BERNSTEIN–WALSH INEQUALITIES AND THE EXPONENTIAL CURVE IN \( \mathbb{C}^2 \)

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(Communicated by Juha M. Heinonen)

Abstract. It is shown that for the pluripolar set \( K = \{ (z, e^z) : |z| \leq 1 \} \) in \( \mathbb{C}^2 \) there is a global Bernstein–Walsh inequality: If \( P \) is a polynomial of degree \( n \) on \( \mathbb{C}^2 \) and \( |P| \leq 1 \) on \( K \), this inequality gives an upper bound for \( |P(z, w)| \) which grows like \( \exp(\frac{1}{4} n^2 \log n) \). The result is used to obtain sharp estimates for \( |P(z, e^z)| \).

1. Introduction

If \( X \) is a non-pluripolar compact set in \( \mathbb{C}^k \) and \( P \) is a polynomial of degree \( n \) on \( \mathbb{C}^k \), the Bernstein–Walsh inequality is (see [K])
\[
|P(z)| \leq \|P\|_{X} e^{nV_X(z)},
\]
where \( \|P\|_{X} \) is the uniform norm of \( P \) on \( X \) and \( V_X(z) \) is the extremal function of \( X \). For example, if \( z = (z_1, \ldots, z_k) \) and \( X = \Delta^k = \{ z \in \mathbb{C}^k : |z_j| \leq 1, 1 \leq j \leq k \} \) is the unit polydisk, then
\[
V_X(z) = L(z) = \max\{ \log^+ |z_1|, \ldots, \log^+ |z_k| \}.
\]

If \( X \) is pluripolar, then, in general, such estimates are impossible. For example, if \( X \) is any piece of an algebraic curve \( \Gamma = \{ (z, w) \in \mathbb{C}^2 : P(z, w) = 0 \} \), where \( P \) is a polynomial, then \( \| eP + 1 \|_X = 1 \) for every \( c > 0 \) and there are no upper bounds on \( eP + 1 \).

We consider the case when \( \Gamma = \{ (z, w) \in \mathbb{C}^2 : w = f(z) \} \) and the compact set
\[
K = \{ (z, f(z)) \in \mathbb{C}^2 : |z| \leq 1 \},
\]
where \( f \) is an entire transcendental function. Then any non-trivial polynomial is not identically equal to 0 on \( K \). Therefore a compactness argument shows that, for every \( n \), there is a number \( c_n > 0 \) such that for any polynomial \( P(z, w) \) of degree at most \( n \) the norm \( \|P\|_{\Delta^2} \leq c_n \|P\|_K \). Hence for every \( (z, w) \in \mathbb{C}^2 \)
\[
|P(z, w)| \leq \|P\|_K E_n(f)e^{nL(z, w)},
\]
where \( E_n(f) \) is the least value of \( c_n \). (See also Section 2.)

Inequality (2) can be viewed as a transcendental global version of the Bernstein–Walsh inequality ([I]), provided that one can obtain good estimates for \( E_n(f) \). Moreover, the numbers \( E_n(f) \) can serve as a measure of transcendency of \( f \): A "less
transcendental” function $f$ has larger numbers $E_n(f)$. Note that if $f$ was algebraic, hence a polynomial of degree $l$, then $E_n(f) = +\infty$ for every $n \geq l$.

In this paper we study the classical case of $f(z) = e^z$ and we let $E_n = E_n(e^z)$. For this function we prove the following global Bernstein–Walsh inequality:

**Theorem 1.1.** If $f(z) = e^z$, then there exists a constant $C_1 > 0$ so that

$$
\exp \left( \frac{n^2 \log n}{2} - C_1 n^2 \right) \leq E_n \leq \exp \left( \frac{n^2 \log n}{2} + C_1 n^2 \right),
$$

for all $n \geq 1$. If $P$ is a polynomial of degree $n$ on $\mathbb{C}^2$, then

$$
|P(z, w)| \leq \|P\|_K \exp \left( \frac{n^2 \log n}{2} + C_1 n^2 + nL(z, w) \right).
$$

Thus, despite the pluripolarity of $K$, there is an upper estimate for the absolute value of polynomials, which grows asymptotically as $\exp(n^2 \log n)$. This is not much worse than $\exp(n)$ in the classical Bernstein–Walsh inequality $\mathcal{I}$. Moreover, this estimate is asymptotically sharp.

