

OSCILLATION OF LINEAR HAMILTONIAN SYSTEMS

FANWEI MENG AND ANGELO B. MINGARELLI

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ABSTRACT. We establish new oscillation criteria for linear Hamiltonian systems using monotone functionals on a suitable matrix space. In doing so we develop new criteria for oscillation involving general monotone functionals instead of the usual largest eigenvalue. Our results are new even in the particular case of self-adjoint second order differential systems.

1. INTRODUCTION

Consider the linear Hamiltonian system

$$(1.1) \quad \begin{cases} x' = A(t)x + B(t)y, \\ y' = C(t)x - A^*(t)y, \end{cases} \quad t \geq t_0,$$

where $A(t)$, $B(t)$, $C(t)$ are $n \times n$ -matrices, and B , C are Hermitian, *i.e.*, $B^*(t) = B(t)$, $C^*(t) = C(t)$. The space of such matrices will be denoted by \mathcal{S} in the sequel. As usual, M^* is the conjugate transpose of the matrix M .

A Hermitian matrix $M \in C^{n \times n}$ is positive semi-definite (positive definite) if for all $u \in C^n$, $u \neq 0$, $u^*Mu \geq 0$ (> 0). A positive semi-definite (positive definite) Hermitian matrix M will be denoted by $M \geq 0$ ($M > 0$) and the usual ordering of the eigenvalues of M by $\lambda_1(M) \geq \lambda_2(M) \geq \dots \geq \lambda_n(M)$.

We also consider the corresponding matrix equation

$$(1.2) \quad \begin{cases} X' = A(t)X + B(t)Y, \\ Y' = C(t)X - A^*(t)Y, \end{cases} \quad t \geq t_0.$$

For any two solutions X_1 , Y_1 and X_2 , Y_2 of (1.2) the “Wronskian” $X_1^*(t)Y_2(t) - Y_1^*(t)X_2(t)$ is a constant matrix. In particular, for any solution X , Y of (1.2), $X^*(t)Y(t) - Y^*(t)X(t)$ is a constant matrix. The solution X , Y of (1.2) is said to be *conjoined* if

$$X^*(t)Y(t) - Y^*(t)X(t) = 0.$$

A conjoined solution X , Y of (1.2) is said to be a conjoined basis of (1.1) or (1.2) if the rank of the $2n \times n$ -matrix $(X(t); Y(t))$ is n .

Two distinct points a and b in $[0, \infty)$ are said to be (mutually) conjugate with respect to (1.1) if there exists a solution $z(t) = (x(t), y(t))$ of (1.1) with $x(a) = 0 =$

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$x(b)$ and $x(t) \neq 0$ on the subinterval with end-points a and b . The system (1.1) is said to be disconjugate on a subinterval J of $[0, \infty)$ if no two distinct points of J are conjugate. If (1.1) is disconjugate on J and X, Y is the conjoined basis of (1.1) satisfying $X(a) = 0, Y(a) = I$, the identity $n \times n$ matrix, $a \in J$, then $\det X(t) \neq 0$ for $t \in J, t \neq a$. A conjoined basis X and Y of (1.2) is said to be oscillatory on $[0, \infty)$ if $\det X(t)$ has arbitrarily large zeros.

Let $\Phi(t)$ be a fundamental matrix for the linear equation $x' = A(t)x$. The pair $(A(t), B(t))$ is said to be *controllable* if the rows of $\Phi^{-1}(t)B(t)$ are linearly independent over any subinterval of $[0, \infty)$ (see [5, pp. 36-37], [9, p. 107]). Suppose that $B(t) \geq 0$ and the pair (A, B) is controllable, and there exists an oscillatory conjoined basis of (1.2). Then from Sturm's separation theorem [5, Theorem 16, p. 71] and [9, Theorem 7.3.5], it follows that every conjoined basis of (1.2) is oscillatory, and hence, the system (1.1) or (1.2) is called oscillatory. In this case, the definition of *oscillation* that we use agrees with the *non-disconjugacy* of (1.1) or (1.2) on any neighborhood of $+\infty$.

