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GAUSSIAN CURVATURE IN THE NEGATIVE CASE

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ABSTRACT. In this paper, we reinvestigate an old problem of prescribing Gaussian curvature in the negative case.

In 1974, Kazdan and Warner considered the equation

$$-\triangle u+\alpha=R(x)e^{u},\ \ x\in M,$$

on any compact two dimensional manifold M with $\alpha<0$. They showed that there exists a number α_o, such that the equation is solvable for every $0>\alpha>\alpha_o$ and it is not solvable for $\alpha<\alpha_o$.

Then one may naturally ask:

Is the equation solvable for $\alpha=\alpha_o$?

In this paper, we answer the question affirmatively. We show that there exists at least one solution for $\alpha=\alpha_o$.

1. Introduction

Prescribing Gaussian curvature is an interesting problem in Riemannian geometry. Given a function $K(x)$ on a two dimensional manifold M with metric g_o, can it be realized as the Guassian curvature of some metric g which is pointwise conformal to g_o? This is equivalent to the solvability of the semilinear elliptic equation

$$-\triangle u+k_o(x)=K(x)e^{2u}$$

where $\triangle$ is the Laplace operator and $k_o(x)$ the Gaussian curvature associated with the metric g_o.

By making a proper change of variables, one can easily arrive at the following simpler equation:

$$-\triangle u+\alpha=R(x)e^{u},\ \ x\in M. \tag{1}$$

In [6], Kazdan and Warner considered this equation on any compact two dimensional manifold M with $\alpha<0$. This corresponds to prescribing Gaussian curvature on negatively curved manifolds.

Let $\bar{R}$ be the average of R. Kandan and Warner showed that

i) *A necessary condition for* (1) *to have a solution is* $\bar{R}<0$.

ii) *If* $\bar{R}<0$, *then there is a constant* $-\infty\leq\alpha_o<0$, *such that one can solve* (1) *if* $\alpha_o<\alpha$, *but cannot solve* (1) *if* $\alpha<\alpha_o$.

iii) $\alpha_o=-\infty$ *if and only if* $R\leq 0$.

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One can see from their results that in the case when R is nonpositive, the problem has been solved completely. However in the case when R changes signs, one has $\alpha_o > -\infty$, hence there are some remaining questions. A natural one is:

Can one solve (1) *when* $\alpha = \alpha_o$?

The main purpose of this paper is to answer the above question. We prove

Theorem 1. *Let* $\alpha_o > -\infty$ *be defined above. Then equation* (1) *has at least one solution when* $\alpha = \alpha_o$.

We will prove the theorem in the next section. The main idea is the following.

Let α_k be a sequence of numbers such that

$$\alpha_k > \alpha_o, \quad \text{and } \alpha_k \to \alpha_o, \text{ as } k \to \infty.$$

For each α_k, we minimize the functional

$$J_k(u) = \frac{1}{2} \int_M |\bigtriangledown u|^2 + \alpha_k \int_M u - \int_M R(x) e^u$$

in a class of functions that are between a sub and a super solution. Let the minimizer be u_k. We show that the sequence $\{u_k\}$ so chosen is bounded and converges to a solution corresponding to $\alpha = \alpha_o$.

2. The proof of Theorem 1

In this section, we prove the existence of a solution for $\alpha = \alpha_o$. The proof is divided into three steps.

In Step 1, we minimize the functional $J_k(\cdot)$ for each $\alpha_k > \alpha_o$. Then let $\alpha_k \to \alpha_o$.

In Step 2, we show that the sequence of minimizers $\{u_k\}$ is bounded in the region where $R(x)$ is positively bounded away from 0.

In Step 3, we prove that $\{u_k\}$ is bounded in $H^1(M)$ and hence converges to a desired solution.

Step 1. Let $\{\alpha_k\}$ be a sequence of numbers such that

$$0 > \alpha_k > \alpha_o, \text{ and } \alpha_k \to \alpha_o.$$

Consider the functional

$$J_k(u) = \frac{1}{2} \int_M |\bigtriangledown u|^2 + \alpha_k \int_M u - \int_M R(x) e^u$$

in Sobolev space $H^1(M)$.

Choose a sufficiently negative constant A to be the subsolution of equation (1) for all α_k, that is,

$$-\triangle A + \alpha_k \le R(x) e^A.$$

This is obviously possible.

For each fixed α_k, choose $\alpha_o < \tilde{\alpha}_k < \alpha_k$. By the result of Kazdan and Warner [6], there exists a solution of equation (1) for $\alpha = \tilde{\alpha}_k$. Call it ψ_k. Obviously, ψ_k is a super solution for equation

$$-\triangle u + \alpha_k = R(x) e^u.$$

Illuminated by an idea of Ding and Liu [4], we minimize the functional $J_k(\cdot)$ in the class of functions

$$H = \{u \in C^1(M) \mid A \le u \le \psi_k\}.$$

Through some standard variational argument (for example, see [4] or [2]), one can conclude that a minimizer u_k of $J_k(\cdot)$ exists and is in the interior of H. Obviously, u_k is the solution of equation (1) for $\alpha = \alpha_k$.