Inequality $\mathcal{I}$ improves when $(z, w) \in \Gamma$. In $\mathcal{T}$ (see also $\mathcal{B}$) it was proved that

$$
|P(z, e^z)| \leq \|P\|_K e^{n^2 \log^+ |z| + 6n|z|}.
$$

This inequality was used to prove deep theorems concerning the algebraic independence of values of $e^z$.

For a general transcendental function $f$ we introduce the function

$$
m_n(r) = \sup \{ \log |P(z, f(z))| : \deg P \leq n, \|P\|_K \leq 1, |z| \leq r \}.
$$

The numbers $m_n(r)$ can also serve as a measure of transcendency of $f$. Let $(|z| - 1)^+ = \max\{|z| - 1, 0\}$. As a consequence of Theorem 1.1, we prove the following restricted Bernstein–Walsh inequality for $f(z) = e^z$:

**Theorem 1.2.** There exists an absolute constant $C_2 > 0$ such that for every polynomial $P$ of degree $n \geq 1$ on $\mathbb{C}^2$ and every $z \in \mathbb{C}$ we have

$$
|P(z, e^z)| \leq \|P\|_K \exp \left[ \frac{n^2}{2} \left( \log^+ |z| + C_2 \frac{(|z| - 1)^+}{1 + \log n} \right) \right].
$$

Moreover $\lim_{n \to \infty} m_n(r)/n^2 = \frac{1}{2} \log r$, locally uniformly for $r \geq 1$.

This theorem provides the exact asymptotic behavior of the functions $m_n(r)$. It also improves Tijdeman’s estimate if one fixes $z$ and lets $n \to \infty$. On the other hand, if one fixes $n$ and lets $z \to \infty$, then Tijdeman’s estimate is better (at least if $n$ is large).

There is a fundamental difference between classical and transcendental Bernstein–Walsh inequalities. In the classical case $\mathcal{I}$ the extremal function $V_X$ is given by (see $\mathcal{K}$)

$$
V_X(z, w) = \lim_{n \to \infty} \frac{1}{n} \sup \{ \log^+ |P(z, w)| : \deg P = n, \|P\|_X \leq 1 \}.
$$

In the transcendental case $\mathcal{I}$ it follows from Theorem 1.1 and the Hartogs lemma that

$$
\limsup_{n \to \infty} \frac{2}{n^2 \log n} \sup \{ \log^+ |P(z, w)| : \deg P = n, \|P\|_K \leq 1 \} = 1.$$
everywhere on \( \mathbb{C}^2 \) except a pluripolar set. Moreover, by Theorem 1.2
\[
\limsup_{n \to \infty} \frac{2}{n^2 \log n} \sup \{ \log^+ |P(z,e^z)| : \deg P = n, \|P\|_K \leq 1 \} = 0.
\]

The next proposition holds for all entire transcendental functions.

**Proposition 1.3.** If \( f \) is an entire transcendental function, then
\[
m_n(r) \geq \frac{n^2 + 3n}{2} \log r
\]
for every \( r \geq 1 \). Moreover for \( r \geq 1 \)
\[
E_n(f) \geq \exp \left( \frac{n^2 + 3n}{2} \log r - nL(r, M_f(r)) \right).
\]
If \( f \) is of finite order of growth \( < \rho \), or of finite order \( \rho \) and finite type, then
\[
E_n(f) \geq \exp \left( \frac{n^2 \log n}{2\rho} - Cn^2 \right)
\]
for all \( n \geq 1 \), where \( C = C(f) > 0 \).

Proposition 1.3 and the previous theorem imply that the function \( e^z \) provides asymptotically the smallest possible functions \( m_n(r) \).

We are grateful to Norm Levenberg for the introduction to the problem and discussions.

2. Preliminaries

We use the following notation. If \( g \) is an entire holomorphic function we let \( M_g(r) = \max \{|g(z)| : |z| = r\} \).