In the case when $A(t) \equiv 0, B(t) > 0$, (1.2) reduces to the second order self-adjoint matrix differential system

$$(1.3) \quad (P(t)X')' + Q(t)X = 0$$

with $P(t) = B^{-1}(t), Q(t) = -C(t)$. The oscillatory and non-oscillatory behavior of (1.3) has been extensively studied by many authors [1]–[8], [10]–[13]. A discrete version of (1.3) is studied in [14]. In particular, for the case when $P(t) \equiv I, i.e.,$ for the equation

$$(1.4) \quad X'' + Q(t)X = 0,$$

there was a conjecture (see Hinton and Lewis [8]) that (1.4) is oscillatory if

$$\lim_{t \rightarrow \infty} \lambda_1 \left[\int_{t_0}^t Q(s) ds \right] = \infty.$$

This conjecture was settled with some additional assumptions on the rate of growth of the trace of $\int_0^t Q(s) ds$ by Mingarelli [12], Kwong *et al.* [10], Butler and Erbe [1], [2], and Butler *et al.* [3]. This conjecture was proved by Kwong and Kaper [10] in the two-dimensional case, and by Byers *et al.* [4] in the n -dimensional case. Here we list the main results of [6] as follows:

Theorem A ([6, Theorem 1]). *Let $H(t, s)$ and $h(t, s)$ be continuous on $D = \{(t, s) : t \geq s \geq t_0\}$ such that $H(t, t) = 0$ for $t \geq t_0$, and $H(t, s) > 0$ for $t > s \geq t_0$. We may assume further that the partial derivative $\frac{\partial}{\partial s} H(t, s) \equiv H_s(t, s)$ is non-positive and continuous for $t \geq s \geq t_0$ and $h(t, s)$ is defined by*

$$H_s(t, s) = -h(t, s) \sqrt{H(t, s)}, \quad (t, s) \in D.$$

Finally, we assume that

$$(1.5) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \lambda_1 \left[\int_{t_0}^t \left(H(t, s) Q(s) - \frac{1}{4} h^2(t, s) P(s) \right) ds \right] = \infty;$$

then (1.3) is oscillatory. However, if $n = 2$, $Q(t) = \text{diag}(\beta t^{-2}, \gamma t^{-2})$ with $\beta \geq \gamma > 0$ in (1.4), then

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \lambda_1 \left[\int_1^t \left(H(t, s)Q(s) - \frac{1}{4}h^2(t, s)I \right) ds \right] \\ & \leq \limsup_{t \rightarrow \infty} \int_1^t \frac{\beta}{s^2} ds = \beta < \infty, \end{aligned}$$

where $H(t, s)$ and $h(t, s)$ are defined as in Theorem A. On the other hand, one can show directly that the 2-dimensional Euler differential system

$$(1.6) \quad X'' + \text{diag} \left(\frac{\beta}{t^2}, \frac{\gamma}{t^2} \right) X = 0, \quad t \geq 1,$$

is oscillatory when $\beta > \frac{1}{4}$ since for such a β , the corresponding vector equation (1.1) has the oscillatory solution $x(t) = \text{col}(x_1(t), 0)$ where $x_1(t)$ is an oscillatory solution of the scalar Euler equation $x'' + \beta t^{-2} x = 0$. Therefore, the results in [6] do not apply to (1.6).

In this paper we investigate the oscillation of (1.1)–(1.2) using criteria involving general monotone functionals on the space of $n \times n$ Hermitian matrices and a generalized Riccati transformation. Our results will improve the above-mentioned oscillation results even for the self-adjoint differential system (1.3) as well as move in the direction of more general functionals other than the largest eigenvalue.

2. MAIN RESULTS

As we pointed out above we will be replacing the functional $\lambda_1(A)$ on \mathcal{S} by more general functionals. A functional $q : \mathcal{S} \rightarrow \mathbf{R}$ is said to be *subhomogeneous of degree c* if there exists a $c \in \mathbf{R}$ such that for any $A \in \mathcal{S}$ and any $\lambda \geq 1$, $q(\lambda A) \leq \lambda^c q(A)$. Such a functional is said to be *monotone* (or *nondecreasing*) if whenever $A - B \geq 0$ we have $q(A) \geq q(B)$ for $A, B \in \mathcal{S}$.

In particular, we observe that because of the classical minimax characterization of the eigenvalues of a matrix in \mathcal{S} , the functional $q(A) = \lambda_1(A)$ is a monotone subhomogeneous functional of degree 1 traditionally called the “largest eigenvalue” functional. On the other hand, it is readily verified that the nonlinear trace functional on \mathcal{S} defined by $q(A) = \text{tr}(A + I)$ is also monotone and subhomogeneous of degree 1. Finally, it is readily verified that if $P \geq 0$ in \mathcal{S} , then the nonlinear functional $q(A) = \lambda_1(A + P)$ is monotone and subhomogeneous of degree 1 and $\lambda_1(A + P) \geq \lambda_1(A)$ on \mathcal{S} .

