

THE WHITEHEAD PRODUCTS AND POWERS IN W -TOPOLOGY

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ABSTRACT. We initiate a study of W -topology. This is a modification of ordinary topology in which morphisms exist after smashing with a fixed space W . Several classical topics are considered in this setting, *inter alia* Whitehead products, Hopf invariants and the E - H - Δ sequence. The main emphasis is on detection of non-trivial elements in the W -homotopy groups of spheres.

0. INTRODUCTION

In recent years a considerable body of literature has accumulated concerning homotopy theory in categories of spaces under a fixed space, over a fixed space, over and under, ex-spaces, et cetera. While these ideas have been fruitful, it is certainly the case that computation of relevant homotopy sets and of the results of specific operations can pose formidable problems. We study here a simpler (but related) modification of the homotopy category of **Top**. (We assume that all spaces are well-pointed, compactly generated and Hausdorff.)

Let W be a fixed space. The objects of the homotopy category in W -topology are the same as those of **Top** but the morphism sets are the sets

$$(0.1) \quad \pi^W(X, Y) = \pi(X \wedge W, Y \wedge W)$$

of homotopy classes between the respective smash products. If X is a space we say that the groups $\pi(S^r \wedge W, X \wedge W)$ are W -homotopy groups and denote them $\pi_r^W(X)$. The endofunctor $-\wedge W$ of **Top** clearly induces a functor $W : \mathbf{Top} \rightarrow W\text{-}\mathbf{Top}$. Since $-\wedge W$ has a right adjoint, W preserves colimits and, in particular, mapping cones. In S^1 -topology $X \times Y$ and $X \vee Y \vee X \wedge Y$ are indistinguishable; the Hopf class η_2 is of order 2. The stable homotopy category and S^1 -**Top** are, of course, distinct. S^1 -topology has the pleasant feature that fundamental groups are always abelian and $\pi_0^{S^1}(X)$ has a natural (non-abelian) group structure.

Since information about W -**Top** also concerns ordinary homotopy theory it will always be of interest; nevertheless part of the motivation is that the particular features of homotopy theory in W -topology are properties of the space W itself and provide a sensitive tool for its study. This point is well brought out by Toda's study [T3] of the *suspension order* of a space, which we recognise retrospectively as

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a sustained contribution to W -topology. In the present paper the main emphasis will be on detection of non-trivial elements in the W -homotopy groups of spheres.

Relevant to (but not part of) W -topology is the notion of W -track group. If X is a space, the groups

$$(0.2) \quad \pi(\Sigma^m W, X) \approx \pi_m(X^W) \quad (m \geq 1)$$

will be called W -track groups. The notion of a (2-fold) Whitehead product for W -track groups has been studied in [HJ1], [HJ2] (see also [OS2]). This product is a pairing

$$(0.3) \quad [-, -]^W : \pi(\Sigma^{m_1} W, X) \times \pi(\Sigma^{m_2} W, X) \longrightarrow \pi(\Sigma^{m_1+m_2-1} W, X) .$$

Not a large amount of information is as yet available concerning the behaviour of the product (0.3), but, in [HJ1], it was shown that if $[\alpha_1, \alpha_2]^W = 0$, then a Hopf construction coset

$$(0.4) \quad c(\alpha_1, \alpha_2) \subseteq \pi(\Sigma^{m_1+m_2+1} W, \Sigma X)$$

can be defined satisfying the equation

$$(0.5) \quad Hc(\alpha_1, \alpha_2) = \pm \Sigma(\alpha_1 \wedge \alpha_2) \circ \Sigma^{m_1+m_2+1} \chi_2 ,$$

where $\chi_2 : W \rightarrow W \wedge W$ refers to the smash diagonal and H a version of the Hopf-James invariant.

In section 1, we define various notions of a higher-order Whitehead product for W -track groups. A characteristic of these generalised Whitehead products is that n -fold products vanish in the case of a space W whose weak Lusternik-Schnirelmann category satisfies *weak cat* $W \leq n$.

It is also possible to define Whitehead products in W -topology proper. For example the 2-fold product is a pairing

$$(0.6) \quad \pi^W(S^{m_1}, Y) \times \pi^W(S^{m_2}, Y) \longrightarrow \pi^W(S^{m_1+m_2-1}, Y) .$$

We define corresponding n -fold products, W -Hopf constructions and W -Hopf invariants. A version (unfortunately only partially exact in general) of the E - H - Δ sequence in W -topology is described. One may conjecture that, eventually, exactness principles of the sequence for individual spaces W will be found (as for the case $W = S^0$) enabling a full application of Toda’s composition method. Nevertheless, sufficiently many classes can be detected at present in the W -homotopy groups of spheres, for example when W is the real projective plane, to indicate that their structure differs substantially from the classical case.