For \( n \geq 0 \) we denote by \( \mathcal{P}_n \) the space of polynomials \( P \in \mathbb{C}[z,w] \) of degree at most \( n \). Then \( \dim \mathcal{P}_n = (n+1)(n+2)/2 = N+1 \), where \( N = (n^2 + 3n)/2 \).

Let \( f \) be an entire transcendental function. For any polynomial \( P \in \mathcal{P}_n \) we denote by \( P_* \) the entire function
\[
P_*(z) = P(z,f(z)), \; z \in \mathbb{C},
\]
so \( \|P\|_K = M_{P_*}(1) \). Since \( f \) is transcendental, it follows that \( \| \cdot \|_K \) is a norm on each vector space \( \mathcal{P}_n \). As \( \mathcal{P}_n \) are finite dimensional we have
\[
E_n(f) = \sup \{ \|P\|_{\Delta^2} : P \in \mathcal{P}_n, \|P\|_K \leq 1 \} < +\infty,
\]
for each \( n \geq 0 \). Note that \( E_0(f) = 1 \) and \( E_n(f) \leq E_{n+1}(f) \).

Inequality (2) implies that the function
\[
u_n(z) = \sup \{ \log |P_*(z)| : P \in \mathcal{P}_n, \|P\|_K \leq 1 \}
\]
is well defined. It is easy to see by a normal family argument that \( u_n \) is a non-negative continuous subharmonic function on \( \Delta \) and \( u_n = 0 \) on \( \Delta \). We have \( m_n(r) = \max \{ u_n(z) : |z| \leq r \} \), hence \( m_n(r) \) is a continuous increasing convex function of \( \log r \).
We need the following simple lemma:

**Lemma 2.1.** The following inequalities hold for every integer \( m > 0 \):

\[
\log(m + 1) \leq \sum_{j=1}^{m} \frac{1}{j} \leq \log m + 1,
\]

\[
m \log m - m + 1 \leq \sum_{j=1}^{m} \log j \leq (m + 1) \log m - m + 1,
\]

\[
\frac{m^2 \log m}{2} - \frac{m^2}{4} + \frac{1}{4} \leq \sum_{j=1}^{m} j \log j \leq \frac{m^2 \log m}{2} - \frac{m^2}{4} + m \log m + \frac{1}{4}.
\]

**Proof.** The proof is elementary. For instance, the third inequality follows using \( \int_{j-1}^{j} x \log x \, dx \leq j \log j \leq \int_{j}^{j+1} x \log x \, dx \).

\[\square\]

3. Proofs

We first prove Proposition 1.3 which was stated for arbitrary entire transcendental functions \( f \). Recall the notations \( N = (n^2 + 3n)/2 \) and \( P_\ast(z) = P(z, f(z)) \).

**Proof of Proposition 1.3.** Since \( \dim \mathcal{P}_n = N + 1 \), there exists \( P \in \mathcal{P}_n \), \( P \neq 0 \), such that the vanishing order of \( P_\ast \) at 0 is at least \( N \). We let \( g(z) = P_\ast(z)/z^N \), so

\[M_{P_\ast}(1) = M_g(1) \leq M_g(r) = M_{P_\ast}(r)/r^N,\]

provided that \( r \geq 1 \). This and the definition of \( m_n(r) \) clearly imply (4). Using (2) with \( w = f(z) \) and \( |z| \leq r \) we get

\[r^N \leq M_{P_\ast}(r)/M_{P_\ast}(1) \leq E_n(f) \exp [nL(r, M_f(r))],\]

so (5) follows.

In the case when \( f \) is of finite order of growth, we have \( \log^+ M_f(r) \leq C r^\rho \) for every \( r \geq 1 \). The conclusion follows by taking \( r = n^{1/\rho} \) in (5) and by using the above estimate on \( \log^+ M_f(r) \).

\[\square\]

In the following proofs we have \( f(z) = e^z \), so \( P_\ast(z) = P(z, e^z) \).

