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BERNSTEIN–WALSH INEQUALITIES AND THE EXPONENTIAL CURVE IN $\mathbb{C}^2$

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ABSTRACT. It is shown that for the pluripolar set $K = \{(z, e^z) : |z| \leq 1\}$ in $\mathbb{C}^2$ there is a global Bernstein–Walsh inequality: If P is a polynomial of degree n on $\mathbb{C}^2$ and $|P| \leq 1$ on K, this inequality gives an upper bound for $|P(z, w)|$ which grows like $\exp(\frac{1}{2}n^2 \log n)$. The result is used to obtain sharp estimates for $|P(z, e^z)|$.

1. INTRODUCTION

If X is a non-pluripolar compact set in $\mathbb{C}^k$ and P is a polynomial of degree n on $\mathbb{C}^k$, the Bernstein–Walsh inequality is (see [K])

$$|P(z)| \leq \|P\|_X e^{nV_X(z)}, \tag{1}$$

where $\|P\|_X$ is the uniform norm of P on X and $V_X(z)$ is the extremal function of X. For example, if $z = (z_1, \dots, z_k)$ and $X = \Delta^k = \{z \in \mathbb{C}^k : |z_j| \leq 1, 1 \leq j \leq k\}$ is the unit polydisk, then

$$V_X(z) = L(z) = \max\{\log^+ |z_1|, \dots, \log^+ |z_k|\}.$$

If X is pluripolar, then, in general, such estimates are impossible. For example, if X is any piece of an algebraic curve $\Gamma = \{(z, w) \in \mathbb{C}^2 : P(z, w) = 0\}$, where P is a polynomial, then $\|cP + 1\|_X = 1$ for every $c > 0$ and there are no upper bounds on $cP + 1$.

We consider the case when $\Gamma = \{(z, w) \in \mathbb{C}^2 : w = f(z)\}$ and the compact set

$$K = \{(z, f(z)) \in \mathbb{C}^2 : |z| \leq 1\},$$

where f is an entire transcendental function. Then any non-trivial polynomial is not identically equal to 0 on K. Therefore a compactness argument shows that, for every n, there is a number $c_n > 0$ such that for any polynomial $P(z, w)$ of degree at most n the norm $\|P\|_{\Delta^2} \leq c_n \|P\|_K$. Hence for every $(z, w) \in \mathbb{C}^2$

$$|P(z, w)| \leq \|P\|_K E_n(f) e^{nL(z,w)}, \tag{2}$$

where $E_n(f)$ is the least value of c_n. (See also Section 2.)

Inequality (2) can be viewed as a transcendental global version of the Bernstein–Walsh inequality (1), provided that one can obtain good estimates for $E_n(f)$. Moreover, the numbers $E_n(f)$ can serve as a measure of transcendency of f: A "less

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transcendental" function f has larger numbers $E_n(f)$. Note that if f was algebraic, hence a polynomial of degree l, then $E_n(f) = +\infty$ for every $n \geq l$.

In this paper we study the classical case of $f(z) = e^z$ and we let $E_n = E_n(e^z)$. For this function we prove the following global Bernstein–Walsh inequality:

Theorem 1.1. *If $f(z) = e^z$, then there exists a constant $C_1 > 0$ so that*

$$\exp\left(\frac{n^2 \log n}{2} - C_1 n^2\right) \leq E_n \leq \exp\left(\frac{n^2 \log n}{2} + C_1 n^2\right),$$

for all $n \geq 1$. If P is a polynomial of degree n on $\mathbb{C}^2$, then

$$|P(z,w)| \leq \|P\|_K \exp\left(\frac{n^2 \log n}{2} + C_1 n^2 + nL(z,w)\right). \tag{3}$$

Thus, despite the pluripolarity of K, there is an upper estimate for the absolute value of polynomials, which grows asymptotically as $\exp(n^2 \log n)$. This is not much worse than $\exp(n)$ in the classical Bernstein–Walsh inequality (1). Moreover, this estimate is asymptotically sharp.

