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SUBSYMMETRIC SEQUENCES AND MINIMAL SPACES

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Abstract. We show that every Banach space saturated with subsymmetric basic sequences contains a minimal subspace.

1. Introduction

W. T. Gowers proved in [G1] the celebrated dichotomy theorem concerning unconditional basic sequences and hereditarily indecomposable spaces using Ramsey-type arguments. In [G2] the reasoning was generalized and, as an application, a dichotomy concerning quasi-minimal spaces, i.e. those spaces for which any two infinite dimensional subspaces contain two further infinite dimensional subspaces which are isomorphic, was obtained. Putting these results together Gowers obtained the following "classification" theorem (the definition of minimal spaces will be given at the beginning of Section 3):

Theorem 1.1 ([G2]). *Let E be an infinite dimensional Banach space. Then E has an infinite dimensional subspace G with one of the following properties. The properties are mutually exclusive and all can and do occur:*

1. *G is a hereditarily indecomposable space,*
2. *G has an unconditional basis and every isomorphism between block subspaces of G is a strictly singular perturbation of the restriction of some invertible diagonal operator on G,*
3. *G has an unconditional basis and is strictly quasi-minimal (i.e. is quasi-minimal and does not contain a minimal subspace),*
4. *G has an unconditional basis and is minimal.*

In this paper we prove that every Banach space saturated with subsymmetric basic sequences contains a minimal subspace. It follows that the class (3) can be restricted to strictly quasi-minimal spaces not containing subsymmetric basic sequences and one could split (4) into minimal spaces with a subsymmetric basis or minimal spaces not containing a subsymmetric basic sequences. An example of a minimal space not containing any subsymmetric sequence is the dual to Tsirelson's space ([LT], [CJT]), whereas Tsirelson's example is a strictly quasi-minimal space ([CO]).

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The method used here extends the technique applied in [P], which reflects the technique of Maurey's proof of Gowers' dichotomy theorem for unconditional sequences and HI spaces ([M]). The same method also provides extensions in the class (1) by examining unconditional-like sequences introduced in [T2] ([P]).

We begin with some basic notation and definitions. Let E be a Banach space. Given a set $A \subseteq E$ by $\langle A \rangle$ denote the vector subspace spanned by A. We will denote by Θ the origin in the space E in order to distinguish it from the number zero.

For standard Banach space notations we refer the reader to [LT]. We say that two Banach spaces E_1, E_2 are $C-$*isomorphic*, for $C \geq 1$, if there is an isomorphism $T : E_1 \to E_2$ satisfying $\frac{1}{C}\|x\| \leq \|Tx\| \leq C\|x\|$ for $x \in E_1$. We say that sequences $\{x_n\}_n$, $\{y_n\}_n$ of vectors of a Banach space are $C-$*equivalent*, for $C \geq 1$, if for any scalars $a_1, \dots, a_n$ and $n \in \mathbb{N}$ we have

$$\frac{1}{C}\|a_1y_1 + \dots + a_ny_n\| \leq \|a_1x_1 + \dots + a_nx_n\| \leq C\|a_1y_1 + \dots + a_ny_n\|.$$

Assume now that E is a Banach space with a basis $\{e_n\}_{n=1}^{\infty}$.

The *support* of a vector $x = \sum_{n=1}^{\infty} x_ne_n$ is the set $\operatorname{supp} x = \{n \in \mathbb{N} : \ x_n \neq 0\}$. We use the notation $x < y$ for vectors $x, y \in E$, if every element of $\operatorname{supp} x$ is smaller than every element of $\operatorname{supp} y$. We write $x < A$ for a vector $x \in E$ and $A \subseteq E$, if $x < y$ for all $y \in A$, and so forth in this manner. A *block sequence* with respect to $\{e_n\}$ is any sequence of non-zero finitely supported vectors $x_1 < x_2 < \dots$. A *block subspace* is a closed subspace spanned by a block sequence. We will use letters $x, y, z, \dots$ to denote vectors of a Banach space, letters x,y,z,... to denote finite block sequences and capital letters W,X,Y,Z, ... for infinite block sequences. For any finite block sequence x, by |x| we denote the length of x, i.e. the number of elements of x. Given any two block sequences $\{x_1, \dots, x_n\} < \{y_1, y_2, \dots\}$ let