Theorem 1. *We assume that $B(t) > 0$ for all $t \geq t_0$. Let $H(t, s)$ and $h(t, s)$ be continuous on $D = \{(t, s) : t \geq s \geq t_0\}$ such that $H(t, t) = 0$ for $t \geq t_0$, $H(t, s) > 0$ for $t > s \geq t_0$, and that for all sufficiently large $s \in \mathbf{R}$, $\liminf_{t \rightarrow \infty} H(t, s) \geq 1$. We assume further that the partial derivative $\frac{\partial}{\partial s} H(t, s) = H_s(t, s)$ is non-positive and is continuous for $t \geq s \geq t_0$, and we define a function $h(t, s)$ by*

$$H_s(t, s) = -h(t, s)\sqrt{H(t, s)}, \quad (t, s) \in D.$$

If there exists a function $f \in C^1[t_0, \infty)$ and a monotone subhomogeneous functional q of degree c on \mathcal{S} such that

$$(2.1) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)^c} q \left(\int_{t_0}^t (H(t, s)R_1(s) - \frac{1}{4}a(s)h^2(t, s)B_1^{-1}(s)) ds \right) = \infty,$$

where

$$a(t) = \exp \left\{ -2 \int^t f(s) ds \right\}, \quad B_1(t) = \Phi^{-1}(t)B(t)[\Phi^*(t)]^{-1}, \\ R_1(t) = \Phi^*(t)R(t)\Phi(t);$$

here $\Phi(t)$ is a fundamental matrix of the linear equation $v' = A(t)v$, and

$$R(t) = a(t) \left\{ -C(t) - f(t)(A^*(t)B^{-1}(t) + B^{-1}(t)A(t)) \right. \\ \left. + f^2(t)B^{-1}(t) - (f(t)B^{-1}(t))' \right\}.$$

Then (1.2) is oscillatory.

Proof. Assume to the contrary that (1.2) is nonoscillatory. Then $X(t)$ is nonsingular for all sufficiently large t , say $t \geq t_0$. For any conjoined basis $X(t)$, $Y(t)$ of (1.2), define

$$(2.2) \quad V(t) = a(t)\{Y(t)X^{-1}(t) + f(t)B^{-1}(t)\}, \quad t \geq t_0.$$

This and (1.2) imply

$$(2.3) \quad V'(t) = -2f(t)V(t) + a(t)\{Y'(t)X^{-1}(t) - Y(t)X^{-1}(t)X'(t)X^{-1}(t) + (f(t)B^{-1}(t))'\} \\ = -R(t) - \frac{1}{a(t)}V(t)B(t)V(t) - A^*(t)V(t) - V(t)A(t).$$

If we let $W(t) = \Phi^*(t)V(t)\Phi(t)$, then from (2.3), we have

$$(2.4) \quad W'(t) = \Phi^*(t)A^*(t)V(t)\Phi(t) + \Phi^*(t)V(t)A(t)\Phi(t) + \Phi^*(t)V'(t)\Phi(t) \\ = -\Phi^*(t)R(t)\Phi(t) - \frac{1}{a(t)}\Phi^*(t)V(t)\Phi(t)\Phi^{-1}(t)B(t)[\Phi^*(t)]^{-1}\Phi^*(t)V(t)\Phi(t) \\ = -R_1(t) - \frac{1}{a(t)}W(t)B_1(t)W(t), \quad t \geq t_0.$$

Multiplying (2.4) with t replaced by s , by $H(t, s)$ and integrating from t_0 to t , we obtain

$$\int_{t_0}^t H(t, s)R_1(s)ds \\ = - \int_{t_0}^t H(t, s)W'(s)ds - \int_{t_0}^t \frac{H(t, s)}{a(s)}W(s)B_1(s)W(s)ds \\ = -H(t, s)W(s) \Big|_{t_0}^t - \int_{t_0}^t \left[-H_s(t, s)W(s) + \frac{H(t, s)}{a(s)}W(s)B_1(s)W(s) \right] ds \\ = H(t, t_0)W(t_0) - \int_{t_0}^t \left[h(t, s)\sqrt{H(t, s)}W(s) + \frac{H(t, s)}{a(s)}W(s)B_1(s)W(s) \right] ds.$$

Since $B_1(t) > 0$, letting $U(t) = \sqrt{B_1(t)}$, we have

$$\begin{aligned} & \int_{t_0}^t H(t, s)R_1(s)ds \\ &= H(t, t_0)W(t_0) - \int_{t_0}^t h(t, s)\sqrt{H(t, s)}U^{-1}(s)[U(s)W(s)U(s)]U^{-1}(s)ds \\ & \quad - \int_{t_0}^t \frac{H(t, s)}{a(s)}U^{-1}(s)[U(s)W(s)U(s)]^2U^{-1}(s)ds \\ &= H(t, t_0)W(t_0) + \frac{1}{4} \int_{t_0}^t h^2(t, s)a(s)B_1^{-1}(s)ds \\ & \quad - \int_{t_0}^t U^{-1}(s) \left\{ \sqrt{\frac{H(t, s)}{a(s)}}[U(s)W(s)U(s)] + \frac{1}{2}\sqrt{a(s)}h(t, s)I \right\}^2 U^{-1}(s)ds. \end{aligned}$$