**Proof of Theorem 1.1.** The lower estimate of \( E_n \) follows from Proposition 1.3 since \( f(z) = e^z \) has order of growth \( \rho = 1 \) and type 1 with respect to this order. Moreover, (3) follows from (2) and the upper bound of \( E_n \). To prove the upper bound, we introduce the following notation. Let \( d = \frac{d}{dz} \). For any polynomial \( R(\lambda) = \sum_{j=0}^{m} c_j \lambda^j \) we denote by \( D_R \) the constant-coefficient differential operator

\[D_R = R(d) = \sum_{j=0}^{m} c_j \frac{d^j}{dz^j}.\]

Then for any integer \( t \geq 0 \) and any \( \alpha \in \mathbb{C} \) we have

\[D_R[e^t e^{\alpha z}] \big|_{z=0} = \sum_{j=t}^{\infty} c_j \frac{j!}{(j-t)!} \alpha^{j-t} = \frac{d^t R}{dz^t} \big|_{\lambda=\alpha} = R^{(t)}(\alpha).\]

Now fix \( P \in \mathcal{P}_n \), \( n \geq 1 \), with \( \|P\|_K \leq 1 \). By Cauchy’s estimates we have

\[P^{(l)}(0) \leq l, \forall l \geq 0.\]
We write

\[ P(z, w) = \sum_{k=0}^{n} P_{n-k}(z)w^k, \quad P_{n-k}(z) = \sum_{j=0}^{n-k} c_{kj} z^j. \]

In the sequel, we denote by \( C \) all absolute constants involved in our estimates.
(They may change from one inequality to the next.)

**Proposition 3.1.** There exists a constant \( C \) such that

\[ M_{P_{n-k}(1)} \leq \exp\left( \frac{n^2 \log n}{2} + Cn^2 \right), \quad \forall k \in \{0, \ldots, n\}. \]

Note that this proposition implies Theorem 1. Indeed, \( \|P\|_{\Delta^2} \leq n \sum_{k=0}^{n} M_{P_{n-k}(1)} \leq \exp\left( \frac{n^2 \log n}{2} + Cn^2 \right) \), \( \forall k \in \{0, \ldots, n\} \). So the same estimate holds for \( E_n \) since \( P \) is arbitrary with \( \|P\|_K \leq 1 \).

In order to prove Proposition 3.1, we fix \( k \in \{0, \ldots, n\} \). We will estimate the coefficients \( c_{kj} \) of \( P_{n-k} \) by using the differential operators given by the polynomials

\[ R_k^* (\lambda - l) = \prod_{l=0, l \neq k}^{n} (\lambda - l)^{n-l+1}, \]

\[ R_k (\lambda) = R_k^* (\lambda - k)^j, \quad j = 0, \ldots, n - k. \]

Note that \( \deg R_{k,n-k} = N \). By (6) we have

\[ \alpha_{kj} := D_{R_k^*} P^*_k (z) \big|_{z=0} = (D_{R_k^*} (d - k)^j) \left[ \sum_{l=0}^{n-k} c_{kl} z^l e^{kz} \right] \big|_{z=0} = D_{R_k} \left[ \sum_{l=j}^{n-k} c_{kl} \frac{l!}{(l-j)!} z^{l-j} e^{kz} \right] \big|_{z=0} = \sum_{l=j}^{n-k} c_{kl} \frac{l!}{(l-j)!} R_k^{(l-j)} (k). \]

We write

\[ c_{kl}' = l! c_{kl}, \quad l = 0, \ldots, n - k, \]

\[ r_{kt} = R_k^{(t)} (k)/t!, \quad t = 0, \ldots, n - k. \]

Then \( c_{kl}' \) are the unique solution of the triangular system

\[ \sum_{l=j}^{n-k} r_{k,l-j} c_{kl}' = \alpha_{kj}, \quad j = 0, \ldots, n - k, \]

which yields

\[ c_{k,n-k} = \frac{\alpha_{k,n-k-j}}{r_{k0}} - \sum_{l=1}^{j} \frac{r_{kl}}{r_{k0}} c_{k,n-k-j+l}, \quad j = 0, \ldots, n - k. \]

In order to estimate the coefficients \( c_{kj}' \), we obtain first bounds for \( \alpha_{kj}, r_{k0} \) and \( r_{kl}/r_{k0} \). This is done in a sequence of lemmas.
Lemma 3.2. For all \( k = 0, \ldots, n \) and \( j = 0, \ldots, n - k \) we have

\[
|\alpha_{kj}| \leq e^{n^2 \log n + C n \log n},
\]

where \( C > 0 \) is an absolute constant.