Inequality (3) improves when $(z,w) \in \Gamma$. In [T] (see also [B]) it was proved that

$$|P(z, e^z)| \leq \|P\|_K e^{n^2 \log^+ |z| + 6n|z|}.$$

This inequality was used to prove deep theorems concerning the algebraic independence of values of e^z.

For a general transcendental function f we introduce the function

$$m_n(r) = \sup\{\log |P(z, f(z))| : \deg P \leq n, \|P\|_K \leq 1, |z| \leq r\}.$$

The numbers $m_n(r)$ can also serve as a measure of transcendency of f. Let $(|z|-1)^+ = \max\{|z|-1, 0\}$. As a consequence of Theorem 1.1, we prove the following restricted Bernstein–Walsh inequality for $f(z) = e^z$:

Theorem 1.2. *There exists an absolute constant $C_2 > 0$ such that for every polynomial P of degree $n \geq 1$ on $\mathbb{C}^2$ and every $z \in \mathbb{C}$ we have*

$$|P(z, e^z)| \leq \|P\|_K \exp\left[\frac{n^2}{2}\left(\log^+ |z| + C_2 \frac{(|z|-1)^+}{1 + \log n}\right)\right].$$

Moreover $\lim_{n\to\infty} m_n(r)/n^2 = \frac{1}{2} \log r$, *locally uniformly for $r \geq 1$.*

This theorem provides the exact asymptotic behavior of the functions $m_n(r)$. It also improves Tijdeman's estimate if one fixes z and lets $n \to \infty$. On the other hand, if one fixes n and lets $z \to \infty$, then Tijdeman's estimate is better (at least if n is large).

There is a fundamental difference between classical and transcendental Bernstein–Walsh inequalities. In the classical case (1) the extremal function V_X is given by (see [K])

$$V_X(z,w) = \limsup_{n\to\infty} \frac{1}{n} \sup\{\log^+ |P(z,w)| : \deg P = n,\ \|P\|_X \leq 1\}.$$

In the transcendental case (3) it follows from Theorem 1.1 and the Hartogs lemma that

$$\limsup_{n\to\infty} \frac{2}{n^2 \log n} \sup\{\log^+ |P(z,w)| : \deg P = n,\ \|P\|_K \leq 1\} = 1$$

everywhere on $\mathbb{C}^2$ except a pluripolar set. Moreover, by Theorem 1.2,

$$\limsup_{n\to\infty} \frac{2}{n^2 \log n} \sup\{\log^+ |P(z, e^z)| : \deg P = n, \ \|P\|_K \le 1\} = 0.$$

The next proposition holds for all entire transcendental functions.

Proposition 1.3. *If f is an entire transcendental function, then*

$$m_n(r) \ge \frac{n^2 + 3n}{2} \log r \tag{4}$$

for every $r \ge 1$. Moreover for $r \ge 1$

$$E_n(f) \ge \exp\left(\frac{n^2 + 3n}{2} \log r - nL(r, M_f(r))\right). \tag{5}$$

If f is of finite order of growth $< \rho$, or of finite order ρ and finite type, then $E_n(f) \ge \exp\left(\frac{n^2 \log n}{2\rho} - Cn^2\right)$ for all $n \ge 1$, where $C = C(f) > 0$.

Proposition 1.3 and the previous theorem imply that the function e^z provides asymptotically the smallest possible functions $m_n(r)$.

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2. Preliminaries

We use the following notation. If g is an entire holomorphic function we let

$$M_g(r) = \max\{|g(z)| : \ |z| = r\}.$$

For $n \ge 0$ we denote by $\mathcal{P}_n$ the space of polynomials $P \in \mathbb{C}[z, w]$ of degree at most n. Then $\dim \mathcal{P}_n = (n + 1)(n + 2)/2 = N + 1$, where $N = (n^2 + 3n)/2$.