$$\{x_1, \dots, x_n\} \cup \{y_1, y_2, \dots\} = \{x_1, \dots, x_n, y_1, y_2, \dots\}.$$

For convenience in the reasoning presented in the next sections we will treat $\{\Theta\}$ as a block sequence and adopt the following convention: $|\{\Theta\}| = 0$, $\Theta < x$, for any $x \neq \Theta$, $\{\Theta\} \cup \{y_1, y_2, \dots\} = \{y_1, y_2, \dots\}$ for any block sequence $\{y_1, y_2, \dots\}$.

While restricting our consideration to the family of block sequences we will use the following fact (see e.g. [LT], 1.a.12). Recall that a sequence $\{x_n\}_n$ of vectors of a Banach space is called seminormalized if $0 < \inf \|x_n\|$ and $\sup \|x_n\| < \infty$.

Lemma 1.2. *Let E be a Banach space with a basis $\{e_i\}_i$. Let $\{x_n\}_n \subseteq E$ be a seminormalized sequence satisfying $\lim_{n\to\infty} e_i^*(x_n) = 0$, $i \in \mathbb{N}$, where $\{e_i^*\}_i$ is the sequence of biorthogonal functionals of $\{e_i\}_i$. Then for any $\varepsilon > 0$ there is a block sequence $\{y_n\}_n$ which is $(1+\varepsilon)-$equivalent to some subsequence of the sequence $\{x_n\}_n$.*

2. The "stabilizing" lemma

In this section we present some more terminology and a stabilizing lemma. It reflects some combinatorial techniques used in [M], [Z], [T1] and other papers.

First we need some more notation. Let E be a Banach space with a basis $\{e_n\}$. Let $\mathbf{Q}$ denote the set of all vectors of the form $\sum_{i=1}^{n} a_ie_i$ for $n \in \mathbb{N}$, $\{a_i\}_1^n \subseteq \mathbb{Q}$ where $\mathbb{Q}$ denotes rationals. Thus $\mathbf{Q}$ is a countable vector space over $\mathbb{Q}$ and $\mathbf{Q}$ is dense in E. Most of our arguments shall take place in $\mathbf{Q}$.

If Z and W are block sequences of $\{e_n\}$ in $\mathbf{Q}$ we write $\mathrm{Z} \leq \mathrm{W}$ if Z is a block sequence of W and $\mathrm{Z} \preceq \mathrm{W}$ if except for finitely many vectors, Z is a block sequence of W. $\mathrm{Z} \doteq \mathrm{W}$ denotes $\mathrm{Z} \preceq \mathrm{W}$ and $\mathrm{W} \preceq \mathrm{Z}$. Given a block sequence W in $\mathbf{Q}$ let $\Sigma(\mathrm{W})$ (resp. $\Sigma_f(\mathrm{W})$) be the set of all infinite (resp. finite) block sequences of W in $\mathbf{Q}$. We let Σ denote the set of all infinite block sequences of $\{e_n\}$ in $\mathbf{Q}$ and let Σ_f denote the set of all finite block sequences of $\{e_n\}$ in $\mathbf{Q}$.

