert space with norm $\|\cdot\|_\lambda$. It is called the *one-mode interacting Fock space* associated with λ [1, 2].

Define a *number vector* $\Phi_n, n \geq 0$, by

$$\Phi_n = (0, \dots, 0, \overset{n+1}{\underset{\vee}{1}}, 0, \dots).$$

The vector Φ_0 is called a *vacuum vector*. Let A be a densely defined operator on $\Gamma(\lambda)$ such that

$$A\Phi_0 = 0, \quad A\Phi_n = \omega_n \Phi_{n-1}, \quad n \geq 1.$$

The adjoint operator A^* of A is easily checked to be given by

$$A^*\Phi_n = \Phi_{n+1}, \quad n \geq 0.$$

The operators A and A^* are called the *annihilation* and *creation operators* on $\Gamma(\lambda)$, respectively. The number operator N is defined by

$$N\Phi_n = n\Phi_n, \quad n \geq 0.$$

In addition, we define an operator α_N on $\Gamma(\lambda)$ by

$$\alpha_N \Phi_n = \alpha_n \Phi_n, \quad n \geq 0.$$

Now we can state the result of Accardi and Bożejko [1]: There exists a unitary isomorphism $U : \Gamma(\lambda) \rightarrow L^2(\mu)$ satisfying the following conditions:

- (1) $U\Phi_0 = 1$,
- (2) $UA^*U^*P_n = P_{n+1}$,
- (3) $U(A + A^* + \alpha_N)U^* = Q$, where Q is the multiplication operator by x on $L^2(\mu)$.

3. SEGAL-BARGMANN TRANSFORM

Let $\{\lambda_n\}_{n=0}^\infty$ be the sequence defined in Equation (2.2) and consider the following series of a complex number z :

$$(3.1) \quad G_\lambda(z) = \sum_{n=0}^\infty \frac{1}{\lambda_n} z^n.$$

Note that by Condition (\star) in Equation (2.3) this series has a positive radius of convergence, denoted by r_λ .

Let $\Omega_\lambda = \{z \in \mathbb{C} : |z| < \sqrt{r_\lambda}\}$. For each $z \in \Omega_\lambda$, Asai [3] has introduced a *coherent vector* $E_\lambda(\cdot, z)$ with respect to the family $\{P_n\}$ in Equation (2.1) by

$$(3.2) \quad E_\lambda(x, z) = \sum_{n=0}^\infty \frac{P_n(x)}{\lambda_n} z^n, \quad x \in \mathbb{R}.$$

It is easy to see

$$\|E_\lambda(\cdot, z)\|_{L^2(\mu)} = G_\lambda(|z|^2)^{1/2}$$

and so $E_\lambda(\cdot, z) \in L^2(\mu)$ for all $z \in \Omega_\lambda$. Moreover, the set $\{E_\lambda(\cdot, z) : z \in \Omega_\lambda\}$ is linearly independent and spans a dense subspace of $L^2(\mu)$.

For $f \in L^2(\mu)$, let $S_\mu f$ be the function defined by:

$$(3.3) \quad (S_\mu f)(z) = \langle E_\lambda(\cdot, \bar{z}), f \rangle_{L^2(\mu)} = \int_{\mathbb{R}} E_\lambda(x, z) f(x) d\mu(x), \quad z \in \Omega_\lambda.$$

The mapping S_μ defined on $L^2(\mu)$ is called the *Segal-Bargmann transform*. Asai has shown in [3] that S_μ is a unitary operator from $L^2(\mu)$ onto \mathcal{H}_λ . Here \mathcal{H}_λ is given by

$$(3.4) \quad \mathcal{H}_\lambda = \left\{ F(z) = \sum_{n=0}^\infty a_n z^n : F \text{ is analytic on } \Omega_\lambda \text{ and } \sum_{n=0}^\infty \lambda_n |a_n|^2 < \infty \right\}.$$

It is a Hilbert space with the norm

$$(3.5) \quad \|F\|_{\mathcal{H}_\lambda} = \left(\sum_{n=0}^\infty \lambda_n |a_n|^2 \right)^{1/2}.$$

Now, let us introduce operators \tilde{A} and \tilde{A}^* acting on \mathcal{H}_λ by

$$\tilde{A}1 = 0, \quad \tilde{A}z^n = \omega_n z^{n-1}, \quad n \geq 1,$$

and

$$\tilde{A}^* z^n = z^{n+1}, \quad n \geq 0.$$

Operators \tilde{A} and \tilde{A}^* satisfy the commutation relation $[\tilde{A}, \tilde{A}^*]z^n = (\omega_{n+1} - \omega_n)z^n$.