Let f be an entire transcendental function. For any polynomial $P \in \mathcal{P}_n$ we denote by $P_\star$ the entire function

$$P_\star(z) = P(z, f(z)), \ z \in \mathbb{C},$$

so $\|P\|_K = M_{P_\star}(1)$. Since f is transcendental, it follows that $\|\cdot\|_K$ is a norm on each vector space $\mathcal{P}_n$. As $\mathcal{P}_n$ are finite dimensional we have

$$E_n(f) = \sup\{\|P\|_{\Delta^2} : \ P \in \mathcal{P}_n, \ \|P\|_K \le 1\} < +\infty,$$

for each $n \ge 0$. Note that $E_0(f) = 1$ and $E_n(f) \le E_{n+1}(f)$.

Inequality (2) implies that the function

$$u_n(z) = \sup\{\log |P_\star(z)| : \ P \in \mathcal{P}_n, \ \|P\|_K \le 1\}$$

is well defined. It is easy to see by a normal family argument that u_n is a non-negative continuous subharmonic function on $\mathbb{C}$ and $u_n = 0$ on Δ. We have $m_n(r) = \max\{u_n(z) : \ |z| \le r\}$, hence $m_n(r)$ is a continuous increasing convex function of $\log r$.

We need the following simple lemma:

Lemma 2.1. *The following inequalities hold for every integer $m > 0$:*

$$\log(m+1) \le \sum_{j=1}^{m} \frac{1}{j} \le \log m + 1,$$

$$m \log m - m + 1 \le \sum_{j=1}^{m} \log j \le (m+1)\log m - m + 1,$$

$$\frac{m^2 \log m}{2} - \frac{m^2}{4} + \frac{1}{4} \le \sum_{j=1}^{m} j \log j \le \frac{m^2 \log m}{2} - \frac{m^2}{4} + m \log m + \frac{1}{4}\,.$$

Proof. The proof is elementary. For instance, the third inequality follows using $\int_{j-1}^{j} x \log x\, dx \le j \log j \le \int_{j}^{j+1} x \log x\, dx$. □

3. Proofs

We first prove Proposition 1.3, which was stated for arbitrary entire transcendental functions f. Recall the notations $N = (n^2+3n)/2$ and $P_\star(z) = P(z, f(z))$.

Proof of Proposition 1.3. Since $\dim \mathcal{P}_n = N+1$, there exists $P \in \mathcal{P}_n$, $P \not\equiv 0$, such that the vanishing order of $P_\star$ at 0 is at least N. We let $g(z) = P_\star(z)/z^N$, so

$$M_{P_\star}(1) = M_g(1) \le M_g(r) = M_{P_\star}(r)/r^N,$$

provided that $r \ge 1$. This and the definition of $m_n(r)$ clearly imply (4). Using (2) with $w = f(z)$ and $|z| \le r$ we get

$$r^N \le M_{P_\star}(r)/M_{P_\star}(1) \le E_n(f) \exp\left[nL(r, M_f(r))\right],$$

so (5) follows.

In the case when f is of finite order of growth, we have $\log^+ M_f(r) \le Cr^\rho$ for every $r \ge 1$. The conclusion follows by taking $r = n^{1/\rho}$ in (5) and by using the above estimate on $\log^+ M_f(r)$. □

In the following proofs we have $f(z) = e^z$, so $P_\star(z) = P(z, e^z)$.

Proof of Theorem 1.1. The lower estimate of E_n follows from Proposition 1.3, since $f(z) = e^z$ has order of growth $\rho = 1$ and type 1 with respect to this order. Moreover, (3) follows from (2) and the upper bound of E_n. To prove the upper bound, we introduce the following notation. Let $d = \frac{d}{dz}$. For any polynomial $R(\lambda) = \sum_{j=0}^{m} c_j \lambda^j$ we denote by D_R the constant-coefficient differential operator

$$D_R = R(d) = \sum_{j=0}^{m} c_j \frac{d^j}{dz^j}\,.$$

Then for any integer $t \ge 0$ and any $\alpha \in \mathbb{C}$ we have

$$(6) \qquad D_R[z^t e^{\alpha z}]\big|_{z=0} = \sum_{j \ge t} c_j \frac{j!}{(j-t)!}\, \alpha^{j-t} = \frac{d^t R}{d\lambda^t}\bigg|_{\lambda=\alpha} = R^{(t)}(\alpha).$$