Lemma 2.1 ([P]). *Let E be a Banach space with a basis $\{e_n\}$ and let Σ be as defined above w.r.t. $\{e_n\}$. Let A be a countable set and let $\tau : \Sigma \to 2^A$ be monotone w.r.t. $\preceq$ and inclusion, i.e.,*

$$\forall\, \mathrm{Z}, \mathrm{W} \in \Sigma, \quad \mathrm{Z} \preceq \mathrm{W} \ \Rightarrow\ \tau(\mathrm{Z}) \subseteq \tau(\mathrm{W})$$
$$\textit{or} \quad \forall\, \mathrm{Z}, \mathrm{W} \in \Sigma, \quad \mathrm{Z} \preceq \mathrm{W} \ \Rightarrow\ \tau(\mathrm{Z}) \supseteq \tau(\mathrm{W}).$$

Then there exists $\mathrm{W}_0 \in \Sigma$ so that

$$\forall\, \mathrm{W} \in \Sigma(\mathrm{W}_0), \quad \tau(\mathrm{W}) = \tau(\mathrm{W}_0).$$

Proof. Without loss of generality we may assume that τ is increasing (otherwise consider $\tau'(\mathrm{W}) = A \setminus \tau(\mathrm{W})$). If the conclusion is false, then by transfinite induction and diagonalization we can construct $\{\mathrm{W}_\alpha\}_{\alpha<\omega_1}$ so that if $\alpha < \beta$, then $\mathrm{W}_\alpha < \mathrm{W}_\beta$ and $\tau(\mathrm{W}_\alpha) \subsetneq \tau(\mathrm{W}_\beta)$. But this is impossible since A is countable. □

3. Subsymmetric sequences and minimal spaces

Definition 3.1. A Banach space E is called $C-$*minimal*, for $C \geq 1$ if any infinite dimensional closed subspace of E contains a subspace which is $C-$isomorphic to E.

A Banach space E is called *minimal* if any infinite dimensional closed subspace of E contains a subspace which is isomorphic to E.

Definition 3.2. A basic sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ is called $C-$*subsymmetric*, for $C \geq 1$, if it is $C-$equivalent to any of its infinite subsequences.

A basic sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ is called *subsymmetric*, if it is $C-$subsymmetric, for some $C \geq 1$.

Some authors also require a subsymmetric sequence to be unconditional. However if $\{e_n\}$ is a bounded subsymmetric sequence, then it is either equivalent to the unit vector basis of ℓ_1 by [R] or it is weak Cauchy and hence $\{e_1 - e_2, e_3 - e_4, \dots\}$ is subsymmetric and unconditional. Thus we prefer to use the definition above.

Lemma 3.3. *Let E be a Banach space with a basis. If E contains a $C-$subsymmetric basic sequence, for some constant $C \geq 1$, then for any $\delta > 0$ the space E contains a $(C+\delta)-$subsymmetric block sequence.*

Proof of Lemma 3.3. Let $\{e_i\}_i$ be a basis for E. By $\{e_i^*\}_i$ denote the biorthogonal functionals for $\{e_i\}_i$. Let $\{x_n\}_n \subseteq E$ be a $C-$subsymmetric basic sequence, for $C \geq 1$. We can assume, picking a subsequence of $\{x_n\}_n$ if needed by diagonalization, that for some scalars $\{a_i\}_i$ we have $\lim_{n\to\infty} e_i^*(x_n) = a_i$, $i \in \mathbb{N}$. Put $z_n = x_{2n} - x_{2n-1}$ for $n \in \mathbb{N}$. Then $\{z_n\}_n$ is clearly $C-$subsymmetric.

Fix $\delta > 0$. Pick $\eta > 0$ satisfying $(1+\eta)^2 C < C + \delta$. Since $\lim_{n\to\infty} e_i^*(z_n) = 0$, $i \in \mathbb{N}$, by Lemma 1.2 there is a block sequence $\{y_n\}_n$ which is $(1+\eta)-$equivalent to some subsequence of $\{z_n\}_n$. Thus by the choice of η the sequence $\{y_n\}_n$ is $(C+\delta)-$subsymmetric. □

We say that a Banach space is *saturated* with sequences of a given type, if every infinite dimensional subspace contains a sequence of this type.