We make systematic use of a class of *distributivity transformations* used previously by Oda and Shimizu [OS1] and others. Examples that feature in the theory are the following natural maps (definitions will be given later):

$$\begin{aligned} \delta : (\Pi S^{m_i}) \wedge W &\longrightarrow \Pi(S^{m_i} \wedge W) && \text{smash/fat-wedge distributivity,} \\ \delta_\star : (A \star B) \wedge W &\longrightarrow (A \wedge W) \star (B \wedge W) && \text{smash/join distributivity,} \\ \delta_\infty : X_\infty \wedge W &\longrightarrow (X \wedge W)_\infty && \text{smash/James space distributivity,} \\ \delta_\Omega : (\Omega X) \wedge W &\longrightarrow \Omega(X \wedge W) && \text{smash/loop distributivity,} \\ \delta : \Pi(X_i^W) &\longrightarrow (\Pi X_i)^W && \text{power/fat-wedge codistributivity.} \end{aligned}$$

The first author acknowledges conversations with the late Arnold Jansen on the theme of W -topology.

1. EXTERIOR PRODUCTS

Let X_i ($1 \leq i \leq n$) be pointed spaces, let $\alpha_i \in \pi_{m_i}(X_i)$ ($1 \leq i \leq n$) and let $m = \Sigma m_i$. The n -fold exterior Whitehead product of the α_i

$$[\alpha_1, \alpha_2, \dots, \alpha_n] \in \pi_{m-1}(\Pi X_i)$$

may be defined as follows; cf. [H1]. The universal example of the n -fold product is the attaching class (with suitably chosen orientation)

$$(1.1) \quad [\iota_{m_1}, \iota_{m_2}, \dots, \iota_{m_n}] = \mu \in \pi_{m-1}((S^{m_1} \times S^{m_2} \times \dots \times S^{m_n}))$$

of the m -dimensional cell of the n -fold product ΠS^{m_i} , where ι_{m_i} refers to the homotopy class of the identity map in $\pi_{m_i}(S^{m_i})$. In particular, this means that for representative maps f_i of α_i , we have

$$(\Pi f_i)_* \mu = [\alpha_1, \alpha_2, \dots, \alpha_n].$$

The corresponding n -fold exterior Whitehead product for W -track groups is defined via the adjunction correspondence

$$\theta : \pi(\Sigma^m W, X) \xrightarrow{\cong} \pi_m(X^W)$$

and the power/wedge codistributivity map

$$\delta : \Pi(X_i^W) \rightarrow (\Pi X_i)^W :$$

if $(f_1, f_2, \dots, f_n) \in \Pi(X_i^W)$, at least one of the maps f_1, f_2, \dots, f_n is the constant map from W into the base-point of its codomain, so we may define

$$(1.2) \quad \psi(f_1, f_2, \dots, f_n)(w) = (f_1(w), f_2(w), \dots, f_n(w))$$

to obtain a point $\psi(f_1, f_2, \dots, f_n)$ of $(\Pi X_i)^W$. Certainly δ is continuous. Then, given $\beta_i \in \pi(\Sigma^{m_i} W, X_i)$ ($1 \leq i \leq n$), we have $\delta_*[\theta\beta_1, \theta\beta_2, \dots, \theta\beta_n] \in \pi_{m-1}((\Pi X_i)^W)$ and we may define

$$(1.3) \quad [\beta_1, \beta_2, \dots, \beta_n]^W = \theta^{-1} \delta_*[\theta\beta_1, \dots, \theta\beta_n] \in \pi(\Sigma^{m-1} W, \Pi X_i).$$

The properties of the n -fold exterior Whitehead product for W -track groups are determined by its universal example class

$$[\iota_{m_1} \wedge W, \iota_{m_2} \wedge W, \dots, \iota_{m_n} \wedge W]^W \in \pi(\Sigma^{m-1} W, \Pi \Sigma^{m_i} W).$$

To identify it we first need to define the smash/fat-wedge distributivity

$$\delta : (\Pi S^{m_i}) \wedge W \longrightarrow \Pi(S^{m_i} \wedge W)$$

given by

$$(1.4) \quad \delta((x_1, x_2, \dots, x_n), w) = ((x_1, w), (x_2, w), \dots, (x_n, w)) \quad (x_i \in S^{m_i}, w \in W).$$

Then we obtain the following generalisation of [HJ2, Proposition 0.4].