Now fix $P \in \mathcal{P}_n$, $n \ge 1$, with $\|P\|_K \le 1$. By Cauchy's estimates we have

$$(7) \qquad \left|P_\star^{(l)}(0)\right| \le l!,\ \forall\, l \ge 0.$$

We write

$$P(z,w)=\sum_{k=0}^{n} P_{n-k}(z)w^k,\ P_{n-k}(z)=\sum_{j=0}^{n-k} c_{kj}z^j. \tag{8}$$

In the sequel, we denote by C all absolute constants involved in our estimates. (They may change from one inequality to the next.)

Proposition 3.1. *There exists a constant C such that*

$$M_{P_{n-k}}(1)\le \exp\left(\frac{n^2\log n}{2}+Cn^2\right),\ \forall\, k\in\{0,\dots,n\}.$$

Note that this proposition implies Theorem 1.1. Indeed,

$$\|P\|_{\Delta^2}\le \sum_{k=0}^{n} M_{P_{n-k}}(1)\le \exp\left(\frac{n^2\log n}{2}+Cn^2\right),$$

so the same estimate holds for E_n since P is arbitrary with $\|P\|_K\le 1$.

In order to prove Proposition 3.1, we fix $k\in\{0,\dots,n\}$. We will estimate the coefficients c_{kj} of P_{n-k} by using the differential operators given by the polynomials

$$\begin{aligned} R_k(\lambda) &= \prod_{l=0,l\ne k}^{n} (\lambda-l)^{n-l+1}, \\ R_{kj}(\lambda) &= R_k(\lambda)(\lambda-k)^j\ ,\ j=0,\dots,n-k. \end{aligned} \tag{9}$$

Note that $\deg R_{k,n-k}=N$. By (6) we have

$$\begin{aligned} \alpha_{kj} := D_{R_{kj}}P_\star(z)\big|_{z=0} &= \left(D_{R_k}(d-k)^j\right)\left[\sum_{l=0}^{n-k} c_{kl}z^le^{kz}\right]\Bigg|_{z=0} \\ &= D_{R_k}\left[\sum_{l=j}^{n-k} c_{kl}\frac{l!}{(l-j)!}\,z^{l-j}e^{kz}\right]\Bigg|_{z=0} \\ &= \sum_{l=j}^{n-k} c_{kl}\frac{l!}{(l-j)!}\,R_k^{(l-j)}(k). \end{aligned}$$

We write

$$c'_{kl}=l!\,c_{kl},\ l=0,\dots,n-k, \tag{10}$$

$$r_{kt}=R_k^{(t)}(k)/t!,\ t=0,\dots,n-k. \tag{11}$$

Then c'_{kl} are the unique solution of the triangular system

$$\sum_{l=j}^{n-k} r_{k,l-j}c'_{kl}=\alpha_{kj},\ j=0,\dots,n-k,$$

which yields

$$c'_{k,n-k-j}=\frac{\alpha_{k,n-k-j}}{r_{k0}}-\sum_{l=1}^{j}\frac{r_{kl}}{r_{k0}}\,c'_{k,n-k-j+l},\ j=0,\dots,n-k. \tag{12}$$

In order to estimate the coefficients c'_{kj}, we obtain first bounds for α_{kj}, r_{k0} and r_{kl}/r_{k0}. This is done in a sequence of lemmas.

Lemma 3.2. *For all $k = 0, \dots, n$ and $j = 0, \dots, n-k$ we have*

$$|\alpha_{kj}| \le e^{n^2 \log n + Cn \log n},$$

where $C > 0$ is an absolute constant.