The number operator \tilde{N} acting on \mathcal{H}_λ is defined by

$$\tilde{N}z^n = nz^n, \quad n \geq 0.$$

In addition, we can define an operator $\tilde{\alpha}_N$ acting on \mathcal{H}_λ by

$$\tilde{\alpha}_N z^n = \alpha_n z^n, \quad n \geq 0.$$

The operators \tilde{A} , \tilde{A}^* , \tilde{N} and $\tilde{\alpha}_N$ correspond to the operators A , A^* , N and α_N on $\Gamma(\lambda)$, respectively.

4. MAIN RESULTS

Our main results (Theorems 4.6 and 4.8 below) in this papers are concerned with the special cases when μ is a Gaussian or Poisson measure.

Case 1. $\mu =$ Gaussian measure μ_g with mean m and variance σ^2 .

From [19, 25] we have the following equalities:

$$(4.1) \quad \begin{aligned} &H_{n+1}(x; \sigma^2) - xH_n(x; \sigma^2) + \sigma^2 n H_{n-1}(x; \sigma^2) = 0, \\ &\int_{\mathbb{R}} H_n(x; \sigma^2) H_m(x; \sigma^2) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} dx = \delta_{n,m} \sigma^{2n} n!, \end{aligned}$$

where $H_n(x; \sigma^2)$ is the Hermite polynomial of degree n with parameter σ^2 . Thus the three quantities for the Gaussian measure μ_g in Equation (2.1) are given by

$$\begin{aligned} P_n(x) &= H_n(x - m; \sigma^2), \\ \omega_n &= \sigma^2 n, \\ \alpha_n &= m \quad \forall n \geq 1. \end{aligned}$$

Moreover, we have the associated quantities in Equations (2.2), (3.1), and (3.2):

$$(4.2) \quad \lambda_n = \sigma^{2n} n!,$$

$$(4.3) \quad \begin{aligned} G_\lambda(z) &= \exp\left[\frac{z}{\sigma^2}\right], \quad r_\lambda = \infty, \\ E_\lambda(x, z) &= \exp\left[\frac{z}{\sigma^2}(x - m) - \frac{z^2}{2\sigma^2}\right]. \end{aligned}$$

Obviously, Condition (\star) in Equation (2.3) is satisfied.

Case 2. $\mu =$ Poisson measure μ_p with parameter a .

From [11] we have the following equalities:

$$\begin{aligned} C_{n+1}(x; a) &= (x - n - a)C_n(x; a) - a n C_{n-1}(x; a), \\ \int_{\mathbb{R}} C_n(x; a) C_m(x; a) d\mu_p(x) &= \delta_{n,m} a^n n!, \end{aligned}$$

where $C_n(x; a)$ is the Charlier polynomial of degree n with parameter a . Thus the three quantities for the Poisson measure μ_p in Equation (2.1) are given by

$$\begin{aligned} P_n(x) &= C_n(x; a), \\ \omega_n &= a n, \\ \alpha_n &= n + a. \end{aligned}$$

Moreover, we have the associated quantities in Equations (2.2), (3.1), and (3.2):

$$\begin{aligned} \lambda_n &= a^n n!, \\ G_\lambda(z) &= \exp\left[\frac{z}{a}\right], \quad r_\lambda = \infty, \\ E_\lambda(x, z) &= e^{-z} \left(1 + \frac{z}{a}\right)^x. \end{aligned}$$

Obviously, Condition (\star) in Equation (2.3) is satisfied.

