

THE NUMBER OF CONNECTED COMPONENTS IN DOUBLE BRUHAT CELLS FOR NONSIMPLY-LACED GROUPS

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(Communicated by John R. Stembridge)

ABSTRACT. We compute the number of connected components in a generic real double Bruhat cell for series B_n and C_n and an exceptional group F_4 .

1. INTRODUCTION AND MAIN RESULT

Let G be a simply connected semisimple algebraic group. Let B and B_- be two \mathbb{R} -split opposite Borel subgroups, N and N_- their unipotent radicals, $H = B \cap B_-$ an \mathbb{R} -split maximal torus of G , and $W = \text{Norm}_G(H)/H$ the Weyl group of G .

The group G has two *Bruhat decompositions*, with respect to B and B_- :

$$G = \bigcup_{u \in W} BuB = \bigcup_{v \in W} B_-vB_- .$$

The *double Bruhat cells* $G^{u,v}$ are defined by $G^{u,v} = BuB \cap B_-vB_-$. The maximal torus H acts freely on $G^{u,v}$ by left (or right) translations. The quotient of $G^{u,v}$ by this action is called the *reduced double Bruhat cell* $L^{u,v} \subset G^{u,v}$ (see [SSVZ], [Z] for a more rigorous definition). Thus, $G^{u,v}$ is biregularly isomorphic to $H \times L^{u,v}$, and all properties of $G^{u,v}$ can be translated in a straightforward way into the corresponding properties of $L^{u,v}$ (and vice versa). In particular, Theorem 1.1 in [FZ] implies that $L^{u,v}$ is biregularly isomorphic to a Zariski open subset of an affine space.

Let $L^{u,v}(\mathbb{R})$ denote the real part of $L^{u,v}$, that is, $L^{u,v}(\mathbb{R}) = L^{u,v} \cap G(\mathbb{R})$, where $G(\mathbb{R})$ is the real part of G . Consider the case when $u = e$ and $v = w_0$, the longest element in W . In this case $L^{u,v}$ is biregularly isomorphic to the intersection of two open opposite Schubert cells $C_{w_0} \cap w_0C_{w_0}$, where $C_{w_0} = (Bw_0B)/B$ is the open Schubert cell in the flag variety G/B . These opposite cells appeared in the literature in various contexts (see e.g. [BFZ], [R1]). Let \sharp denote the number of connected components in $L^{e,w_0}(\mathbb{R})$. Following [Z] we write $\sharp = \sharp(X_n)$, where $X_n = A_n, B_n, \dots, G_2$ runs over all types of simple Lie groups in the Cartan–Killing classification.

The numbers $\sharp(A_n)$ were determined in [SSV97], [SSV98]: it turns out that $\sharp(A_1) = 2$, $\sharp(A_2) = 6$, $\sharp(A_3) = 20$, $\sharp(A_4) = 52$, and $\sharp(A_n) = 3 \cdot 2^n$ for $n \geq 5$. The numbers $\sharp(D_n)$ were determined in [Z]; namely, $\sharp(D_n) = 3 \cdot 2^n$ for $n \geq 4$. It is also

Received by the editors May 8, 2001 and, in revised form, October 25, 2001.

2000 *Mathematics Subject Classification*. Primary 20F55; Secondary 05E15, 14M15.

Key words and phrases. Double Bruhat cells, Coxeter graphs, groups generated by transvections.

shown in [Z] that $\sharp(E_n) = 3 \cdot 2^n$ for $n = 6, 7, 8$. The case G_2 was treated in [R2]: $\sharp(G_2) = 11$ (see also [Z] for another proof of this result). For nonsimply-laced series B_n and C_n , only the simplest case $n = 2$ is known; in this case $\sharp(B_2) (= \sharp(C_2)) = 8$ (see [R2], [Z]).

In this note we calculate $\sharp(X_n)$ for the remaining simple Lie groups of types B_n , C_n , and F_4 , and thus provide a complete solution for the problem posed in [Z, Remark 5.3].

Theorem 1. *For any $n \geq 4$ one has $\sharp(B_n) = \sharp(C_n) = (n + 5) \cdot 2^{n-1}$. Besides, $\sharp(B_3) = \sharp(C_3) = 30$, $\sharp(B_2) = \sharp(C_2) = 8$ and $\sharp(F_4) = 80$.*

In fact, we prove a more general result, and find the number of connected components of $L^{u,v}(\mathbb{R})$ for any generic pair $(u, v) \in W \times W$ (see Theorem 4 below).