Proposition 1.5. *The universal example class of the exterior Whitehead product for W -track groups satisfies*

$$[\iota_{m_1} \wedge W, \iota_{m_2} \wedge W, \dots, \iota_{m_n} \wedge W]^W = \delta_*(\mu \wedge W) \in \pi(\Sigma^{m-1} W, \Pi \Sigma^{m_i} W),$$

where $\mu = [\iota_{m_1}, \iota_{m_2}, \dots, \iota_{m_n}]$.

Proof. The class $\theta(\iota_{m_i} \wedge W)$ has a representative map

$$(1.5.1) \quad g_i : S^{m_i} \rightarrow (S^{m_i} \wedge W)^W, \quad \text{where } g_i(x)(w) = (x, w) \quad (x \in S^{m_i}, w \in W).$$

Then, for the composite map

$$\delta(g_1 \times \dots \times g_n) : (S^{m_1} \times \dots \times S^{m_n}) \longrightarrow ((S^{m_1} \wedge W) \times \dots \times (S^{m_n} \wedge W))^W,$$

we have, applying (1.2) and (1.5.1),

$$\delta(g_1(x_1), \dots, g_n(x_n))(w) = ((x_1, w), \dots, (x_n, w)) \quad (x_i \in S^{m_i}, w \in W).$$

Then the desired result follows by taking an adjoint and applying (1.4). □

We recall [BH] that *weak-cat* W , the *weak pointed category of* W , is *less than or equal to* n if and only if the composite map

$$\chi_n : W \xrightarrow{\Delta_n} W^n \xrightarrow{p} W^{(n)}$$

is nullhomotopic, where Δ_n refers to the diagonal map into the n -fold product and p denotes the map into the n -fold *smash* product of W with itself that shrinks the fat wedge. The map χ_n is the n -fold *smash diagonal*. (Note that in [BH] I. Berstein and P.J. Hilton used the renormalised version of the definition, i.e. replacing n by $n - 1$). The case $n = 2$ of the following result may be found in [OS3].

Theorem 1.6. *If weak-cat* $W \leq n$, *then*

$$[\iota_{m_1} \wedge W, \iota_{m_2} \wedge W, \dots, \iota_{m_n} \wedge W]^W = 0 \in \pi(\Sigma^{m-1}W, \Pi \Sigma^{m_i}W).$$

To prove the theorem we shall need to recall the definition of the *generalised Whitehead product* given by G. Porter [P]. It will be convenient to use the formulation given in [H4].

Let C, Σ denote, respectively, the reduced cone and reduced suspension endofunctors of **Top**. We recall that CV and ΣV are obtained from the cylinder $V \times I$ by identifying with the base point $*$, respectively, the subspaces

$$\{(v, t) \mid v = * \text{ or } t = 1\} \quad \text{and} \quad \{(v, t) \mid v = * \text{ or } t^2 = t\}.$$

If **Top** ^{n} is the category of n -tuples of members of **Top**, we also denote by C and Σ the endofunctors they induce in **Top** ^{n} . Let Π denote the n -fold product and let $J(V_1, V_2, \dots, V_n)$ and $\Pi \Sigma(V_1, V_2, \dots, V_n)$ denote the subspaces

$$\{(v_1, t_1), (v_2, t_2), \dots, (v_n, t_n) \mid \text{at least one of the } t_i \text{ is zero}\}$$

of $\Pi C(V_1, V_2, \dots, V_n)$ and $\Pi \Sigma(V_1, V_2, \dots, V_n)$, respectively. The transformation $\Pi \sigma$ clearly induces by restriction a natural transformation

$$(1.6.1) \quad \sigma : J(V_1, V_2, \dots, V_n) \rightarrow \Pi \Sigma(V_1, V_2, \dots, V_n).$$