For the Poisson case, we have

$$(4.4) \quad \alpha_N = \frac{1}{a} A^* A + a = N + a,$$

which implies that

$$(4.5) \quad A^* + A + \alpha_N = \left(\frac{1}{\sqrt{a}} A^* + \sqrt{a}\right) \left(\frac{1}{\sqrt{a}} A + \sqrt{a}\right).$$

Proposition 4.1. *For the Gaussian measure μ_g on \mathbb{R} with mean m and variance σ^2 , the following equalities hold:*

- (a) $S_{\mu_g} H_n(\cdot - m; \sigma^2) = z^n$,
- (b) $S_{\mu_g} UAU^* H_n(\cdot - m; \sigma^2) = \sigma^2 n z^{n-1}$,
- (c) $S_{\mu_g} UA^*U^* H_n(\cdot - m; \sigma^2) = z^{n+1}$,
- (d) $S_{\mu_g} ((x - m)H_n(\cdot - m; \sigma^2)) = z^{n+1} + \sigma^2 n z^{n-1}$.

Proof. Conclusion (a) follows from Equations (3.2), (3.3), (4.1), and (4.2). (b) follows from (a) and the fact that $UAU^* H_n(\cdot - m; \sigma^2) = \sigma^2 n H_{n-1}(\cdot - m; \sigma^2)$. (c) follows from (a) and the fact that $UA^*U^* H_n(\cdot - m; \sigma^2) = H_{n+1}(\cdot - m; \sigma^2)$. (d) follows from (a) and Equation (4.1). \square

Proposition 4.2. *For the Poisson measure μ_p on \mathbb{R} with parameter a , the following equalities hold:*

- (a) $S_{\mu_p} C_n(\cdot; a) = z^n$,
- (b) $S_{\mu_p} UAU^* C_n(\cdot; a) = a n z^{n-1}$,
- (c) $S_{\mu_p} UA^*U^* C_n(\cdot; a) = z^{n+1}$,
- (d) $S_{\mu_p} (x C_n(\cdot; a)) = z^{n+1} + (n + a) z^n + a n z^{n-1}$.

Proof. The idea is similar to the proof of Proposition 4.1 with the Hermite polynomials being replaced by the Charlier polynomials. \square

We point out that for the Poisson case, we have the equalities for operators acting on \mathcal{H}_λ

$$\tilde{\alpha}_N = \frac{1}{a} \tilde{A}^* \tilde{A} + a = \tilde{N} + a$$

and

$$\tilde{A}^* + \tilde{A} + \tilde{\alpha}_N = \left(\frac{1}{\sqrt{a}} \tilde{A}^* + \sqrt{a}\right) \left(\frac{1}{\sqrt{a}} \tilde{A} + \sqrt{a}\right)$$

which correspond to the Equations (4.4) and (4.5), respectively.

Then we can apply Propositions 4.1 and 4.2 to get the next result.

Corollary 4.3. *Let μ_g be the Gaussian measure on \mathbb{R} with mean m and variance σ^2 . Let μ_p be the Poisson measure on \mathbb{R} with the parameter a . Then*

- (1) $S_{\mu_g}((x-m)H_n(\cdot-m; \sigma^2)) = (\tilde{A}^* + \tilde{A})S_{\mu_g}H_n(\cdot-m; \sigma^2)$.
 (2) $S_{\mu_p}(xC_n(\cdot; a)) = (\tilde{A}^* + \tilde{A} + \tilde{\alpha}_N)S_{\mu_p}C_n(\cdot; a)$.

Now, suppose μ is a probability measure on \mathbb{R} with finite moments of all orders. Let λ be the sequence associated with μ as given in Equation (2.2). With this λ , we have a Hilbert space \mathcal{H}_λ of analytic functions on Ω_λ in Equation (3.4) with norm $\|\cdot\|_{\mathcal{H}_\lambda}$ in Equation (3.5). Consider the following

Question. Does there exist a unique measure $\tilde{\mu}$ on Ω_λ such that $F \in \mathcal{H}_\lambda$ if and only if F is analytic on Ω_λ and $F \in L^2(\Omega_\lambda, \tilde{\mu})$ with

$$(4.6) \quad \|F\|_{\mathcal{H}_\lambda}^2 = \int_{\Omega_\lambda} |F(z)|^2 d\tilde{\mu}(z)?$$

Bargmann [9] considered the equality in Equation (4.6) for the multidimensional standard Gaussian case. See also the paper by Gross and Malliavin [13].