The authors would like to thank B. Shapiro and A. Zelevinsky for valuable discussions and encouragement. They express their gratitude to Volkswagen-Stiftung for the financial support of their stay at the Mathematischen Forschungsinstitut Oberwolfach in Summer 2000 (under the program “Research in Pairs”). The first and third authors are also grateful to the Gustafsson Foundation for the financial support of their visits to KTH in the Fall 2000 and in the Spring 2001.

2. PROOFS

We start by recalling the following important construction from [SSVZ], [Z]; in fact, this is not the original construction itself, but rather its version reduced modulo 2.

Let Π be the Coxeter graph of G , and let $s_i (i \in \Pi)$ be the system of simple reflections that generate W . A word $\mathbf{i} = (i_1, \dots, i_m)$ in the alphabet Π is a *reduced word* for $w \in W$ if $w = s_{i_1} \cdots s_{i_m}$, and m is the smallest length of such a factorization. The length of any reduced word for w is called the *length* of w and denoted by $\ell(w)$.

Let \mathfrak{g} be the Lie algebra of G , \mathfrak{h} be the Cartan subalgebra of \mathfrak{g} and $A = (a_{ij})$ be the Cartan matrix. Recall that for $i \neq j$ the indices i and j are adjacent in Π if and only if $a_{ij}a_{ji} \neq 0$; we shall denote this by $\{i, j\} \in \Pi$.

Let us consider the group $W \times W$. It corresponds to a graph $\tilde{\Pi}$ given by the union of two disconnected copies of Π . We identify the vertex set of $\tilde{\Pi}$ with $\{+1, -1\} \times \Pi$, and write a vertex $(\pm 1, i) \in \tilde{\Pi}$ simply as $\pm i$. For each $i \in \Pi$, we set $\varepsilon(\pm i) = \pm 1$ and $|\pm i| = i$. Thus, two vertices i and j of $\tilde{\Pi}$ are adjacent if and only if $\varepsilon(i) = \varepsilon(j)$ and $\{i, j\} \in \Pi$. In this notation, a reduced word for a pair $(u, v) \in W \times W$ is an arbitrary shuffle of a reduced word for u written in the alphabet $-\Pi$ and a reduced word for v written in the alphabet Π . The set of all reduced words for a given pair $(u, v) \in W \times W$ is denoted by $R(u, v)$.

Now let us fix a pair $(u, v) \in W \times W$, and let $d = \ell(u) + \ell(v)$. Let $\mathbf{i} = (i_1, \dots, i_d) \in R(u, v)$ be any reduced word for (u, v) . We associate to \mathbf{i} a $d \times d$ matrix (Ω_{kl}) over the two-element field \mathbb{F}_2 in the following way: set $\Omega_{kl} = 1$ if $|i_k| = |i_l|$ and $\Omega_{kl} = a_{|i_k|, |i_l|} \bmod 2$ if $|i_k| \neq |i_l|$.

Next, we associate with \mathbf{i} a graph $\Sigma(\mathbf{i})$ on the set of vertices $[1, d] = \{1, 2, \dots, d\}$. For $l \in [1, d]$, we denote by $l^- = l_{\mathbf{i}}^-$ the maximal index k such that $1 \leq k < l$ and $|i_k| = |i_l|$; if $|i_k| \neq |i_l|$ for $1 \leq k < l$; then we set $l^- = 0$. The edges of $\Sigma(\mathbf{i})$ are now defined as follows.

A pair $\{k, l\} \subset [1, d]$ with $k < l$ is an edge of $\Sigma(\mathbf{i})$ if it satisfies one of the following three conditions:

- (i) $k = l^-$;
- (ii) $k^- < l^- < k$, $\{|i_k|, |i_l|\} \in \Pi$, and $\varepsilon(i_{l^-}) = \varepsilon(i_k)$;
- (iii) $l^- < k^- < k$, $\{|i_k|, |i_l|\} \in \Pi$, and $\varepsilon(i_{k^-}) = -\varepsilon(i_k)$.

The edges of type (i) are called *horizontal*, and those of types (ii) and (iii) *inclined*. Each inclined edge corresponds to an edge of the graph Π . We shall write $\{k, l\} \in \Sigma(\mathbf{i})$ if $\{k, l\}$ is an edge of $\Sigma(\mathbf{i})$.