Hence, we have

$$\int_{t_0}^t \left[H(t, s)R_1(s) - \frac{1}{4}a(s)h^2(t, s)B_1^{-1}(s) \right] ds \leq H(t, t_0)W(t_0), \quad t \geq t_0.$$

Since q is monotone on \mathcal{S} it follows that for $t \geq t_0$,

$$(2.5) \quad q \left(\int_{t_0}^t (H(t, s)R_1(s) - \frac{1}{4}a(s)h^2(t, s)B_1^{-1}(s)) ds \right) \leq q(H(t, t_0)W(t_0)).$$

Now we choose t sufficiently large so that $H(t, t_0) \geq 1$, for $t \gg t_0$. Since q is subhomogeneous of degree c on \mathcal{S} we find

$$(2.6) \quad \frac{1}{H(t, t_0)^c} q \left(\int_{t_0}^t (H(t, s)R_1(s) - \frac{1}{4}a(s)h^2(t, s)B_1^{-1}(s)) ds \right) \leq q(W(t_0)),$$

for $t \gg t_0$. Taking the upper limit as $t \rightarrow \infty$ in both sides of (2.6) we see that the right-hand side is always bounded, which contradicts condition (2.1). This completes the proof. \square

Remark 1. This result may be strengthened somewhat in the case where $q(A) = \lambda_1(A)$ (and $c = 1$). In this case we note that the proof may be modified slightly so that the liminf hypothesis is unnecessary since the subhomogeneity condition actually holds for all $\lambda > 0$ and not only for those $\lambda \geq 1$.

Corollary 1. *Let $H(t, s) = (t - s)^\alpha$, $t \geq s \geq t_0$, where $\alpha > 1$, let $P \geq 0$ be any fixed matrix in \mathcal{S} and let $q(A) = \lambda_1(A + P)$ be a bounded perturbation of the largest eigenvalue functional. Suppose that there exists a function $f \in C^1[t_0, \infty)$, satisfying*

$$(2.7) \quad \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \lambda_1 \left[\int_{t_0}^t \left((t - s)^\alpha R_1(s) - \frac{\alpha^2}{4}(t - s)^{\alpha-2} a(s) B_1^{-1}(s) \right) ds + P \right] = \infty,$$

where $R_1(t)$, $a(t)$, $B_1(t)$ are the same as Theorem 1. Then (1.2) is oscillatory.

Modifying the hypotheses of Theorem 1 we can obtain the following result as a special case.

Theorem 2. Let $q(A) = \lambda_1(A)$ be subhomogeneous of degree 1. If (2.1) is replaced by the two conditions

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \lambda_1 \left[\int_{t_0}^t a(s) h^2(t, s) B_1^{-1}(s) ds \right] < \infty$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \lambda_1 \left[\int_{t_0}^t H(t, s) R_1(s) ds \right] = \infty,$$

then (1.2) is oscillatory.

Example 1. Theorems 1 and 2 generalize and improve [6, Theorems 1 and 2]. To see that our results improve the results in [6] and this in a new direction, let us return to the Euler differential system (1.6). In the Hamiltonian system corresponding to (1.6) we set $A(t) \equiv 0$, $B(t) = I$, $C(t) = -diag(\frac{\beta}{t^2}, \frac{\gamma}{t^2})$, and $\Phi(t) \equiv I$ for $t \geq 1$. Choose $f(t) = -\frac{1}{2t}$; then $a(t) = t$, $f'(t) = \frac{1}{2t^2}$ and $B_1(t) = B(t) = I$,

$$R_1(t) = R(t) = t \left\{ diag\left(\frac{\beta}{t^2}, \frac{\gamma}{t^2}\right) + f^2(t)I - f'(t)I \right\}.$$

Since $\alpha > 1$, applying the Mean Value Theorem to $g(x) = (t - x)^\alpha$ for $0 \leq x \leq s (\leq t)$ we deduce that

$$(t - s)^\alpha \geq t^\alpha - \alpha s t^{\alpha-1}, \text{ for } t \geq s \geq 0.$$