Our main results answer the above question for Gaussian and Poisson measures. For convenience, let $\mathcal{HL}^2(\Omega_\lambda, \tilde{\mu})$ denote the Hilbert space of analytic functions F on Ω_λ which are square integrable with respect to $\tilde{\mu}$. The norm on $\mathcal{HL}^2(\Omega_\lambda, \tilde{\mu})$ is the $L^2(\tilde{\mu})$ -norm.

To answer the above question, we consider a criterion to check whether a measure ν satisfying the equation

$$(4.7) \quad \int_{\mathbb{C}} \bar{z}^m z^n d\nu(z) = \gamma_{m,n}$$

is unique for given moments $\{\gamma_{m,n}\}$.

Proposition 4.4. *Suppose $\{\gamma_{m,n}\}$ satisfies the condition*

$$(4.8) \quad \lim_{n \rightarrow \infty} \frac{\gamma_{n,n}^{1/n}}{n^2} = 0.$$

Then the measure ν satisfying Equation (4.7) is unique.

Proof. Apply the Schwartz inequality to get

$$|\gamma_{m,n}| = \left| \int_{\mathbb{C}} \bar{z}^m z^n d\nu(z) \right| \leq (\gamma_{m,m} \gamma_{n,n})^{\frac{1}{2}}.$$

Therefore the function

$$g(t, s) = \sum_{m,n=0}^{\infty} \gamma_{m,n} \frac{t^m s^n}{m!n!}$$

converges absolutely for $t, s \in \mathbb{C}$ by Equation (4.8). In fact, we see that

$$(4.9) \quad \begin{aligned} \int_{\mathbb{C}} \sum_{m,n=0}^{\infty} \frac{|(t\bar{z})^m (sz)^n|}{m!n!} d\nu(z) &\leq \sum_{m,n=0}^{\infty} \frac{(\gamma_{m,m} \gamma_{n,n})^{\frac{1}{2}}}{m!n!} |t|^m |s|^n \\ &\leq \left(\sum_{n=0}^{\infty} \frac{\gamma_{n,n}^{\frac{1}{2}}}{n!} R^n \right)^2 < \infty \end{aligned}$$

for $t, s \in \mathbb{C}$, $|t|, |s| \leq R$. Equation (4.9) implies that $\exp[t\bar{z} + sz]$ is integrable with respect to a measure ν and

$$\int_{\mathbb{C}} \exp[t\bar{z} + sz] d\nu(z) = g(t, s)$$

holds. Therefore the characteristic function of a measure $d\nu(x, y) = d\nu(z)$, $z = x + iy$, satisfies the equality

$$\int_{\mathbb{R}^2} \exp[i\xi x + i\eta y] d\nu(x, y) = g\left(\frac{i\xi + \eta}{2}, \frac{i\xi - \eta}{2}\right)$$

for any $\xi, \eta \in \mathbb{R}$. Hence ν is unique. □

Remark 4.5. We are unable to find any literature dealing with the moment problem for measures on the complex plane \mathbb{C} . The above proposition gives a sufficient condition for the uniqueness of a measure on \mathbb{C} in the moment problem. On the other hand, many authors have studied the moment problem for measures on \mathbb{R} .

Theorem 4.6. *Let μ_g be the Gaussian measure with mean m and variance σ^2 . Let \mathcal{H}_λ be the Hilbert space associated with μ_g as in Equation (3.4), i.e., $\Omega_\lambda = \mathbb{C}$ and $\lambda_n = \sigma^{2n}n!$. Then there exists a unique measure $\tilde{\mu}$ on \mathbb{C} such that $\mathcal{H}_\lambda = \mathcal{HL}^2(\mathbb{C}, \tilde{\mu})$ and Equation (4.6) holds.*

Proof. Consider the above question for the Gaussian measure μ_g with mean m and variance σ^2 . In this case, we have $\Omega_\lambda = \mathbb{C}$ by Equation (4.3) and $\lambda_n = \sigma^{2n}n!$ by Equation (4.2).

