

REDUCTIVE EMBEDDINGS ARE COHEN-MACAULAY

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ABSTRACT. In this paper we prove that in positive characteristics normal embeddings of connected reductive groups are Frobenius split. As a consequence, they have rational singularities and are thus Cohen–Macaulay varieties.

As an application, we study the particular case of reductive monoids, which are affine embeddings of their unit group. In particular, we show that the algebra of regular functions of a normal irreducible reductive monoid M has a good filtration for the action of the unit group of M .

1. INTRODUCTION

Let G be a complex connected reductive algebraic group and X a complex spherical G -variety. In [1] it is proved that if p is a prime number big enough, then the reduction mod p of X is Frobenius split and, in particular, has rational singularities. Thus, in characteristic zero $G \times G$ -equivariant normal embeddings of G —where $G \times G$ acts by left and right multiplication—have rational singularities. However, some spherical varieties in characteristic $p > 0$ are not Frobenius split; examples are quotients of a connected semi-simple group by non-reduced parabolic subgroup-schemes (see [9]). In this work we show that, in arbitrary positive characteristic, if G is reductive, then any normal G -embedding is Frobenius split, and as a consequence, it has rational singularities; moreover, we show that there exists a *canonical Frobenius splitting*. This can be done because of the good control of the geometry of these varieties, which allows us to follow the argumental lines used in the case of the complete symmetric varieties (see [15] and [2]). In particular, normal irreducible *reductive monoids*, i.e. algebraic monoids whose unit group is connected and reductive, have rational singularities. A first step in the resolution of this problem for reductive monoids can be found in [13], where it is shown that reductive monoids with unit group $\mathrm{GL}_2(k)$, $\mathrm{PGL}_2(k) \times k^*$ and $\mathrm{SL}_2(k) \times k^*$ are Cohen-Macaulay varieties.

As an application, we generalize a result by Doty ([3]) about the existence of good filtrations for normal reductive monoids with one-dimensional center to arbitrary reductive monoids.

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2. PRELIMINARIES

In this section we recall the basic facts about the classification of *reductive embeddings*—i.e. embeddings of a reductive group—to be used in this work.

Definition 1. Let G be a connected reductive group over a field k of arbitrary characteristic. A homogeneous space G/H is *spherical* if there exists a Borel subgroup B of G , such that BH is open in G . If X is a G -variety with an open orbit isomorphic to G/H , then X is called a G/H -*embedding*. If moreover X is normal, we say that X is a *spherical variety*. We say that X is *simple* if the action of G over X has only one closed orbit.

Note that $G \cong (G \times G)/\Delta(G)$, where $\Delta(G)$ is the diagonal in $G \times G$, is a $(G \times G)$ -spherical homogeneous space. Thus, a G -*embedding* is a $(G \times G)/\Delta(G)$ -embedding.

A reductive monoid is an algebraic monoid with unit group a reductive group. They are reductive embeddings; moreover we have the following:

Theorem 1 ([14]). *The irreducible reductive monoids are exactly the affine embeddings of reductive groups. The commutative irreducible reductive monoids are exactly the affine embeddings of tori.* \square

Notation. From now on, we restrict ourselves to the study of normal reductive embeddings and irreducible normal reductive monoids, unless stated otherwise. All reductive groups are supposed to be connected.

Spherical varieties, and thus reductive monoids, are classified in combinatorial terms (cf. [14] for the case of reductive monoids and [8] for the general case of spherical varieties). For reductive monoids, this classification is dual to that made by Vinberg in [17] in characteristic zero, in terms of the decomposition of their algebra of regular functions for the action of the unit group.