Now we present the main results:

Theorem 3.4. *Let E be a Banach space saturated with $C-$subsymmetric basic sequences, for some constant $C \geq 1$. Then for any $\varepsilon > 0$, the space E contains a $(C^2+\varepsilon)-$minimal subspace.*

Corollary 3.5. *A Banach space saturated with subsymmetric basic sequences contains a minimal space.*

Proof of Corollary 3.5. We may assume that E is a Banach space with a basis. It sufficies to show that for some constant $C \geq 1$ there exists a block subspace so that all further block subspaces contain a $C-$subsymmetric block sequence. If not, one can construct a block sequence $\{x_i\}_{i=1}^\infty$ so that for all n no block sequence of $\{x_i\}_{i=n}^\infty$ is $n-$subsymmetric. But then no block sequence of $\{x_i\}_{i=1}^\infty$ is subsymmetric. Thus by Lemma 3.3, the block subspace spanned by $\{x_i\}_{i=1}^\infty$ does not contain a subsymmetric basic sequence. □

Notice that we proved above that a Banach space saturated with subsymmetric sequences contains a "uniformly" minimal subspace, i.e. $C-$minimal for some constant $C \geq 1$.

Proof of Theorem 3.4. We can assume that E is a Banach space with a basis. We will use below the notation introduced above. Assume that E is saturated with $C-$subsymmetric sequences, for some $C \geq 1$, and fix $\varepsilon > 0$.

Pick a scalar $\delta > 0$ satisfying $(C+\delta)^2(1+\delta) \leq C^2+\varepsilon$. By Lemma 3.3 and the density of $\mathbf{Q}$ in E the space E is saturated with $(C+\delta)-$subsymmetric block sequences from the family Σ. We shall produce $\mathrm{Z}_0 \in \Sigma$ so that for any $\mathrm{W} \in \Sigma(Z_0)$ there exists $\mathrm{Y} \in \Sigma(\mathrm{W})$ which is $(C+\delta)^2-$equivalent to Z_0. By the choice of δ this will finish the proof of Theorem 3.4.

From now on, unless otherwise stated, we work in $\mathbf{Q}$. Thus e.g. $\langle \mathrm{W} \rangle$ will denote $\langle \mathrm{W} \rangle \cap \mathbf{Q}$. Put $c = C+\delta$.

Recall that a *tree* $\mathcal{T}$ on an arbitrary set A is a subset of the set $\bigcup_{n=1}^\infty A^n$ such that $\{a_1, \dots, a_n\} \in \mathcal{T}$ whenever $\{a_1, \dots, a_n, a_{n+1}\} \in \mathcal{T}$.

A *branch* of a tree $\mathcal{T}$ is an infinite sequence $\{a_n\}_{n \in \mathbb{N}}$ such that $\{a_1, \dots, a_n\} \in \mathcal{T}$ for any $n \in \mathbb{N}$.

Now we introduce some important terminology. We call a tree $\mathcal{T}$ on $\mathbf{Q}$ a *block tree* if $\mathcal{T} \subseteq \Sigma_f$ and for any $\mathrm{x} \in \mathcal{T}$ the set $\mathcal{T}(\mathrm{x}) = \{x \in E : \ \mathrm{x} \cup \{x\} \in \mathcal{T}\}$ contains an infinite block sequence in $\mathbf{Q}$. Any branch of a block tree is a block sequence. Moreover, since for any $\mathrm{x} \in \mathcal{T}$ we have $\mathcal{T}(\mathrm{x}) \neq \emptyset$, every element $\mathrm{x} \in \mathcal{T}$ is a part of some branch of $\mathcal{T}$.