Proof. We write $R_{kj}(\lambda) = \sum_{l=0}^{N_j} s_l \lambda^l$, $N_j = \deg R_{kj} = N - (n-k-j)$, and define

$$|R_{kj}|(\lambda) = \sum_{l=0}^{N_j} |s_l| \lambda^l = (\lambda + k)^j \prod_{l=0, l\neq k}^{n} (\lambda + l)^{n-l+1}.$$

Using Cauchy's estimates (7) and Stirling's formula $l! \le e(l/e)^l \sqrt{l}$, for $l \ge 1$ (see [R]), we get

$$\begin{aligned} |\alpha_{kj}| &\le \sum_{l=0}^{N_j} |s_l| l! \le e \sum_{l=0}^{N_j} |s_l| (l/e)^l \sqrt{l} \\ &\le e\sqrt{N} \sum_{l=0}^{N_j} |s_l| N^l = e\sqrt{N}\, |R_{kj}|(N) \\ &\le e\sqrt{N} \prod_{l=0}^{n} (N+l)^{n-l+1} \le e\sqrt{N} \exp\left((\log(N+n)) \sum_{l=1}^{n+1} l \right). \end{aligned}$$

Since $\log(N+n) \le 2\log n + 2/n$, this yields $|\alpha_{kj}| \le \exp(n^2 \log n + Cn \log n)$. ☐

Lemma 3.3. *If $0 \le k \le n$, then $|r_{k0}| \ge \exp[(n^2 \log n)/2 - Cn^2]$, where $C > 0$ is an absolute constant.*

Proof. We have by (9) and (11) that

$$|r_{k0}| = |R_k(k)| = \prod_{l=0, l\neq k}^{n} |k - l|^{n-l+1},$$

so after a direct calculation we get

$$\log |r_{k0}| = (n-k+1) \left(\sum_{l=1}^{k} \log l + \sum_{l=1}^{n-k} \log l \right) + \sum_{l=1}^{k} l \log l - \sum_{l=1}^{n-k} l \log l.$$

Using Lemma 2.1, with the convention that $x \log x = 0$ for $x = 0$, it follows that for every $k = 0, \dots, n$ we have

$$\begin{aligned} \log |r_{k0}| &\ge (n-k+1)[k \log k + (n-k)\log(n-k) - n] \\ &\quad + \frac{k^2 \log k}{2} - \frac{k^2}{4} - \frac{(n-k)^2 \log(n-k)}{2} \\ &\quad + \frac{(n-k)^2}{4} - (n-k)\log(n-k) \\ &\ge k\left(n - \frac{k}{2}\right) \log k + \frac{(n-k)^2}{2} \log(n-k) - Cn^2 \\ &= \frac{n^2 \log n}{2} + n^2 F\left(\frac{k}{n}\right) - Cn^2 \ge \frac{n^2 \log n}{2} - Cn^2, \end{aligned}$$

where $F(x) = x(1 - x/2)\log x + 0.5(1-x)^2 \log(1-x) > -C$ for all $x \in [0,1]$ and some constant $C > 0$. ☐

Lemma 3.4. *For any* $k = 0, \dots, n$ *and* $j = 0, \dots, n-k$ *we have*

$$|r_{kj}|/|r_{k0}| \le e^2 n^2 (n+1)^j.$$

Proof. We fix $\epsilon \in (0,1)$, to be chosen later in terms of n. Using the definition (11) of r_{kj} and applying Cauchy's estimates to R_k, we obtain that $|r_{kj}| \le M/\epsilon^j$, where $M = \max\{|R_k(\lambda)| : |\lambda - k| = \epsilon\}$. If λ is on the circle $|\lambda - k| = \epsilon$, then

$$|R_k(\lambda)| \le \prod_{l=0}^{k-1} (k+\epsilon-l)^{n-l+1} \prod_{l=k+1}^{n} (l-k+\epsilon)^{n-l+1} = |r_{k0}| F_k(\epsilon),$$

where

$$F_k(\epsilon) = \prod_{l=0}^{k-1} \left(1 + \frac{\epsilon}{k-l}\right)^{n-l+1} \prod_{l=k+1}^{n} \left(1 + \frac{\epsilon}{l-k}\right)^{n-l+1}.$$