Then there is a diagram

$$\begin{array}{ccc} J(V_1, \dots, V_n) & \xrightarrow{\sigma} & \Pi \Sigma(V_1, \dots, V_n) \\ \downarrow h & \nearrow \mu' & \uparrow \bar{h} \\ \Sigma^{n-1} \wedge (V_1, \dots, V_n) & & \end{array}$$

in which h is a homotopy equivalence with homotopy inverse \bar{h} and the dotted arrow $\mu' = \sigma \bar{h}$ is a representative of the universal example of Porter's version of

the generalised Whitehead product [P]. If we now consider the case $V_i = \Sigma^{m_i-1}W$ ($1 \leq i \leq n$) we obtain a commutative diagram of solid arrows

$$\begin{array}{ccc}
 J(S^{m_1-1}, \dots, S^{m_n-1}) \wedge W & \xrightarrow{\sigma' \wedge W} & (\Pi S^{m_i}) \wedge W \\
 \downarrow h \wedge W & \searrow \delta_J & \searrow \delta \\
 & & J(S^{m_1-1} \wedge W, \dots, S^{m_n-1} \wedge W) \xrightarrow{\sigma'} \Pi(S^{m_i} \wedge W) \\
 & & \downarrow \rho \circ h \\
 S^{m-1} \wedge W & \xrightarrow{S^{m-1} \wedge \chi_n} & S^{m-1} \wedge W^{(n)} \xrightarrow{\mu' \circ \rho^{-1}} \Pi(S^{m_i} \wedge W)
 \end{array}$$

in which $\delta_J, \delta.$ denote distributivity transformations and ρ is a homeomorphism permuting smash factors. The clockwise route around the outside of the diagram from $S^{m-1} \wedge W$ to $\Pi(S^{m_i} \wedge W)$ is a representative of $\delta_* \mu \wedge W$ and the diagram shows that it factors up to homotopy through $S^{m-1} \wedge \chi_n$ which, in view of the hypothesis concerning W , is homotopically trivial. An application of Proposition 1.5 completes the proof.

Remark 1.7. It is a consequence of Proposition 1.5 that the class $\mu_W = \mu \wedge W$ is a composition factor of the universal example class of the exterior Whitehead product for W -track groups. It is interesting to ask if *weak-cat* $W \leq n$ implies $\mu_W = 0$. This seems unlikely, but at this moment we do not have a counterexample. (If $n = 2$ the answer is yes, for in that case the distributivity transformation $\delta.$ is a homeomorphism: smash with W distributes over a thin wedge.) The theory of the next two sections will cast some light on this issue since the vanishing of a Whitehead power in W -topology implies that a certain Hopf construction coset is non-trivial.

2. THE WHITEHEAD POWERS

In this section we consider Whitehead product operations taking values in the skeleta of some James reduced product complex.

If $m_i = m$ ($1 \leq i \leq n$), let $\phi : \Pi_{(1 \leq i \leq n)} S^m \rightarrow S_{n-1}^m$ denote the identification map onto the subcomplex S_{n-1}^m of the James reduced product space S_∞^m . Then we define

$$(2.1) \quad \omega(W, m, n) = (\phi \wedge W) \circ \mu_W \in \pi(S^{mn-1} \wedge W, S_{n-1}^m \wedge W) = \pi_{mn-1}^W(S_{n-1}^m)$$

to be the n -fold W -Whitehead power of the class $\{1_{S^m}\}$.

(It might be convenient to also consider the slightly more general form of the above definition, in which $S^m = \Sigma S^{m-1}$ is replaced by ΣX , for an arbitrary pointed space X .)

Remark 2.2. One might also wish to replace the functor Σ by the functor $\Gamma \wedge -$, where Γ is a co-H-space, in order to define the corresponding Γ -Whitehead and Γ - W -Whitehead operations. For $n = 2$ this works well [O], but it is not yet clear what the definitions should be when $n > 2$, for an ‘‘algebraic’’ definition of the products has thus far only been achieved for $n = 2$. ‘‘Geometric’’ definitions of the n -fold operations ($n > 2$) were given in [H1] and in [P]; see also [Bau]. A related construction of Whitehead products (for $n = 2$, but extendible to $n > 2$) utilizing the exterior join functor is given in [M, §4].