Thus,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \lambda_1 \left[\int_1^t \left((t - s)^\alpha R_1(s) - \frac{\alpha^2}{4} (t - s)^{\alpha-2} a(s) I \right) ds \right] \\ &= \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \int_1^t \left((t - s)^\alpha s \left[\frac{\beta}{s^2} + \frac{1}{4s^2} - \frac{1}{2s^2} \right] - \frac{\alpha^2}{4} (t - s)^{\alpha-2} s \right) ds \\ &= \frac{4\beta - 1}{4} \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \int_1^t \frac{(t - s)^\alpha}{s} ds - \frac{\alpha}{4(\alpha - 1)} \\ &\geq \frac{4\beta - 1}{4} \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \int_1^t \frac{t^\alpha - \alpha s t^{\alpha-1}}{s} ds - \frac{\alpha}{4(\alpha - 1)} \\ &= \frac{4\beta - 1}{4} \limsup_{t \rightarrow \infty} \left[\log t - \frac{\alpha(t - 1)}{t} \right] - \frac{\alpha}{4(\alpha - 1)} = \infty \text{ (since } \beta > 1/4 \text{)}. \end{aligned}$$

It follows from Corollary 1 with $P = 0$ that (1.6) is oscillatory if $\beta > \frac{1}{4}$.

Example 2. Consider the differential system

$$(2.8) \quad [t^2 Y'(t)]' + diag(\beta, \gamma) Y(t) = 0,$$

where, $\beta > \gamma > 0$, and $\beta > \frac{1}{4}$ are constants and $Y(t)$ is a 2×2 -matrix. In the corresponding Hamiltonian system to (2.8) we set $A(t) = 0$, $B(t) = \frac{1}{t^2} I$, $C(t) = -diag(\beta, \gamma)$ and $\Phi(t) \equiv I$, for $t \geq 1$. We can easily show that (2.8) is oscillatory by the change of variable $t \rightarrow x = -\frac{1}{t}$ which transforms (2.8) into

$$Y''(x) + diag\left(\frac{\beta}{x^2}, \frac{\gamma}{x^2}\right) Y(x) = 0,$$

and so the result of Example 1 yields the same conclusion, namely that (2.8) is oscillatory at infinity.

We could also have obtained the same conclusion using Corollary 1 with the choices $f(t) = \frac{1}{2t}$, which implies $a(t) = \frac{1}{t}$ and

$$\begin{aligned} B_1(t) = B(t) &= \frac{1}{t^2}I, \\ R_1(t) = R(t) &= \frac{1}{t}\{diag(\beta, \gamma) + f^2(t)B^{-1}(t) - (f(t)B^{-1}(t))'\} \\ &= \frac{1}{t}\{diag(\beta, \gamma) - \frac{1}{4}I\}. \end{aligned}$$

Example 3. Consider the function defined by

$$H(t, s) = \left(\int_s^t \frac{d\tau}{\Theta(\tau)} \right)^\alpha, \text{ for } t \geq s \geq t_0,$$

where $\alpha > 1$ and Θ is a positive continuous function on $[t_0, \infty)$ such that

$$\int_{t_0}^{\infty} \frac{1}{\Theta(\tau)} d\tau = \infty.$$

Clearly $H(t, t) = 0$, for $t \geq t_0$, and $H(t, s) > 0$, for $t > s \geq t_0$. In addition,

$$-\frac{\partial}{\partial s}H(t, s) = \frac{\alpha}{\Theta(s)} \left(\int_s^t \frac{d\tau}{\Theta(\tau)} \right)^{\alpha-1}$$

and

$$h(t, s) = \frac{\alpha}{\Theta(s)} \left(\int_s^t \frac{d\tau}{\Theta(\tau)} \right)^{\alpha/2-1}, \quad t \geq s \geq t_0.$$

One of the important cases to be considered is the one where $\Theta(\tau) = \tau^\beta$ with β real.

Concluding remarks. One can also formulate results in the case where $H(t, s) = e^{t-s} - 1$ for $t \geq s \geq 0$, a case in which Example 3 does not apply but Theorems 1 and 2 do. Thus, by applying Theorems 1 and 2 in the special cases considered, one can derive many new criteria for the oscillation of the system (1.2) by varying the non-linear functionals as described.

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DEPARTMENT OF MATHEMATICS, QUFU NORMAL UNIVERSITY, QUFU, SHANDONG, 273165, PEOPLE'S REPUBLIC OF CHINA

E-mail address: `fwmeng@qfnu.edu.cn`

SCHOOL OF MATHEMATICS AND STATISTICS, CARLETON UNIVERSITY, OTTAWA, ONTARIO, CANADA K1S 5B6