We now associate to each $r \in [1, d]$ a transvection $\tau_r: \mathbb{F}_2^d \rightarrow \mathbb{F}_2^d$ defined as follows: $\tau_r(\xi_1, \dots, \xi_d) = (\xi'_1, \dots, \xi'_d)$, where $\xi'_k = \xi_k$ for $k \neq r$, and

$$(1) \quad \xi'_r = \xi_r + \sum_{\{k,r\} \in \Sigma(\mathbf{i})} \Omega_{kr} \xi_k$$

(note that (1) coincides with the reduction modulo 2 of formula (2.2) in [Z]). We call an index $r \in [1, d]$ *\mathbf{i} -bounded* if $r^- > 0$. The set of all bounded indices (and corresponding vertices of $\Sigma(\mathbf{i})$) is denoted by B and its complement is denoted by C .

Let $\Gamma_{\mathbf{i}}$ denote the group of linear transformations of \mathbb{F}_2^d generated by the transvections τ_r for all \mathbf{i} -bounded indices $r \in [1, d]$. The following result was conjectured in [SSVZ] for a simply laced case, and proved in [Z] in the general case (see also [SSV97] for the case of open cells for type A_n).

Theorem 2. *For every reduced word $\mathbf{i} \in R(u, v)$, the connected components of $L^{u,v}(\mathbb{R})$ are in a natural bijection with the $\Gamma_{\mathbf{i}}$ -orbits in \mathbb{F}_2^d .*

This theorem, together with the description of orbits of groups generated by symplectic transvections presented in [SSV98], [SSVZ], form the basis of the enumerative results in the simply-laced case cited in the Introduction.

However, in the nonsimply-laced case, the transvections generating $\Gamma_{\mathbf{i}}$ are no longer symplectic. To handle this case, we have to extend several results of [SSV98], [SSVZ].

Let W^t , $t \in [1, n - 1]$, be the Coxeter group with n generators s_1, \dots, s_n and relations of the form $s_i^2 = 1$, $(s_i s_j)^2 = 1$ for $j > i + 1$, $(s_i s_{i+1})^3 = 1$ for $i \neq t$, and $(s_t s_{t+1})^4 = 1$. Denote by Π^t the Coxeter graph of W^t . Finally, define the $n \times n$ matrix A^t as follows: $a_{t,t+1} = -2$, $a_{t+1,t} = -1$, $a_{i,i+1} = a_{i+1,i} = -1$ for any $i \in [1, n - 1]$, $i \neq t$, $a_{ij} = 0$ for $|i - j| \neq 1$.

Fix a pair $(u, v) \in W^t \times W^t$, take an arbitrary reduced word \mathbf{i} for the pair (u, v) , and build the graph $\Sigma(\mathbf{i})$ and transvections τ_r exactly as above, with Π replaced by Π^t and A replaced by A^t . Observe that for $t = 1$ the above construction describes the C_n case, for $t = n - 1$ it describes the B_n case, and for $n = 4$, $t = 2$ the F_4 case.

Define Π_U^t to be the subgraph of Π^t induced by the vertices $\{1, 2, \dots, t\}$ and Π_L^t to be the complement to Π_U^t in Π^t . In accordance with this partition of Π^t , we subdivide the vertex set of Σ into $U = \{k \in \Sigma: |i_k| \in \Pi_U^t\}$ and its complement L (we omit in the notation the dependence of Σ and other objects on the reduced word \mathbf{i} which is assumed fixed). Together with the partition into bounded and unbounded vertices described above, this gives four subsets, which we denote B_U , B_L , C_U , and C_L ; the subgraph of Σ induced by a subset $X \subseteq \Sigma$ is denoted Σ_X , and \mathbb{F}_2^X is the linear subspace of \mathbb{F}_2^d defined by the condition that all coordinates that correspond to $\Sigma \setminus X$ vanish. The subgroups Γ_U and Γ_L of Γ are defined in a natural way; clearly, Γ is generated by Γ_U and Γ_L .

For any vector $\nu \in \mathbb{F}_2^L$, the action of Γ_U preserves the affine subspace $\nu + \mathbb{F}_2^U$. Identifying $\nu + \mathbb{F}_2^U$ with \mathbb{F}_2^U with the help of the shift $\xi \mapsto \xi - \nu$, we get an action of Γ_U on \mathbb{F}_2^U ; slightly abusing notation, we call it the $\Gamma_U(\nu)$ -action on \mathbb{F}_2^U . Note that for $\nu \neq 0$ the $\Gamma_U(\nu)$ -action is not linear, but rather affine; the $\Gamma_U(0)$ -action coincides with the usual linear action of Γ_U on \mathbb{F}_2^U .