The uniqueness of $\tilde{\mu}$ follows from Proposition 4.4. Thus we only need to find a measure $\tilde{\mu}$ on \mathbb{C} such that

$$(4.10) \quad \int_{\mathbb{C}} \bar{z}^m z^n d\tilde{\mu}(z) = \delta_{m,n} \sigma^{2n} n!$$

Suppose $\tilde{\mu}$ is given by $d\tilde{\mu}(z) = \frac{1}{2\pi} d\rho(r) d\theta$ for $z = re^{i\theta}$. Then Equation (4.10) becomes

$$\frac{1}{2\pi} \int_0^\infty \left(\int_0^{2\pi} e^{-i(m-n)\theta} d\theta \right) r^{m+n} d\rho(r) = \delta_{m,n} \sigma^{2n} n!$$

This equality is obviously valid when $m \neq n$. Thus we are looking for ρ such that

$$(4.11) \quad \int_0^\infty r^{2n} d\rho(r) = \sigma^{2n} n!$$

But it is easy to see that ρ is given by

$$(4.12) \quad d\rho(r) = \frac{2}{\sigma^2} r \exp\left[-\frac{r^2}{\sigma^2}\right] dr.$$

The resulting measure

$$d\tilde{\mu}_g(z) = \frac{1}{\pi\sigma^2} r \exp\left[-\frac{r^2}{\sigma^2}\right] dr d\theta, \quad z = re^{i\theta},$$

is a *Gaussian measure* on \mathbb{C} . Hence we have proved the existence of a measure $\tilde{\mu}$ satisfying Equation (4.6). □

Remark 4.7. The measure given in Equation (4.12) is the unique measure satisfying Equation (4.11) since the sequence $\lambda_n = \sigma^{2n}n!$ satisfies the condition

$$\sum_{n=1}^\infty (\lambda_n)^{-\frac{1}{2n}} = \infty$$

in Theorem 1.11 of the book by Shohat and Tamarkin [24]. However, in general it is not true that a measure $d\tilde{\mu}(z)$ on \mathbb{C} can be written as $\frac{1}{2\pi} d\rho(r) d\theta$. Thus we really need Proposition 4.4 to show the uniqueness of $\tilde{\mu}$.

Observe that we can also apply the above arguments to the Poisson measure μ_p . Thus we have the corresponding theorem for μ_p .

Theorem 4.8. *Let μ_p be the Poisson measure with parameter a . Let \mathcal{H}_λ be the Hilbert space associated with μ_p as in Equation (3.4), i.e., $\Omega_\lambda = \mathbb{C}$ and $\lambda_n = a^n n!$. Then there exists a unique measure $\tilde{\mu}$ on \mathbb{C} such that $\mathcal{H}_\lambda = \mathcal{HL}^2(\mathbb{C}, \tilde{\mu})$ and Equation (4.6) holds.*

In fact, it is easy to see that the unique measure $\tilde{\mu}$ for μ_p is also a Gaussian measure on \mathbb{C} with a replacing σ^2 in Equation (4.12). In particular, we see that when $a = \sigma^2$ (i.e., μ_g and μ_p have the same variance) the Segal-Bargmann representing spaces for μ_g and μ_p are the same space, namely, the space $\mathcal{HL}^2(\mathbb{C}, \tilde{\mu}_g)$. However, the multiplication operators for μ_g and μ_p are decomposed quite differently on $\mathcal{HL}^2(\mathbb{C}, \tilde{\mu}_g)$.

Remark 4.9. Hudson-Parthasarathy [14] and Ito-Kubo [15] obtained similar results presented in this section from the viewpoint of quantum stochastic calculus and white noise calculus (cf. [19]), respectively. On the relationship between infinite dimensional Gaussian analysis and the Segal-Bargmann transform, see the papers by Asai-Kubo-Kuo [5, 6, 7, 8], Cochran-Kuo-Sengupta [12], Kubo-Yokoi [18], Lee [20], Segal [22, 23], Yokoi [26] and the references therein.

Remark 4.10. Recently, the q -deformed versions of Theorems 4.6 and 4.8 have been discussed by the first author [4] following the technique in [3]. We remark that the case of q -deformed Gaussian measure has been investigated earlier by van Leeuwen-Maassen [21]. However, their method of analysis is slightly different from that in [3, 4]. In addition, Królak examined the Segal-Bargmann space associated with the q -commutation relation for $q > 1$.

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