Definition 3.6. Given sequences $\mathrm{x}, \mathrm{y} \in \Sigma_f \cup \{\Theta\}$, $|\mathrm{x}| \geq |\mathrm{y}|$, a block sequence $\mathrm{Z} \in \Sigma$ and a block tree $\mathcal{T}$ on $\mathbf{Q}$ we write $(\mathrm{x}; \mathrm{Z}) \sim (\mathrm{y}; \mathcal{T})$ if $\mathcal{T} = \bigcup \{\mathcal{T}_\mathrm{X} : \ \mathrm{X} \in \Sigma(\mathrm{Z}), \ \mathrm{X} > \mathrm{x}\}$, where $\{\mathcal{T}_\mathrm{X}\}$ are block trees on $\mathbf{Q}$ satisfying the following conditions:

(1) for every block sequence $\mathrm{X} \in \Sigma(\mathrm{Z}), \mathrm{X} > \mathrm{x}$, and every branch Y of $\mathcal{T}_\mathrm{X}$ we have that $\mathrm{Y} > \mathrm{y}$ and the sequences $\mathrm{x} \cup \mathrm{X}$, $\mathrm{y} \cup \mathrm{Y}$ are $c-$equivalent,
(2) for any block sequences $\mathrm{X}_1, \mathrm{X}_2 \in \Sigma(\mathrm{Z}), \mathrm{X}_1 > \mathrm{x}, \mathrm{X}_2 > \mathrm{x}$, and $n \in \mathbb{N} \cup \{0\}$, if $\mathrm{X}_1 \cap E^n = \mathrm{X}_2 \cap E^n$, then $\mathcal{T}_{\mathrm{X}_1} \cap E^{n+|\mathrm{x}|-|\mathrm{y}|} = \mathcal{T}_{\mathrm{X}_2} \cap E^{n+|\mathrm{x}|-|\mathrm{y}|}$, where $E^0 = \{\Theta\}$.

This means that a tree of block sequences of Z beginning with a finite sequence x can be represented in $\mathcal{T}$ in a special way. In fact we will use the relation defined above only in the case when $|\mathrm{x}| = |\mathrm{y}|$ or $|\mathrm{x}| = |\mathrm{y}| + 1$.

Claim 1. Let $\mathrm{x}, \mathrm{y} \in \Sigma_f \cup \{\Theta\}$, $|\mathrm{x}| \geq |\mathrm{y}|$, $\mathrm{Z} \in \Sigma$, and let $\mathcal{T}$ be a block tree on $\mathbf{Q}$. Assume $(\mathrm{x}; \mathrm{Z}) \sim (\mathrm{y}; \mathcal{T})$.

(1) Let $x_0 \in \langle \mathrm{Z} \rangle$, $\mathrm{x} < x_0$. Then there exists a block subtree $\mathcal{T}' \subseteq \mathcal{T}$ which satisfies $(\mathrm{x} \cup \{x_0\}; \mathrm{Z}) \sim (\mathrm{y}; \mathcal{T}')$.
(2) Let $|\mathrm{x}| > |\mathrm{y}|$, $y_0 \in \mathcal{T} \cap E$. Then $\mathcal{T}[y_0] = \{\{y_1, \ldots, y_n\} : \{y_0, y_1, \ldots, y_n\} \in \mathcal{T}\}$ is a block tree and $(\mathrm{x}; \mathrm{Z}) \sim (\mathrm{y} \cup \{y_0\}; \mathcal{T}[y_0])$.

Proof of Claim 1. For the first case, in the situation as above define $\mathcal{T}'$ by putting

$$\mathcal{T}'_{\mathrm{X}} = \mathcal{T}_{\{x_0\} \cup \mathrm{X}}, \quad \text{for} \quad \mathrm{X} \in \Sigma(\mathrm{Z}),\ \mathrm{X} > x_0.$$

The second case is obvious by the definition of the relation $\sim$, since we can put $(\mathcal{T}[y_0])_{\mathrm{X}} = (\mathcal{T}_{\mathrm{X}})[y_0]$ for any $\mathrm{X} \in \Sigma(\mathrm{Z})$. ☐