It seems to be advantageous to regard the Whitehead (and W -Whitehead) powers as being *operations* on classes of maps between spheres. For example, if $\gamma \in \pi_r(S^{mn-1})$ is an element of the homotopy groups of spheres, then

$$(2.3) \quad \omega(m, n)(\gamma) = \omega(m, n) \circ \gamma \in \pi_r(S_{n-1}^m)$$

may be described as the (m, n) -Whitehead power operation. Note that $\omega(m, n)$ is its universal example. A substantial amount of information concerning these operations is already known in (ordinary) topology, where they play a significant role in the theory of the unstable homotopy groups of spheres; cf. [T2], [H2], [H3]. It is to be expected that the corresponding operations

$$\omega(W, m, n)(\gamma) = (\omega(m, n) \wedge W) \circ \gamma \in \pi_r^W(S_{n-1}^m), \quad \gamma : S^r \wedge W \rightarrow S^{mn-1} \wedge W$$

(the W -Whitehead powers) will be equally significant in W -topology.

As evidence contributory to the foregoing claim, we shall describe a W -Hopf construction element that is defined when a W -Whitehead power vanishes at γ and, further, show that it can be detected by a W -Hopf invariant.

Let $i_n : S_{n-1}^m \rightarrow S_\infty^m$ denote the inclusion map. Suppose that $\omega(W, m, n)(\gamma) = 0$. Then the Toda bracket

$$(2.4) \quad \{^\circ i_n \wedge 1_W, \omega(W, m, n), \gamma\} \subseteq \pi_{r+1}^W(S_\infty^m)$$

is defined. (The little circle decorating the Toda bracket indicates that a preferred nullhomotopy, in this case the constant nullhomotopy of representatives of the left hand composites, has to be used.) We describe an element of (2.4) as a W -Hopf construction element. The use of a preferred nullhomotopy has the effect of reducing the indeterminacy of a Toda bracket (see [H5]). In consequence the bracket (2.4) is a coset of the subgroup

$$(2.4.1) \quad (i_n \wedge W)_* \pi_{r+1}^W(S_{n-1}^m) .$$

3. THE W -HOPF INVARIANTS

Let X be a pointed space, and let $j_n : X_\infty \rightarrow X_\infty^{(n)}$ be its n 'th James map, where

$$X^{(n)} = X \wedge X \wedge \dots \wedge X \quad (n \text{ times}).$$

Consider the James space distributivity map

$$\delta_\infty : X_\infty \wedge W \rightarrow (X \wedge W)_\infty, \quad (x_1 x_2 x_3 \dots, w) \mapsto (x_1, w)(x_2, w)(x_3, w) \dots .$$

Now there is a homomorphism

$$(3.1) \quad \theta_W : \pi_{r+1}^W(X_\infty) \xrightarrow{(\delta_\infty)^*} \pi(S^{r+1} \wedge W, (X \wedge W)_\infty) \xrightarrow{\psi} \pi_{r+2}^W(\Sigma X)$$

induced by composition with δ_∞ followed by the James isomorphism ψ . We may regard θ_W as the natural extension of the James correspondence to W -topology but, of course, there is no reason to expect that θ_W will remain an isomorphism (as it is in the case $W = S^0$). In Example 4.4 we show that there are elements in the kernel of θ_{S^1} . The following can be checked.

Lemma 3.2. *There is a commutative diagram*

$$\begin{array}{ccccc}
 X_\infty \wedge W & \xrightarrow{j_n \wedge W} & X_\infty^{(n)} \wedge W & \xrightarrow{X_\infty^{(n)} \wedge \chi_n} & X_\infty^{(n)} \wedge W^{(n)} \\
 \delta_\infty \downarrow & & & & \downarrow \delta_\infty \\
 (X \wedge W)_\infty & \xrightarrow{j_n} & (X \wedge W)_\infty^{(n)} & \xrightarrow{\rho_\infty} & (X^{(n)} \wedge W^{(n)})_\infty
 \end{array}$$

where the map ρ permutes coordinates and χ_n denotes the n -fold smash diagonal map.

The diagram of Lemma 3.2 induces the following commutative diagram of homomorphisms:

(3.2.1)

$$\begin{array}{ccc}
 \pi_{r+1}^W(X_\infty) & \xrightarrow{\theta_W} & \pi_{r+2}^W(\Sigma X) \\
 \downarrow (j_n \wedge W)_* & \searrow H_n^W & \downarrow H_n \\
 \pi_{r+1}^W(X_\infty^{(n)}) & \xrightarrow{\theta_W} & \pi_{r+2}^W(\Sigma X^{(n)}) \\
 \downarrow (X_\infty^{(n)} \wedge \chi_n)_* & & \searrow (\Sigma X^{(n)} \wedge \chi_n)_* \\
 \pi(S^{r+1}W, X_\infty^{(n)} \wedge W^{(n)}) & \xrightarrow{\psi(\delta_\infty)_*} & \pi(S^{r+2} \wedge W, \Sigma X^{(n)} \wedge W^{(n)})
 \end{array}$$

thus defining the operator H_n^W , which we call the n 'th W -Hopf invariant. The homomorphism labelled H_n is equivalent to the n 'th Hopf-James invariant.