Therefore $|r_{kj}|/|r_{k0}| \le F_k(\epsilon)/\epsilon^j$. Using Lemma 2.1 we get

$$\begin{aligned} \log F_k(\epsilon) &\le 2(n+1) \sum_{l=1}^{n} \log\left(1 + \frac{\epsilon}{l}\right) \\ &\le 2\epsilon(n+1) \sum_{l=1}^{n} \frac{1}{l} \le 2\epsilon(n+1)(\log n + 1). \end{aligned}$$

Choosing $\epsilon = 1/(n+1)$ we obtain $\log F_k(\epsilon) \le 2(\log n + 1)$, hence the lemma follows. □

We can now estimate the coefficients c'_{kj} by using the system of equations (12). Lemmas 3.2 and 3.3 imply that

$$\frac{|\alpha_{kj}|}{|r_{k0}|} \le e^{t_n},\ t_n := \frac{n^2 \log n}{2} + Cn^2 \tag{13}$$

holds for every $k = 0, \dots, n$ and $j = 0, \dots, n-k$, with an absolute constant $C > 0$. For fixed k, we prove by induction on $j = n-k, \dots, 0$ that

$$|c'_{kj}| \le e^{t_n} \left[(n+1)(1+e^2n^2)\right]^{n-k-j}. \tag{14}$$

If $j = n-k$ this holds by (12) and (13). Assuming that the inequality is true for $l = n-k, \dots, n-k-j+1$, we obtain by using Lemmas 3.2, 3.3 and 3.4 that

$$\begin{aligned} |c'_{k,n-k-j}| &\le \frac{|\alpha_{k,n-k-j}|}{|r_{k0}|} + \sum_{l=1}^{j} \frac{|r_{kl}|}{|r_{k0}|} |c'_{k,n-k-j+l}| \\ &\le e^{t_n} \left[1 + \sum_{l=1}^{j} e^2n^2(n+1)^l[(n+1)(1+e^2n^2)]^{j-l}\right] \\ &\le e^{t_n}(n+1)^j \left[1 + e^2n^2 \sum_{l=1}^{j} (1+e^2n^2)^{j-l}\right] \\ &= e^{t_n}(n+1)^j(1+e^2n^2)^j, \end{aligned}$$

so (14) is proved. Therefore, combining (10), (13) and (14), we get for every $j = 0, \dots, n-k$ that

$$|c_{kj}| \le |c'_{kj}| \le e^{t_n}(n+1)^n(1+e^2n^2)^n \le \exp\left(\frac{n^2 \log n}{2} + Cn^2\right),$$

with an absolute constant $C > 0$. Using this and (8) it follows that

$$M_{P_{n-k}}(1) \leq \sum_{j=0}^{n-k} |c_{kj}| \leq \exp\left(\frac{n^2 \log n}{2} + Cn^2\right).$$

The proof of Proposition 3.1, and hence the proof of Theorem 1.1, are complete. □

Remark. Using the main steps and ideas from this proof, together with more careful estimates of the constants involved, we can prove for every $n \geq 1$ the inequalities

$$\exp\left(\frac{n^2 \log n}{2} - n^2\right) < E_n < \exp\left(\frac{n^2 \log n}{2} + n^2 + 63\right).$$

Proof of Theorem 1.2. Let x_n be defined as the unique solution of the equation $n(x_n - 1)e^{x_n} = \log E_n$, for $n \geq 1$. Using the polynomial $P(z, w) = w - 1 - z$ one checks that $E_1 > 1$. Therefore $x_n > 1$ and by Theorem 1.1 $\lim_{n\to\infty} x_n = +\infty$.