Given $\mathrm{W} \in \Sigma$ put

$$\tau(\mathrm{W}) = \{(\mathrm{x}, \mathrm{y}) \in (\Sigma_f \cup \{\Theta\})^2 :\ |\mathrm{x}| \geq |\mathrm{y}|,\ \exists\, \mathrm{Z} \in \Sigma,\ \mathrm{Z} \preceq \mathrm{W},$$
$$\text{and } \exists \text{ a block tree } \mathcal{T} \text{ on } \mathrm{W} \text{ with } (\mathrm{x}; \mathrm{Z}) \sim (\mathrm{y}; \mathcal{T})\}.$$

Take $\mathrm{W}_1 \preceq \mathrm{W}_2$ and a pair $(\mathrm{x}, \mathrm{y}) \in \tau(\mathrm{W}_1)$. Then there exists $\mathrm{Z} \in \Sigma$, $\mathrm{Z} \preceq \mathrm{W}_1$ and a block tree $\mathcal{T}_1$ on W_1 such that $(\mathrm{x}; \mathrm{Z}) \sim (\mathrm{y}; \mathcal{T}_1)$. Put $\mathcal{T}_2 = \mathcal{T}_1 \cap \bigcup_{n \in \mathbb{N}} (\mathrm{W}_2)^n$ (this means cutting off from $\mathcal{T}_1$ sequences containing vectors lying outside W_2). Then $\mathcal{T}_2$ is also a block tree (since $\mathrm{W}_1 \preceq \mathrm{W}_2$) satisfying $(\mathrm{x}; \mathrm{Z}) \sim (\mathrm{y}; \mathcal{T}_2)$. One only has to realize that for any sequence $\mathrm{X} \in \Sigma(\mathrm{Z})$, the tree $(\mathcal{T}_2)_{\mathrm{X}} = \mathcal{T}_{\mathrm{X}} \cap \bigcup_{n \in \mathbb{N}} (\mathrm{W}_2)^n$ satisfies the definition.

Therefore we have shown that the mapping τ is monotone, i.e. if $\mathrm{W}_1 \preceq \mathrm{W}_2$, then $\tau(\mathrm{W}_1) \subseteq \tau(\mathrm{W}_2)$. Hence, on the basis of Lemma 2.1, there exists $\mathrm{W}_0 \in \Sigma$ which is stabilizing for τ.

Claim 2. Let $(\mathrm{x}, \mathrm{y}) \in \tau(\mathrm{W}_0)$, $|\mathrm{x}| > |\mathrm{y}|$. Then for any $\mathrm{W} \in \Sigma(\mathrm{W}_0)$ there is a vector $y_0 \in \langle \mathrm{W} \rangle$ such that $(\mathrm{x}, \mathrm{y} \cup \{y_0\}) \in \tau(\mathrm{W})$.

Proof of Claim 2. In the situation as above, by the stabilization property, for some $\mathrm{Z} \in \Sigma(\mathrm{W})$ and a block tree $\mathcal{T}$ on W we have $(\mathrm{x}; \mathrm{Z}) \sim (\mathrm{y}; \mathcal{T})$ and Claim 1 finishes the proof of Claim 2. ☐

Given $\mathrm{W} \in \Sigma(\mathrm{W}_0)$ let

$$\rho(\mathrm{W}) = \{(\mathrm{x}, \mathrm{y}) \in (\Sigma_f \cup \{\Theta\})^2 :\ |\mathrm{x}| \geq |\mathrm{y}|,\ \exists\, \mathrm{Z} \in \Sigma,\ \mathrm{Z} \doteq \mathrm{W},$$
$$\text{and } \exists \text{ a block tree } \mathcal{T} \text{ on } \mathrm{W}_0 \text{ with } (\mathrm{x}; \mathrm{Z}) \sim (\mathrm{y}; \mathcal{T})\}.$$