The following results record relations between θ_W and the James isomorphism ψ and between H_n and H_n^W .

Proposition 3.3. *For any element $\alpha \in \pi_{r+1}(X_\infty)$ the smash product $\alpha \wedge W$ is an element of $\pi_{r+1}^W(X_\infty)$ and $\theta_W(\alpha \wedge W) = \psi(\alpha) \wedge W \in \pi_{r+2}^W(\Sigma X)$.*

Proof. Let $i : X_\infty \rightarrow \Omega \Sigma X$ and $i' : (X \wedge W)_\infty \rightarrow \Omega \Sigma(X \wedge W)$ be the canonical injections. We define a map

$$\delta_\Omega : (\Omega \Sigma X) \wedge W \rightarrow \Omega \Sigma(X \wedge W)$$

by $\delta_\Omega(f, w)(t) = f(t) \wedge w$ for any $f \in \Omega \Sigma X$, $w \in W$ and $t \in S^1$. Then the following diagram is commutative:

$$\begin{array}{ccc}
 \pi(S^{r+1} \wedge W, X_\infty \wedge W) & \xrightarrow{(i \wedge W)_*} & \pi(S^{r+1} \wedge W, (\Omega \Sigma X) \wedge W) \\
 \downarrow (\delta_\infty)_* & & \downarrow (\delta_\Omega)_* \\
 \pi(S^{r+1} \wedge W, (X \wedge W)_\infty) & \xrightarrow{i'_*} & \pi(S^{r+1} \wedge W, \Omega \Sigma(X \wedge W))
 \end{array}$$

The result now follows by the definition of θ_W . □

The following result gives a relation between H_n^W and H_n for the elements of type $\alpha \wedge W \in \pi_{r+1}^W(X_\infty)$. The case $n = 2$ of the following Theorem is used in the argument of Examples 4.6 and 4.8.

Theorem 3.4. *Let $\alpha \in \pi_{r+1}(X_\infty)$ be any element. Then $\alpha \wedge W \in \pi_{r+1}^W(X_\infty)$ and we have*

$$H_n^W(\alpha \wedge W) = H_n(\psi(\alpha)) \wedge W \in \pi_{r+2}^W(\Sigma X^{(n)}).$$

Proof. Let $\psi^{(n)} : \pi_{r+1}(X_\infty^{(n)}) \rightarrow \pi_{r+2}(\Sigma X^{(n)})$ be the James isomorphism for $X^{(n)}$. Then

$$H_n = \psi^{(n)} \circ j_{n*} \circ \psi^{-1} : \pi_{r+2}(\Sigma X) \rightarrow \pi_{r+1}(X_\infty) \rightarrow \pi_{r+1}(X_\infty^{(n)}) \rightarrow \pi_{r+2}(\Sigma X^{(n)}),$$

where $j_n : X_\infty \rightarrow X_\infty^{(n)}$ is the n 'th James map. By Proposition 3.3 we have

$$\begin{aligned} H_n^W(\alpha \wedge W) &= \theta_W\{(j_n \wedge W) \circ (\alpha \wedge W)\} = \theta_W\{(j_n \circ \alpha) \wedge W\} \\ &= \{\psi^{(n)}(j_n \circ \alpha)\} \wedge W = H_n(\psi(\alpha)) \wedge W, \end{aligned}$$

as required. □

The following result enables one to prove that not all elements of $\pi_{r+1}^W(S^n)$ need be of the above type.

Theorem 3.5. *Suppose that the W -Whitehead power $\omega(W, m, n)(\gamma)$ is equal to zero, where $\gamma \in \pi_r^W(S^{mn-1})$, and let $\xi \in \pi_{r+1}^W(S_\infty^m)$ be an element of the W -Hopf construction coset (2.4). Then $H_n^W \xi$ is defined and*

(a)
$$H_n^W \xi = \pm \Sigma^2 \gamma,$$

(b)
$$H_n \theta_W \xi = \pm (t_{mn+1} \wedge \chi_n) \circ \Sigma^2 \gamma .$$

Remark 3.5.1. In Theorem 3.5, note that $\gamma \in \pi_r^W(S^{mn-1})$, but it is not required that γ is a class in the image of the functor W .