It follows from [SSVZ, Proposition 6.1] that the number of fixed points of the $\Gamma_U(0)$ -action equals 2^t ; the number of nontrivial orbits of this action (those which are not fixed points) we denote by N_U . In a similar fashion, define the number N_L of nontrivial orbits of the action of Γ_L on \mathbb{F}_2^L ; the number of fixed points of this action equals 2^{n-t} . Observe that one can also define the $\Gamma_L(\varkappa)$ -action on \mathbb{F}_2^L for any $\varkappa \in \mathbb{F}_2^U$, but this action does not depend on \varkappa and coincides with the Γ_L -action.

Lemma 1. *For any vector $\nu \in \mathbb{F}_2^L$ there are $2^t + N_U$ orbits of the $\Gamma_U(\nu)$ -action on \mathbb{F}_2^U .*

Proof. Indeed, the $\Gamma_U(\nu)$ -action on \mathbb{F}_2^U is generated by affine transformations of the form $\theta_j(\xi) = \tau_j^U(\xi) + b_j$ for $j \in B_U$, $\xi \in \mathbb{F}_2^U$, where τ_j^U is the symplectic transvection with respect to the restriction of Ω to \mathbb{F}_2^U and b_j depends only on ν . Assume that $\xi^* \in \mathbb{F}_2^U$ is a fixed point of this affine action. Then $\tau_j^U(\xi) - \xi^* = \tau_j^U(\xi - \xi^*)$ for any $j \in B_U$ and $\xi \in \mathbb{F}_2^U$, and hence the orbits of the $\Gamma_U(\nu)$ -action are just the orbits of the $\Gamma_U(0)$ -action shifted by ξ^* . Therefore, the number of affine orbits equals $2^t + N_U$, the number of $\Gamma_U(0)$ -orbits.

It remains to check the existence of a fixed point of the affine action. Such a fixed point should satisfy the equation $M\xi = b(\nu)$ for some $b(\nu) \in \mathbb{F}_2^{B_U}$, where $M: \mathbb{F}_2^U \rightarrow \mathbb{F}_2^{B_U}$ is given by

$$(M\xi)_j = \xi_j + \sum_{\{k,j\} \in \Sigma_U} \xi_k.$$

The kernel of M consists of the fixed points of the $\Gamma_U(0)$ -action. Therefore, its dimension equals $t = |C_U|$, which means that the image of M coincides with $\mathbb{F}_2^{B_U}$. Therefore, equation $M\xi = b$ can be solved for any b , and we are done. \square

The number of Γ -orbits in \mathbb{F}_2^d is determined as follows.

Theorem 3. *Assume that Σ_B is connected. Then the number of Γ -orbits in \mathbb{F}_2^d equals $2^n + 2^{n-t}N_U + 2^tN_L$.*

Proof. First observe that the projections of the orbits of the Γ -action onto \mathbb{F}_2^L are exactly the orbits of the Γ_L -action on \mathbb{F}_2^L . First consider Γ -orbits whose projections onto \mathbb{F}_2^L are fixed points of this Γ_L -action. The number of fixed points of the Γ_L -action is 2^{n-t} , hence by Lemma 1 we see that the number of such Γ -orbits equals $(2^t + N_U)2^{n-t}$.

Next, consider Γ -orbits whose projections onto \mathbb{F}_2^L are nontrivial orbits of the Γ_L -action on \mathbb{F}_2^L . We claim that the number of such Γ -orbits equals 2^tN_L .

Indeed, let us fix a vector $\nu \in \mathbb{F}_2^L$ in such a Γ_L -orbit, and consider the $\Gamma_U(\nu)$ -action on \mathbb{F}_2^U . As before, by Lemma 1 we get an affine action having $2^t + N_U$ orbits for this choice of ν . We shall show that the Γ_L -action can be used to glue these orbits into 2^t Γ -orbits differing only by the values on C_U . To achieve this, it is enough to show that one can change the value ξ_r for any given $r \in B_U$, and to keep all the other ξ_j , $j \in B$, unchanged. This is evidently true if $\tau_r(\xi) \neq \xi$, so in what follows we assume that $\tau_r(\xi) = \xi$.