Proof. We write \( R_{kj}(\lambda) = \sum_{l=0}^{N_j} s_l \lambda^l \), \( N_j = \deg R_{kj} = N - (n - k - j) \), and define

\[
|R_{kj}|(\lambda) = \sum_{l=0}^{N_j} |s_l| \lambda^l = (\lambda + k)^j \prod_{l=0, l \neq k}^{n} (\lambda + l)^{n-l+1}.
\]

Using Cauchy’s estimates (7) and Stirling’s formula \( \log(l!) \leq e(l/e)^l \sqrt{l} \), for \( l \geq 1 \) (see \([R]\)), we get:

\[
|\alpha_{kj}| \leq \sum_{l=0}^{N_j} |s_l|! \leq e \sum_{l=0}^{N_j} |s_l|(l/e)^l \sqrt{l}
\]

\[
\leq e \sqrt{N} \sum_{l=0}^{N_j} |s_l| N^l = e \sqrt{N} |R_{kj}|(N)
\]

\[
\leq e \sqrt{N} \prod_{l=0}^{n} (N + l)^{n-l+1} \leq e \sqrt{N} \exp\left(\log(N + n) \sum_{l=1}^{n+1} l\right).
\]

Since \( \log(N + n) \leq 2 \log n + 2/n \), this yields \( |\alpha_{kj}| \leq \exp(n^2 \log n + C n \log n) \). \( \square \)

Lemma 3.3. If \( 0 \leq k \leq n \), then \( |r_{k0}| \geq \exp\left(\left(n^2 \log n\right)/2 - C n^2\right) \), where \( C > 0 \) is an absolute constant.

Proof. We have by (11) and (11) that

\[
|r_{k0}| = |R_k(k)| = n \prod_{l=0, l \neq k}^{n} |k - l|^{n-l+1},
\]

so after a direct calculation we get

\[
\log |r_{k0}| = (n - k + 1) \left( \sum_{l=1}^{k} \log l + \sum_{l=1}^{n-k} \log l \right) + \sum_{l=1}^{k} l \log l - \sum_{l=1}^{n-k} l \log l.
\]

Using Lemma 2.1 with the convention that \( x \log x = 0 \) for \( x = 0 \), it follows that for every \( k = 0, \ldots, n \) we have

\[
\log |r_{k0}| \geq \left( n - k + 1 \right) \left[ k \log k + (n - k) \log(n - k) - n \right] + \frac{k^2 \log k}{2} - \frac{k^2}{4} - \frac{(n-k)^2 \log(n-k)}{2}
\]

\[
+ \frac{(n-k)^2}{4} - (n-k) \log(n-k)
\]

\[
\geq k \left( n - k \right) \log k + \frac{(n-k)^2}{2} \log(n-k) - C n^2
\]

\[
= \frac{n^2 \log n}{2} + n^2 F\left( \frac{k}{n} \right) - C n^2 \geq \frac{n^2 \log n}{2} - C n^2,
\]

where \( F(x) = x(1 - x/2) \log x + 0.5(1 - x)^2 \log(1 - x) > -C \) for all \( x \in [0, 1] \) and some constant \( C > 0 \). \( \square \)
Lemma 3.4. For any \( k = 0, \ldots, n \) and \( j = 0, \ldots, n - k \) we have
\[
|r_{kj}|/|r_{k0}| \leq e^2 n^2 (n + 1)^j.
\]