A proof of Theorem 3.5 will be given in the next section.

4. THE E - H - Δ SEQUENCES IN W -TOPOLOGY

We shall study a sequence

$$(4.1) \quad \leftarrow \pi_r^W(X_{n-1}) \xrightarrow{E_n} \pi_r^W(X_\infty) \xrightarrow{H_n^W} \pi_{r+1}^W(\Sigma X^{(n)}) \xleftarrow{\Delta_n^-} \pi_{r-1}^W(X_{n-1}) \longrightarrow$$

which is partially exact in the sense of [H5] and in which

$$E_n = (i_n \wedge W)_* : \pi_r^W(X_{n-1}) \rightarrow \pi_r^W(X_\infty) .$$

The sequence (4.1) can be used for purposes of computation (detection of non-trivial classes) in much the same way as the classical E - H - Δ sequence is used for computation of homotopy groups of spheres. (There is a well-known difficulty to compute the (classical) Δ homomorphism: in practice we are only confident in identifying the image of Δ when we can recognise it as a Whitehead power element, but in that case we might as well argue instead with Δ_n^- .) First, note that the composite $H_n^W \circ E_n$ is the zero homomorphism.

The operator Δ_n^- is defined on the kernel of E_n . If $\alpha \in \pi_{r-1}^W(X_{n-1})$ is such that $E_n \circ \alpha = 0$, then

$$(4.2) \quad \Delta_n^- \alpha = \theta_W\{j_n \wedge W, i_n \wedge W, \alpha\} \subseteq \pi_{r+1}^W(\Sigma X^{(n)}) .$$

Note that $\Delta_n^- \alpha$ has an indeterminacy equal to the image of H_n^W . An instance in which Δ_n^- is defined is given in the following Lemma dealing with the case $X = S^m$.

Lemma 4.3. $\Delta_n^{\leftarrow} \omega(W, m, n) = \pm \iota_{mn+1} \wedge W$.

Essentially, this is a consequence of the classical principle $\Delta(\iota_{mn+1}) = \pm \omega(m, n)$ due to James and Toda. For further detail in the present case see [H2, Lemma 3.3].

Proof of Theorem 3.5. We use Toda’s formula $\alpha \circ \{\beta, \gamma, \delta\} = -\{\alpha, \beta, \gamma\} \circ \Sigma \delta$ (cf. [T1, Proposition 1.4]). We have

$$\begin{aligned} H_n^W \xi &\in \theta_W(j_n \wedge W \circ \{^\circ i_n \wedge W, \omega(W, m, n), \gamma\}) \\ &= -\theta_W(\{j_n \wedge W, i_n \wedge W, \omega(W, m, n)^\circ\} \circ \Sigma \gamma), \end{aligned}$$

by (2.3) of [H5]. Comparing the Toda bracket in the last expression with that in (4.2) we see that the two coincide if α is replaced by $\omega(W, m, n)$. (After the replacement each bracket has zero indeterminacy.) The desired result is now a consequence of Lemma 4.3. \square

Example 4.4. In the case $W = S^1$ it is well-known that $\omega(S^1, m, n) = 0$. Hence the hypothesis of Theorem 3.5 is satisfied with $\gamma = \iota_{mn-1} \wedge S^1 \in \pi_{mn-1}^{S^1}(S^{mn-1})$ ($m \geq 1, n \geq 2$). It follows that there exist S^1 -Hopf construction elements

$$\xi_{m,n} \in \pi_{mn}^{S^1}(S_\infty^m) = \pi_{mn+1}(S_\infty^m \wedge S^1) \quad (m \geq 1, n \geq 2)$$

each having an S^1 -Hopf invariant 1. Since $S_\infty^m \wedge S^1 \simeq \vee_{(1 \leq n)} S^{mn+1}$ it is easy to identify these classes as equivalent to injections of spheres into a thin wedge. It is interesting to note that these classes all belong to the kernel of θ_{S^1} .

The following lemma concerning the functor $W : \mathbf{Top} \rightarrow W\text{-}\mathbf{Top}$ will be used in the next example.