Step 2. Since the sequence $\{u_k\}$ is uniformly bounded from below, to show that it is also bounded from above in the region where $R(x)$ is positively bounded away from 0, we use a result of Brezis and Li [1].

Lemma 2.1. *Assume that V is a Lipschitz function satisfying*

$$0 < a \leq V(x) \leq b < \infty$$

and let K be a compact subset of a domain Ω in R^2. Then any solution u of

$$-\triangle u = V(x)e^u, \quad x \in \Omega, \tag{2}$$

satisfies

$$\sup_K u + \inf_K u \leq C(a, b, \|\bigtriangledown V\|_{L^\infty}, K, \Omega). \tag{3}$$

In fact, we [3] recently improved their result so that the constant C is independent of b—the upper bound of $R(x)$.

Let x_o be a point at which $R(x_o) > 0$. Let ϵ be so small such that

$$R(x) > 0 \ \ \forall x \in B_\epsilon(x_o).$$

Let ψ_k be a solution of

$$\begin{cases} -\triangle \psi_k - \alpha_k = 0 & x \in B_\epsilon(x_o), \\ \psi_k(x) = 1 & x \in \partial B_\epsilon(x_o). \end{cases}$$

Let $w_k = u_k + \psi_k$. Then w_k satisfies

$$-\triangle w_k = R(x)e^{-\psi_k}e^{w_k}.$$

It is obvious that the sequence $\{\psi_k\}$ is bounded from above and below in the small ball. Since $\{u_k\}$ is uniformly bounded from below, $\{w_k\}$ is also bounded from below. Locally, the metric is pointwise conformal to the Euclidean metric, so one can apply Lemma 2.1 to $\{w_k\}$ and conclude that the sequence $\{w_k\}$ is also bounded from above in the small ball, and hence bounded in the region where $R(x)$ is positively bounded away from zero. Now the bound for $\{u_k\}$ follows suit.

Step 3. We show that the sequence $\{u_k\}$ is bounded in $H^1(M)$.

On one hand, since u_k is a minimizer of the functional J_k, the second derivative $J_k''(u_k)$ is positively definite. It follows that for any function ϕ,

$$\int_M (|\bigtriangledown \phi|^2 - R(x)e^{u_k}\phi^2) \geq 0.$$

Choosing $\phi = e^{\frac{u_k}{2}}$, we have

$$\frac{1}{4}\int_M |\bigtriangledown u_k|^2 e^{u_k} \geq \int_M R(x)e^{2u_k}. \tag{4}$$

On the other hand, multiplying both sides of the equation

$$-\triangle u_k + \alpha_k = R(x)e^{u_k}$$

by e^{u_k} and integrating, we have

$$\int_M (|\bigtriangledown u_k|^2 e^{u_k} + \alpha_k e^{u_k}) = \int_M R(x)e^{2u_k}. \tag{5}$$

Combining (4) and (5), we arrive at

$$-\alpha_k \int_M e^{u_k} \geq \frac{3}{4} \int_M |\bigtriangledown u_k|^2 e^{u_k}. \tag{6}$$

If we let $v_k = e^{\frac{u_k}{2}}$, then (6) becomes

$$\int_M |\bigtriangledown v_k|^2 \leq C \int_M v_k^2 \tag{7}$$

for some positive constant C.

In Step 2, we have shown that $\{u_k\}$ is bounded in the region where $R(x)$ is bounded away from 0, as does $\{v_k\}$ there. It follows from (7) that $\{v_k\}$ is bounded in $H^1(M)$, therefore, $\{u_k\}$ is also bounded in $H^1(M)$. Consequently, there exists a subsequence of $\{u_k\}$ that converges to a function u_o in $H^1(M)$, which is the desired solution of equation (1) for $\alpha = \alpha_o$.

This completes the proof of the Theorem.

References

1. H.Brezis, Y.Y.Li, A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities, J. Funct. Analysis, 115(1993) 344-358. MR **94g:**35080
2. K.C.Chang, Infinite dimensional Morse theory and multiple solution problems, Birkhauser, 1993. MR **94e:**58023
3. W.Chen, C.Li, Moving planes, moving spheres, and apriori estimates, preprint, 2000.
4. W.Ding, J.Liu, A note on the problem of prescribing Gaussian curvature on surfaces, Trans. AMS, 347(1995) 1059-1065. MR **95e:**53058
5. D.Gilbarg, N.S.Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, 1983. MR **86c:**35035
6. J.Kazdan, F. Warner, Curvature functions for compact two manifolds, Ann. of Math., 99(1974) 14-47. MR **49:**7949

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