Let $\mathrm{W}_1 \succeq \mathrm{W}_2$ and $(\mathrm{x}, \mathrm{y}) \in \rho(\mathrm{W}_1)$. There exists $\mathrm{Z}_1 \in \Sigma$, $\mathrm{Z}_1 \doteq \mathrm{W}_1$ and a block tree $\mathcal{T}_1$ on W_0 such that $(\mathrm{x}; \mathrm{Z}_1) \sim (\mathrm{y}; \mathcal{T}_1)$. Put

$$\mathrm{Z}_2 = \langle \mathrm{Z}_1 \rangle \cap \mathrm{W}_2, \quad \mathcal{T}_2 = \bigcup \{(\mathcal{T}_1)_{\mathrm{X}} :\ \mathrm{X} \in \Sigma(\mathrm{Z}_2)\}.$$

Then obviously $\mathrm{Z}_2 \doteq \mathrm{W}_2$ and $(\mathrm{x}; \mathrm{Z}_2) \sim (\mathrm{y}; \mathcal{T}_2)$, hence $(\mathrm{x}, \mathrm{y}) \in \rho(\mathrm{W}_2)$.

Therefore the mapping ρ is monotone. Let $\mathrm{W}_{00} \in \Sigma(\mathrm{W}_0)$ be stabilizing for ρ, chosen on the basis of Lemma 2.1.

Claim 3. For any $\mathrm{W}, \mathrm{Z} \in \Sigma(\mathrm{W}_{00})$ we have $\tau(\mathrm{W}) = \rho(\mathrm{Z})$.

Proof of Claim 3. By the stabilization property it is enough to prove that $\tau(\mathrm{W}_{00}) = \rho(\mathrm{W}_{00})$. By definition and the stabilization property $\rho(\mathrm{W}_{00}) \subseteq \tau(\mathrm{W}_0) = \tau(\mathrm{W}_{00})$. Now, if $(\mathrm{x},\mathrm{y}) \in \tau(\mathrm{W}_{00})$, then $(\mathrm{x},\mathrm{y}) \in \rho(\mathrm{W})$ for some $\mathrm{W} \leq \mathrm{W}_{00}$, hence again by the stabilization $(\mathrm{x},\mathrm{y}) \in \rho(\mathrm{W}_{00})$. □

By the assumption and Lemma 3.3 there is a $c-$subsymmetric block sequence $\mathrm{Z}_0 = \{z_n\}_{n=1}^{\infty} \in \Sigma(\mathrm{W}_{00})$.

Claim 4. $(\{\Theta\};\mathrm{Z}_0) \sim (\{\Theta\};\Sigma_f(\mathrm{Z}_0))$, in particular $(\{\Theta\},\{\Theta\}) \in \tau(\mathrm{W}_{00})$.

Proof of Claim 4. Take any block sequence $\mathrm{X} = \{x_n\}_{n=1}^{\infty} \in \Sigma(\mathrm{Z}_0)$. Then

$$x_n = \sum_{i=i_n}^{i_{n+1}-1} a_i\, z_i, \quad n \in \mathbb{N},$$

for some scalars $\{a_i\}_{i\in\mathbb{N}}$ and some sequence $\{i_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$. Let

$$\mathcal{T}_{\mathrm{X}} = \left\{ \left\{ \sum_{i=i_1}^{i_2-1} a_i\, z_{\phi(i)}, \ldots, \sum_{i=i_n}^{i_{n+1}-1} a_i\, z_{\phi(i)} \right\} \middle| \phi : \mathbb{N} \to \mathbb{N} \text{ strictly increasing}, n \in \mathbb{N} \right\}.$$