Lemma 4.5. *Let α, β, γ be homotopy classes in ordinary topology with $\alpha \circ \beta = 0$ and $\beta \circ \gamma = 0$. Then*

$$W\{\alpha, \beta, \gamma\} \subseteq \{W\alpha, W\beta, W\gamma\}.$$

Example 4.6. Let $m = 1, 3$ or 7 . Then it is known that $\omega(W, m, 2) = 0$ for arbitrary W . Choose $\gamma = \iota_{2m-1} \wedge W$ and let $\xi_{m,2} \in \pi_{2m}^W(S_\infty^m)$ denote the corresponding elements. Then, if $\bar{\eta} \in \pi_2(S_\infty^1)$ is the preimage of the Hopf class η_2 under the James correspondence, it follows via Lemma 4.5 that $W\bar{\eta} - \xi_{1,2}$ belongs to the indeterminacy of the Hopf construction, and hence, by (2.4.1), is an element of filtration 1 in $\pi_2^W(S_\infty^1)$. Since θ_W clearly defines a bijection between the respective subgroups of filtration 1, $\theta_W \xi_{1,2} = W\eta_2$ modulo a suspension element. Similar remarks can be made concerning the Hopf classes ν and σ .

For the next examples we consider a Moore space $W = P^2(p) = S^1 \cup_p e^2$, where p is a prime. From [Bar, Table 2 on p. 295] we have:

Proposition 4.7 (Barratt). *For $n \geq 2$, $\pi_n^W(S^n) \approx \mathbb{Z}_p$ (p odd), $\pi_n^W(S^n) \approx \mathbb{Z}_4$ ($p = 2$).*

Example 4.8. First we consider the case p odd and m odd. As is well-known, $2[\iota_m, \iota_m] = 0$. Hence, in view of Proposition 4.7, we have $\omega(W, m, 2) = 0$. This leads to W -Hopf construction elements $\xi_m \in \pi_{2m}^W(S_\infty^m)$ satisfying $H_2^W(\xi_m) = \iota_{2m+1} \wedge W$, which is an element of order p . Is $\theta_W(\xi_m) \neq 0$? In this regard we observe that by

commutativity of the upper triangle of diagram (3.2.1),

$$H_2(\theta_W(\xi_m)) = (\Sigma^{2m+1}\chi_2)_*(H_2^W\xi_m) = \Sigma^{2m+1}\chi_2.$$

However $\chi_2 = 0$ by [N, Proposition 9.4].

Next, suppose that $p = 2$, so that W is the real projective plane. Then (with $m = 2$) we have by Proposition 4.7,

$$(4.8.1) \quad 2\omega(W, 2, 2) = ([\iota_2, \iota_2] \wedge W) \circ (2\iota_3 \wedge 1_W) = 4\eta_2 \wedge W = 0.$$

By application of Theorem 3.5 to (4.8.1), we obtain a W -Hopf construction element $\xi \in \pi_4^W(S_\infty^2)$ satisfying

$$H_2^W(\xi) = 2\iota_5 \wedge W \neq 0.$$

Is $\theta_W\xi \neq 0$?

Finally, consider again the case p odd. It is well-known (cf. [T2, (4.2)]) that the element $\alpha \in \pi_{2p}(S^3)$, discovered by J.P. Serre, is such that $\alpha = E_{p-1}\bar{\alpha}$ for an element $\bar{\alpha} \in \pi_{2p-1}(S_{p-1}^2)$ having infinite order and satisfying $p\bar{\alpha} = \omega(2, p)$. It follows that $\omega(W, 2, p) = 0$ and hence there is a W -Hopf construction element $\xi' \in \pi_{2p}^W(S_\infty^2)$ such that $H_p^W(\xi') = \iota_{2p+1} \wedge W \in \pi_{2p+1}^W(S^{2p+1})$. Again we may enquire, is $\theta_W(\xi') \neq 0$? The question would be answered in the affirmative if we knew $\Sigma^{p+1}\chi_p \neq 0$, where $\chi_p : W \rightarrow W^{(p)}$ refers to the p -fold smash diagonal. However the last condition holds only for spaces W whose weak Lusternik-Schnirelmann category is at least p , which is not the case for the present W .

Concluding remark 4.9. The reader will note that our theory is reasonably effective for detecting new classes in $\pi_r^W(S_\infty^n)$ but that transfer of classes to $\pi_{r+1}^W(S^{n+1})$ is less sure. This attaches some urgency to the problem of determining the behaviour of the homomorphism θ_W .

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