Denote by $T(\xi)$ the set of all $j \in B_L$ such that $\tau_j(\xi) \neq \xi$; $T(\xi) \neq \emptyset$, since ν belongs to a nontrivial Γ_L -orbit. The connectivity of Σ_B implies the existence of a path joining r with the set $T(\xi)$. Moreover, since the set of all vertices q having the same height $|i_q|$ is connected in Σ_B , there exists a *monotone* path from r to $T(\xi)$, that is, one for which the height changes monotonously along the path. Let $P = (q_0 \in T(\xi), q_1, \dots, q_k = r)$ be a shortest monotone path between $T(\xi)$ and r ; besides, let q_t be the first vertex at height t in this path. Note that since the path P is monotone, all the vertices $q_j, j \in [l, k]$, belong to B_U .

Assume first that $\tau_{q_j}(\xi) = \xi$ for $j \in [l, k]$. Consequently apply $\tau_{q_0}, \tau_{q_1}, \dots, \tau_{q_k}$; upon applying τ_{q_i} , the value ξ_{q_i} is changed, since the only edge of the type $\{q_i, q_j\}, j < i$, is the edge $\{q_i, q_{i-1}\}$ (otherwise the path is not the shortest possible). Hence, applying the whole sequence results in changing the value ξ_r . To restore the values $\xi_{q_i}, i \neq [0, k - 1]$, consequently apply $\tau_{q_{l-1}}, \tau_{q_{l-2}}, \dots, \tau_{q_0}$ followed by $\tau_{q_l}, \tau_{q_{l+1}}, \dots, \tau_{q_{k-1}}$.

Otherwise, let $q_m, m \in [l, k]$, be the vertex of P closest to r for which $\tau_{q_m}(\xi) \neq \xi$. Consequently apply $\tau_{q_m}, \tau_{q_{m+1}}, \dots, \tau_{q_k}$ to change the value ξ_r . To restore the values $\xi_{q_i}, i \in [m, k - 1]$, we have to solve the same problem as above, but now the length of a shortest monotone path to $T(\xi)$ equals $k - 1$, and we are done by induction.

Proceeding in this way, we see that any Γ -orbit whose projection onto \mathbb{F}_2^L does not coincide with a fixed point of the Γ_L -action on \mathbb{F}_2^L contains a vector that vanishes at any point of B_U . Therefore, the only invariants of such an orbit are the values of ξ at the points of C_U . Since the number of these points equals t , we get 2^t Γ -orbits per each nontrivial Γ_L -orbit, which totals $2^t N_L$ Γ -orbits. \square

To prove our main result, we need the following definition. Let $\Sigma = P$ be a path on m vertices $\{1, 2, \dots, m\}$. Define the Γ_P -action on $\mathbb{F}_2^P = \mathbb{F}_2^m$ as the action generated by symplectic transvections $\tau_j^P, j \in [2, m]$, given by

$$\tau_j^P(\zeta) = \zeta + (\zeta_{j-1} + \zeta_{j+1})e_j,$$

where $\{e_j\}$ is the standard basis of \mathbb{F}_2^m .

Lemma 2. *The number of orbits of the Γ_P -action equals $m + 1$. Exactly two of these orbits are fixed points of the Γ_P -action.*

Proof. Let $\zeta \in \mathbb{F}_2^m$ be of the form

$$\zeta = (\underbrace{0 \dots 0}_{l_1} \underbrace{1 \dots 1}_{m_1} \dots \underbrace{0 \dots 0}_{l_k} \underbrace{1 \dots 1}_{m_k} \underbrace{0 \dots 0}_{l_{k+1}}),$$

where $l_1, l_{k+1} \geq 0, l_2, \dots, l_k, m_1, \dots, m_k > 0$; we put $c(\zeta) = k$. It is easy to see that $c(\tau_j^P(\zeta)) = c(\zeta)$ and that $(\tau_j^P(\zeta))_1 = \zeta_1$. Let us prove that if $c(\zeta) = k$ and $\zeta_1 = 1$ (resp., $\zeta_1 = 0$), then there exists $\gamma \in \Gamma_P$ such that

$$\gamma(\zeta) = (\underbrace{1, 0, 1, 0, \dots, 1, 0, 1, 0, \dots, 0}_{2k-1})$$

(resp., $\gamma(\zeta) = (\underbrace{0, 1, 0, 1, \dots, 0, 1, 0, \dots, 0}_{2k})$).