Obviously every set $\mathcal{T}_{\mathrm{X}}$ is a block tree. Moreover, $\Sigma_f(\mathrm{Z}_0) = \bigcup\{\mathcal{T}_{\mathrm{X}} :\ \mathrm{X} \in \Sigma(\mathrm{Z}_0)\}$ and, by the $c-$subsymmetry of the sequence $\{z_n\}_n$, for any $\mathrm{X} \in \Sigma(\mathrm{Z}_0)$ every infinite branch of $\mathcal{T}_{\mathrm{X}}$ is $c-$equivalent to X. The "uniqueness" condition is also satisfied. □

We will show that for any block sequence $\mathrm{W} \in \Sigma(\mathrm{Z}_0)$ there exists $\mathrm{Y} \in \Sigma(\mathrm{W})$ which is c^2-equivalent to Z_0 which will finish the proof of the Theorem.

Let $\mathrm{W} \in \Sigma(\mathrm{Z}_0)$. We will pick by induction block sequences $\{z_{k_n}\}$ and $\{y_n\} \leq \mathrm{W}$ such that $(\mathrm{z}_n, \mathrm{y}_n) \in \tau(\mathrm{W})$ for $n \in \mathbb{N}$, where $\mathrm{z}_n = \{z_{k_1},\ldots,z_{k_n}\}$ and $\mathrm{y}_n = \{y_1,\ldots,y_n\}$, $n \in \mathbb{N}$. This implies in particular that for any $n \in \mathbb{N}$ sequences $\{z_{k_1},\ldots,z_{k_n}\}$ and $\{y_1,\ldots,y_n\}$ are $c-$equivalent, thus also sequences $\{z_{k_n}\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are $c-$equivalent. By the $c-$subsymmetry of the sequence $\{z_n\}_{n\in\mathbb{N}}$ the sequences $\{z_n\}_n$ and $\{y_n\}_n$ are c^2-equivalent, hence Z_0 works.

Put $k_1 = 1$. By Claims 4 and 1 $(\mathrm{z}_1, \{\Theta\}) \in \tau(\mathrm{W}_0) = \tau(\mathrm{W})$. By Claim 2 there is a vector $y_1 \in \mathrm{W}$ such that $(\mathrm{z}_1, \mathrm{y}_1) \in \tau(\mathrm{W})$.

Assume now that we have picked vectors $z_{k_1},\ldots,z_{k_n}$ and $y_1,\ldots,y_n \in \langle \mathrm{W}\rangle$ such that $(\mathrm{z}_n, \mathrm{y}_n) \in \tau(\mathrm{W})$. By Claim 3 $(\mathrm{z}_n, \mathrm{y}_n) \in \rho(\mathrm{Z}_0)$. Therefore for some $\mathrm{Z} \doteq \mathrm{Z}_0$ there is a tree $\mathcal{T}$ on W_0 such that $(\mathrm{z}_n;\mathrm{Z}) \sim (\mathrm{y}_n;\mathcal{T})$. Let k_{n+1} be such that $z_{k_{n+1}} > z_{k_n}$ and $z_{k_{n+1}} \in \langle \mathrm{Z}\rangle$. Then by Claim 1 $(\mathrm{z}_{n+1}, \mathrm{y}_n) \in \tau(\mathrm{W}_0)$. Hence by Claim 2 there is a vector $y_{n+1} \in \langle \mathrm{W}\rangle$ such that $(\mathrm{z}_{n+1}, \mathrm{y}_{n+1}) \in \tau(\mathrm{W})$, which finishes the inductive step and the proof of Theorem 3.4. □

We should point out that there exist minimal spaces with a subsymmetric basis which do not contain any isomorph of c_0 or any ℓ_p $(1 \leq p \leq \infty)$. One such space is due to Th. Schlumprecht [S]. We do not know if this is the case for symmetric bases. Furthermore we have the following problem raised by H. Rosenthal.

Problem. Let E have a basis $\{e_n\}$ with the property that every normalized block basis admits a subsequence equivalent to $\{e_n\}$. Is $\{e_n\}$ equivalent to the unit vector basis of ℓ_p or c_0?

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