Proof. We fix \( \epsilon \in (0, 1) \), to be chosen later in terms of \( n \). Using the definition \( 11 \) of \( r_{kj} \) and applying Cauchy's estimates to \( R_k \), we obtain that \( |r_{kj}| \leq M/\epsilon^j \), where \( M = \max\{|R_k(\lambda)| : |\lambda - k| = \epsilon\} \). If \( \lambda \) is on the circle \( |\lambda - k| = \epsilon \), then
\[
|R_k(\lambda)| \leq \prod_{l=0}^{k-1} (k + \epsilon - l)^{n-l+1} \prod_{l=k+1}^{n} (l - k + \epsilon)^{n-l+1} = |r_{k0}| F_k(\epsilon),
\]
where
\[
F_k(\epsilon) = \prod_{l=0}^{k-1} \left( 1 + \frac{\epsilon}{k - l} \right)^{n-l+1} \prod_{l=k+1}^{n} \left( 1 + \frac{\epsilon}{l - k} \right)^{n-l+1}.
\]
Therefore \( |r_{kj}|/|r_{k0}| \leq F_k(\epsilon)/\epsilon^j \). Using Lemma 2.1 we get
\[
\log F_k(\epsilon) \leq 2(n + 1) \sum_{l=1}^{n} \log \left( 1 + \frac{\epsilon}{l} \right) \leq 2\epsilon(n + 1) \sum_{l=1}^{n} \frac{1}{l} \leq 2\epsilon n (\log n + 1).
\]
Choosing \( \epsilon = 1/(n + 1) \) we obtain \( \log F_k(\epsilon) \leq 2(\log n + 1) \), hence the lemma follows.

We can now estimate the coefficients \( c_{kj}^{'} \) by using the system of equations \( 12 \). Lemmas 3.2 and 3.3 imply that
\[
\frac{|\alpha_{kj}|}{|r_{k0}|} \leq e^{t_n}, \quad t_n := \frac{n^2 \log n}{2} + C n^2
\]
holds for every \( k = 0, \ldots, n \) and \( j = 0, \ldots, n - k \), with an absolute constant \( C > 0 \). For fixed \( k \), we prove by induction on \( j = n - k, \ldots, 0 \) that
\[
|c_{kj}^{'}| \leq e^{t_n} \left( (n + 1)(1 + e^2 n^2) \right)^{n-k-j}.
\]
If \( j = n - k \) this holds by \( 12 \) and \( 13 \). Assuming that the inequality is true for \( l = n - k, \ldots, n - k - j + 1 \), we obtain by using Lemmas 3.2, 3.3 and 3.4 that
\[
|c_{kj}^{'}| \leq \frac{|\alpha_{k,n-k-j}|}{|r_{k0}|} + \sum_{l=1}^{j} \frac{|r_{k,l}|}{r_{k0}} |c_{k,n-k-j+l}| \leq e^{t_n} \left[ 1 + \sum_{l=1}^{j} e^2 n^2 (n + 1)^l (1 + e^2 n^2)^{j-l} \right] \leq e^{t_n} (n + 1)^j \left[ 1 + e^2 n^2 \sum_{l=1}^{j} (1 + e^2 n^2)^{j-l} \right] = e^{t_n} (n + 1)^j (1 + e^2 n^2)^j,
\]
so \( 14 \) is proved. Therefore, combining \( 10, 13 \) and \( 14 \), we get for every \( j = 0, \ldots, n - k \) that
\[
|c_{kj}| \leq |c_{kj}^{'}| \leq e^{t_n} (n + 1)^n (1 + e^2 n^2)^n \leq \exp \left( \frac{n^2 \log n}{2} + C n^2 \right),
\]
with an absolute constant \( C > 0 \). Using this and (8) it follows that
\[
M_{P_n} (1) \leq \sum_{j=0}^{n-k} |c_{kj}| \leq \exp \left( \frac{n^2 \log n}{2} + Cn^2 \right).
\]

The proof of Proposition [3] and hence the proof of Theorem [1.1] are complete. \( \square \)

**Remark.** Using the main steps and ideas from this proof, together with more careful estimates of the constants involved, we can prove for every \( n \geq 1 \) the inequalities
\[
\exp \left( \frac{n^2 \log n}{2} - n^2 \right) < E_n < \exp \left( \frac{n^2 \log n}{2} + n^2 + 63 \right).
\]

**Proof of Theorem [1.2]** Let \( x_n \) be defined as the unique solution of the equation \( n(x_n - 1)e^{x_n} = \log E_n \), for \( n \geq 1 \). Using the polynomial \( P(z, w) = w - 1 - z \) one checks that \( E_1 > 1 \). Therefore \( x_n > 1 \) and by Theorem [1.1] \( \lim_{n \to \infty} x_n = +\infty \).