Indeed, if $\zeta = (\zeta_1, \dots, \zeta_{j-1}, \underbrace{1, \dots, 1}_l, 0, \zeta_{j+l+1}, \dots)$, then

$$\tau_{j+1}^P \cdots \tau_{j+l-1}^P(\zeta) = (\zeta_1, \dots, \zeta_{j-1}, \underbrace{1, 0, \dots, 0}_l, \zeta_{j+l+1}, \dots).$$

Similarly, if $\zeta = (\zeta_1, \dots, \zeta_{j-1}, \underbrace{0, \dots, 0}_l, \zeta_{j+l+1}, \dots)$, then

$$(\tau_{j+2}^P \tau_{j+1}^P) \cdots (\tau_{j+l}^P \tau_{j+l-1}^P)(\zeta) = (\zeta_1, \dots, \zeta_{j-1}, 0, 1, \underbrace{0, \dots, 0}_{l-1}, \zeta_{j+l+1}, \dots).$$

Combining transformations of these two types, we can eventually bring ζ to the required form.

Since the number of these forms equals $m + 1$, and any two of them differ either at $c(\zeta)$ or at ζ_1 (or at both of them), we conclude that the number of Γ_P -orbits equals $m + 1$. Evidently, if m is even, then $(0, 1, \dots, 0, 1)$ is a fixed point of the Γ_P -action, while if m is odd, the $(1, 0, \dots, 1, 0, 1)$ is such a fixed point. The only other fixed point is $(0, \dots, 0)$. □

Now we return to the cases B_n and C_n . We say that a pair (u, v) is *generic* if there exists $\mathbf{i} \in R(u, v)$ such that the subgraph $\Sigma_B(\mathbf{i})$ is connected, and the subgraph $\Sigma_{B_L}(\mathbf{i})$ (in the C_n case) or $\Sigma_{B_U}(\mathbf{i})$ (in the B_n case) is E_6 -compatible. One can easily prove that almost all pairs (u, v) are generic, that is, the ratio of the number of generic pairs to the number of all pairs tends to 1 as n tends to ∞ (cp. with a similar result in the A_n -case proved in [SSV99]). Recall that in the C_n (respectively, B_n) case, the graph Σ_U (respectively, Σ_L) is a path. Let m denote the number of vertices in U for the C_n case, and the number of vertices in L for the B_n case. It is easy to see that this number depends only on the pair (u, v) and does not depend on the reduced word $\mathbf{i} \in R(u, v)$.

Theorem 4. *Let (u, v) be a generic pair. Then the number of connected components in $L^{u,v}(\mathbb{R})$ equals $(m + 5) \cdot 2^{n-1}$ for both types B_n and C_n .*

Proof. Since the pair (u, v) is generic, there exists $\mathbf{i} \in R(u, v)$ such that the subgraph of $\Sigma(\mathbf{i})$ induced by B is connected. Hence, by Theorem 3, the number of $\Gamma_{\mathbf{i}}$ -orbits for type C_n equals $2^n + 2^{n-1}N_U + 2N_L$, and for type B_n equals $2^n + 2^{n-1}N_L + 2N_U$. Besides, by [SSVZ, Th. 7.2], the E_6 -compatibility condition implies that $N_L = 2^n$ for type C_n , and $N_U = 2^n$ for type B_n . Moreover, by Lemma 2, $N_U = m - 1$ for type C_n , and $N_L = m - 1$ for type B_n . Therefore, in both cases the total number of orbits equals $2^n + (m - 1) \cdot 2^{n-1} + 2^{n+1} = (m + 5) \cdot 2^{n-1}$. By Theorem 2, this number equals the number of connected components in $L^{u,v}(\mathbb{R})$. □

To prove Theorem 1 stated in the Introduction one has to check that the pair (e, w_0) is generic for $n \geq 4$. This fact follows immediately from Figure 1 presenting the graph $\Sigma(\mathbf{i})$ and its corresponding subgraphs for $n = 4$ and $\mathbf{i} = 1234123412341234$.