Let $\alpha_n = (\log E_n)/x_n$. We estimate first x_n and α_n. By Theorem 1.1

$$(x_n - 1)e^{x_n} = \frac{\log E_n}{n} = \frac{n \log n}{2} + O(n),$$

hence $x_n + \log(x_n - 1) = \log n - \log 2 + \log(\log n + O(1))$. This implies $\lim_{n\to\infty} x_n / \log n = 1$, so $\lim_{n\to\infty} 2\alpha_n/n^2 = 1$. Since

$$x_n = \log n - \log 2 + \log\left(\frac{\log n + O(1)}{x_n - 1}\right)$$

and since $x_n/\log n \to 1$ we get $x_n = \log n + O(1)$ and

$$\frac{2\alpha_n}{n^2} - 1 = \frac{2\log E_n}{n^2 x_n} - 1 = \frac{\log n + O(1)}{\log n + O(1)} - 1 = \frac{O(1)}{\log n}. \tag{15}$$

We can now prove the estimate in the statement. Let $P \in \mathcal{P}_n$ with $\|P\|_K = 1$ and consider the subharmonic function $u(z) = \log^+ |P_\star(z)|$. Then $u \leq 0$ on Δ and $u(z) \leq \log E_n + n|z|$, for every $z \in \mathbb{C}$, by (2). Let $\widetilde{u}(x) = \max\{u(z) : |z| = e^x\}$, where $x \geq 0$. Then $\widetilde{u}$ is a convex increasing function, $\widetilde{u}(0) = 0$ and

$$\widetilde{u}(x) \leq \phi(x) = \log E_n + ne^x, \ \forall\, x \geq 0.$$

Note that x_n verifies $x_n \phi'(x_n) = \phi(x_n)$, which means that the tangent line to the graph of ϕ at $(x_n, \phi(x_n))$ passes through the origin. Using the convexity of $\widetilde{u}$ it follows that

$$\widetilde{u}(x) \leq \begin{cases} \phi'(x_n)x \text{ , if } 0 \leq x \leq x_n, \\ \phi(x) \text{ , if } x \geq x_n. \end{cases}$$

Since $x \leq e^x - 1$ and $ne^{x_n} = (\log E_n)/(x_n - 1)$ we obtain

$$\begin{aligned} \phi'(x_n)x &= (\log E_n + ne^{x_n})\frac{x}{x_n} \\ &\leq \alpha_n x + \frac{ne^{x_n}}{x_n}(e^x - 1) = \alpha_n\left(x + \frac{e^x - 1}{x_n - 1}\right), \end{aligned}$$

for all $x \geq 0$. Moreover, it is easy to see that

$$\phi(x) = \log E_n + ne^x \leq \alpha_n x + \frac{ne^{x_n}}{x_n}(e^x - 1)$$

holds for $x \geq x_n$. We conclude that for all $x \geq 0$ we have

$$\widetilde{u}(x) \leq \alpha_n\left(x + \frac{e^x - 1}{x_n - 1}\right) = \frac{n^2}{2}\left[x + \left(\frac{2\alpha_n}{n^2} - 1\right)x + \frac{2\alpha_n}{n^2}\,\frac{e^x - 1}{x_n - 1}\right].$$

Using (15) and the asymptotics of x_n we obtain

$$\widetilde{u}(x) \leq \frac{n^2}{2}\left(x + \frac{Cx}{\log n} + C'\frac{e^x-1}{x_n}\right) \leq \frac{n^2}{2}\left(x + C_2\frac{e^x-1}{1+\log n}\right),$$

where $C_2 > 0$ is an absolute constant. If $x = \log|z|$, $|z| \geq 1$, this gives the desired inequality.

Combining the inequality we have just proved with (4), we get for all $n \geq 1$ and $r \geq 1$ that

$$\frac{\log r}{2} + \frac{3\log r}{2n} \leq \frac{m_n(r)}{n^2} \leq \frac{\log r}{2} + \frac{C_2(r-1)}{2(1+\log n)}.$$

Therefore $\lim_{n\to\infty} m_n(r)/n^2 = \frac{1}{2}\log r$, locally uniformly for $r \geq 1$. □

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