Let \( \alpha_n = (\log E_n)/x_n \). We estimate first \( x_n \) and \( \alpha_n \). By Theorem [1.1]
\[
(x_n - 1)e^{x_n} = \frac{\log E_n}{n} = \frac{n \log n}{2} + O(n),
\]
hence \( x_n + \log(x_n - 1) = \log n - \log 2 + \log(\log(n + O(1))) \). This implies \( \lim_{n \to \infty} x_n / \log n = 1 \), so \( \lim_{n \to \infty} 2\alpha_n/n^2 = 1 \). Since
\[
x_n = \log n - \log 2 + \log \left( \frac{\log n + O(1)}{x_n - 1} \right)
\]
and since \( x_n / \log n \to 1 \) we get \( x_n = \log n + O(1) \) and
\[
(15) \quad \frac{2\alpha_n}{n^2} - 1 = \frac{2 \log E_n}{n^2x_n} - 1 = \frac{\log n + O(1)}{\log n + O(1)} - 1 = \frac{O(1)}{\log n}.
\]

We can now prove the estimate in the statement. Let \( P \in \mathcal{P}_n \) with \( \|P\|_K = 1 \) and consider the subharmonic function \( u(z) = \log^+ |P_n(z)| \). Then \( u \leq 0 \) on \( \Delta \) and \( u(z) \leq \log E_n + n|z| \), for every \( z \in \mathbb{C} \), by (2). Let \( \bar{u}(x) = \max\{u(z) : |z| = e^x\} \), where \( x \geq 0 \). Then \( \bar{u} \) is a convex increasing function, \( \bar{u}(0) = 0 \) and
\[
\bar{u}(x) \leq \phi(x) = \log E_n + ne^{x^2}, \quad \forall \ x \geq 0.
\]

Note that \( x_n \) verifies \( x_n \phi'(x_n) = \phi(x_n) \), which means that the tangent line to the graph of \( \phi \) at \( (x_n, \phi(x_n)) \) passes through the origin. Using the convexity of \( \bar{u} \) it follows that
\[
\bar{u}(x) \leq \left\{ \begin{array}{ll}
\phi'(x_n)x, & \text{if } 0 \leq x \leq x_n, \\
\phi(x), & \text{if } x \geq x_n.
\end{array} \right.
\]

Since \( x \leq e^x - 1 \) and \( ne^{x_n} = (\log E_n)/(x_n - 1) \) we obtain
\[
\phi'(x_n)x = (\log E_n + ne^{x_n})\frac{x}{x_n} \leq \alpha_n x + \frac{ne^{x_n}}{x_n} (e^x - 1) = \alpha_n \left( x + \frac{e^x - 1}{x_n - 1} \right),
\]
for all \( x \geq 0 \). Moreover, it is easy to see that
\[
\phi(x) = \log E_n + ne^{x} \leq \alpha_n x + \frac{ne^{x_n}}{x_n} (e^x - 1)
\]
holds for \( x \geq x_n \). We conclude that for all \( x \geq 0 \) we have
\[
\bar{u}(x) \leq \alpha_n \left( x + \frac{e^x - 1}{x_n - 1} \right) = \frac{n^2}{2} \left[ x + \left( \frac{2\alpha_n}{n^2} - 1 \right) x + \frac{2\alpha_n}{n^2} \frac{e^x - 1}{x_n - 1} \right].
\]
Using (15) and the asymptotics of \(x_n\) we obtain
\[
\tilde{u}(x) \leq \frac{n^2}{2} \left( x + \frac{Cx}{\log n} + C'e^{-1} \frac{1}{x_n} \right) \leq \frac{n^2}{2} \left( x + C_2 e^{-1} \frac{1}{1 + \log n} \right),
\]
where \(C_2 > 0\) is an absolute constant. If \(x = \log |z|, |z| \geq 1\), this gives the desired inequality.

Combining the inequality we have just proved with (4), we get for all \(n \geq 1\) and \(r \geq 1\) that
\[
\frac{\log r}{2} + 3 \frac{\log r}{2n} \leq m_n(r) \leq \frac{\log r}{2} + \frac{C_2(r - 1)}{2(1 + \log n)}.
\]
Therefore \(\lim_{n \to \infty} m_n(r)/n^2 = \frac{1}{2} \log r\), locally uniformly for \(r \geq 1\).

References


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