Consider now the cases $n = 2, 3$. One can easily check that the subgraphs Σ_B remain connected, although the pair (e, w_0) is no longer generic; therefore, Theorem 3 remains valid. Besides, one gets $N_U = N_L = 1$ for types B_2 and C_2 , $N_U = 2, N_L = 7$ for type C_3 , and $N_U = 7, N_L = 2$ for type B_3 . Thus, Theorem 3 yields $\sharp_2 = 4 + 2 + 2 = 8$ and $\sharp_3 = 8 + 8 + 14 = 30$.

To treat the case of F_4 , we first consider the graph $S = S(m)$ defined as follows: S contains vertices $\{1, \dots, 2m\}$ arranged into two levels, the lower (resp. upper) level is formed by odd-numbered (resp. even-numbered) vertices. Horizontal edges are of the form $(2i, 2i + 2)$ and $(2i - 1, 2i + 1)$, and inclined edges are of the form $(2i + 1, 2i + 2)$ and $(2i, 2i + 1)$, where i runs from 1 to $m - 1$ (see Figure 1e). It is convenient to represent elements of \mathbb{F}_2^S by vectors $\zeta = (\zeta_i)_{i=1}^{2m} \in \mathbb{F}_2^{2m}$. The Γ_S -action

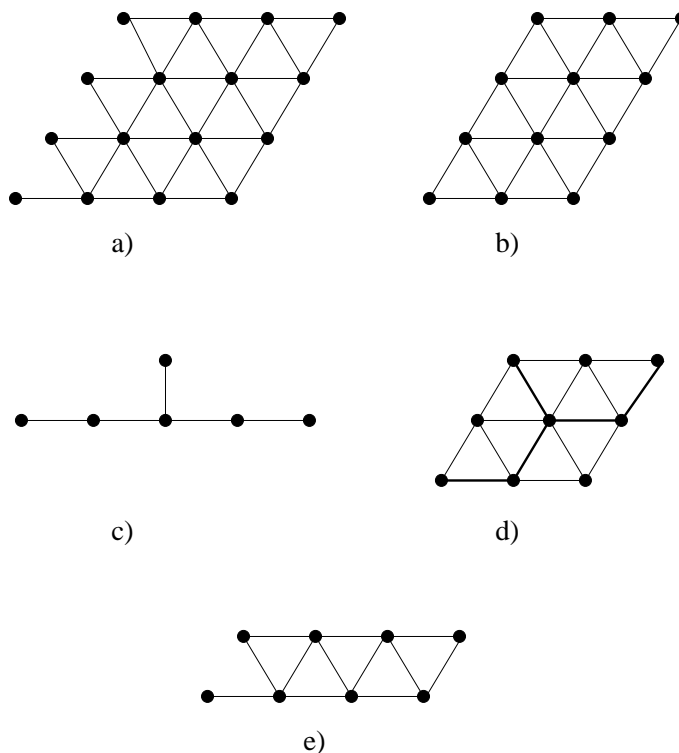


FIGURE 1. To the proof of Theorem 1: a) graph $\Sigma(\mathbf{i})$; b) graph $\Sigma_B(\mathbf{i})$; c) graph E_6 ; d) graph $\Sigma_{B_L}(\mathbf{i})$ and an induced E_6 in it; e) graph $S(4)$.

on \mathbb{F}_2^S is generated by transvections τ_j^S , $j \in [3, 2m]$, defined by

$$(2) \quad \tau_j^S(\zeta) = \zeta + (\zeta_{j-2} + \zeta_{j-1} + \zeta_{j+1} + \zeta_{j+2})e_j,$$

where $\zeta_i = 0$ if $i > 2m$.

Lemma 3. *Let $m > 2$. Then every nontrivial orbit of the Γ_S -action contains either an element of the form $(\zeta_1, \zeta_2, \zeta_3, \zeta_4, 0, \dots, 0)$ where not all ζ_i are equal to zero, or the element $\bar{\zeta} = (0, 0, 1, 1, 1, 0, \dots, 0)$.*

Proof. Let us fix a nontrivial orbit \mathcal{O} of the Γ_S -action. To prove the statement, it suffices to show that for any $\xi \in \mathcal{O}$ of the form

$$\xi = (\xi_1, \dots, \xi_{j-1}, 1, \underbrace{0, \dots, 0}_{2m-j})$$

such that $j > 4$ and $\xi \neq \bar{\zeta}$, there exists $\gamma \in \Gamma_S$ such that $\gamma(\xi)_i = 0$ for $i \geq j$.

If the set $T = \{i: 3 \leq i \leq j, \tau_i^S(\xi) \neq \xi\}$ is not empty (this is clearly the case for $j = 2m$), we denote by k the largest element in T and define γ as the product of τ_i^S along any shortest path from k to j . Then $\gamma(\xi)_i = 0$ for $i \geq j$.

Otherwise, $T = \emptyset$ and the smallest i such that $\tau_i^S(\xi) \neq \xi$ is equal either to $j + 1$ or to $j + 2$. In the first case, ξ has to be of the form

$$\xi = (\xi_1, \dots, \xi_{j-5}, 0, 1, 0, 0, 1, 0, \underbrace{0 \dots 0}_{2m-j-1}).$$

Define $\gamma = \tau_{j+1}^S \tau_j^S \tau_{j-2}^S \tau_{j-1}^S \tau_{j+1}^S$. Then

$$\gamma(\xi) = (\xi_1, \dots, \xi_{j-5}, 0, 1, 1, 1, \underbrace{0 \dots 0}_{2m-j+1}).$$

In the second case, either $\xi = \bar{\zeta}$ and we are done, or

$$\xi = (\xi_1, \dots, \xi_{j-6}, 0, 0, 0, 1, 1, 1, 0, 0, \underbrace{0 \dots 0}_{2m-j-2}),$$

in which case we put $\gamma = \tau_{j+1}^S \tau_{j-1}^S \tau_{j-3}^S \tau_{j+2}^S \tau_j^S \tau_{j-1}^S \tau_{j+1}^S \tau_{j+2}^S$. Then

$$\gamma(\xi) = (\xi_1, \dots, \xi_{j-5}, 0, 0, 1, 1, 1, \underbrace{0 \dots 0}_{2m-j+1}).$$

This finishes the proof. □

Corollary. *If $m > 2$, then the number of orbits of the Γ_S -action is equal to 12. Four of these orbits are fixed points of the action.*

Proof. It follows from (2) that for every choice of $\alpha, \beta \in \mathbb{F}_2$ there is exactly one fixed point of the Γ_S -action with $\zeta_{2n-1} = \alpha, \zeta_{2n} = \beta$. Thus, we have four orbits that are fixed points of the action.

By the previous lemma, any other orbit is either the orbit through $\bar{\zeta}$ or the orbit through an element of the form $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, 0, \dots, 0)$, where ζ_i cannot all be equal to zero. It is easy to see that if $\zeta_3 \neq \zeta_2$, then $\tau_4^S(\zeta) \neq \zeta$; moreover, either $\tau_3^S(\zeta) \neq \zeta$, or $\tau_3^S \tau_4^S(\zeta) \neq \tau_4^S(\zeta)$. Besides, if $\zeta_3 = \zeta_2$ and $\zeta_4 = \zeta_1 + \zeta_2$, then $\tau_3^S(\zeta) = \tau_4^S(\zeta) = \zeta$. This means that the number of nontrivial orbits does not exceed 8.

A nonhomogeneous quadratic form

$$Q_S(\xi) = \sum_{i \in S} \xi_i + \sum_{(i,j) \in S} \xi_i \xi_j$$

is an invariant of the Γ_S -action (see [SSVZ]) along with the values of ξ_1, ξ_2 . Now, to finish the proof it is sufficient to notice that the triple $(\xi_1, \xi_2, Q_S(\xi))$ takes different values on the following eight elements:

$$(1, 1, 1, 1, 0, \dots, 0), (1, 1, 1, 0, 0, \dots, 0), (1, 0, 0, 1, 0, \dots, 0), (1, 0, 0, 0, 0, \dots, 0), \\ (0, 1, 1, 1, 0, \dots, 0), (0, 1, 1, 0, 0, \dots, 0), (0, 0, 1, 0, 0, \dots, 0), (0, 0, 1, 1, 1, 0, \dots, 0).$$

□

We are now in a position to finish the proof of Theorem 1.

Theorem 5. $\sharp(F_4) = 80$.

Proof. Recall that $\mathbf{i} = (1234)^6$ is a reduced word for w_0 in the Weyl group that corresponds to F_4 . We can use Theorem 3 again. In this case, $n = 4, t = 2$ and both subgraphs Σ_L and Σ_U coincide with $S(6)$. Then, by Theorem 3 and Corollary to Lemma 3, $\sharp(F_4) = 2^4 + 2 * 2^2 * 8 = 80$. □

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