We will use some properties of the classification of spherical varieties in order to describe the geometry of reductive embeddings, in particular in order to find a desingularisation for any reductive embedding, so we summarize it in order to fix notations. Let G/H be a spherical homogeneous space, and B be a Borel subgroup of G such that $BH \subset G$ is open. We denote $k(G/H)^{(B)}$ to be the set of B -eigenvectors of $k(G/H)$, the field of rational functions of G/H , and $\Lambda_{G/H}$ to be the set of weights of $k(G/H)^{(B)}$. We consider the space $\text{Hom}_{\mathbb{Z}}(\Lambda_{G/H}, \mathbb{Z}) \otimes \mathbb{Q} = \mathcal{Q}(G/H)$; the restriction to $k(G/H)^{(B)}$ induces an injection of the set of G -invariant valuations of the field $k(G/H)$, into $\mathcal{Q}(G/H)$. Its image is a rational polyhedral cone, the *valuation cone*, denoted $\mathcal{V}(G/H)$. On the other hand, if we denote $\mathcal{D}(G/H)$ to be the set of the irreducible B -stable divisors of G/H , the *colors*, then there exists a map $\rho : \mathcal{D}(G/H) \rightarrow \mathcal{Q}(G/H)$.

To each simple spherical variety X with open orbit G/H corresponds a unique *colored cone*, namely a pair $(\mathcal{C}(X), \mathcal{F}(X))$ constructed as follows: let Y be the unique closed G -orbit of X , and put $\mathcal{F}(X) = \{D \in \mathcal{D}(G/H) \mid \overline{D} \supset Y\}$. Let $\mathcal{B}(X)$ be the subset of $\mathcal{V}(G/H)$ consisting of the valuations associated to the irreducible G -stable divisors of X . Then $\mathcal{C}(X)$ is the strictly convex polyhedral cone generated by $\mathcal{B}(X)$ and $\rho(\mathcal{F}(X))$. This cone verifies that $\mathcal{C}(X)^\circ$, the relative interior of $\mathcal{C}(X)$, intersects the valuation cone.

Conversely, a pair $(\mathcal{C}, \mathcal{F})$ verifying the properties described above determines a unique G/H -spherical variety such that $(\mathcal{C}(X), \mathcal{F}(X)) = (\mathcal{C}, \mathcal{F})$. Non-simple spherical varieties can be classified by means of *colored fans*, which are collections

of colored cones with some compatibility restrictions. If the set of colors is empty, we say that the associated spherical variety is *without colors*, and we write \mathcal{C} instead of (\mathcal{C}, \emptyset) if no confusion is possible.

Finally, we recall that if X and X' are spherical varieties with open orbits G/H and G/H' respectively, dominant G -equivariant morphisms $\varphi : X \rightarrow X'$ are in bijection with linear maps $\varphi_* : \mathcal{Q}(G/H) \rightarrow \mathcal{Q}(G/H')$ such that $\varphi_*(\mathcal{E}_X) \subset \mathcal{E}_{X'}$, where \mathcal{E}_X denotes the colored fan associated to X (cf. [8]).

Notation. Let us fix some notations. Let G be a reductive group, and let T be a maximal torus of G . Let B be a Borel subgroup containing T , and B^- its opposite Borel subgroup. We denote $\Xi(T)$ as the set of weights and $\Xi_+(T)$ the semigroup of dominant weights with respect to B . We denote W to be the Weyl group associated to T , and $C = C(G)$ to be the Weyl chamber associated to (B, T) . We denote $\alpha_1, \dots, \alpha_l$ and $\omega_1, \dots, \omega_l$ to be the simple roots and fundamental weights associated to (B, T) , respectively.

Finally, we denote $\Xi_*(T)$ to be the set of one parameter subgroups (1-PS) of T . We identify $\Xi_*(T)$ with $\Xi(T)$ by means of a W -invariant scalar product $\langle \cdot, \cdot \rangle$, in such a way that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$.

From the Bruhat decomposition we get that the subvarieties $D_i = \overline{Bs_{\alpha_i}B^-}$, $s_{\alpha_i} \in W$, the reflection associated to the simple root α_i , $i = 1, \dots, l$, are exactly the irreducible $(B \times B^-)$ -stable divisors of G (the colors).

The combinatorial data associated to the spherical homogeneous space $G = (G \times G)/\Delta(G)$ are the space $\mathcal{Q}(G) = \Xi(T) \otimes \mathbb{Q}$, with valuation cone $\mathcal{V}(G) = -C$, and colors α_i^\vee , $i = 1, \dots, l$ ([14]).

3. FROBENIUS SPLITTING FOR G -EMBEDDINGS

In this section we prove that any normal reductive embedding has a rational resolution, and thus is Cohen-Macaulay.

Definition 2. We say that a birational proper morphism $\varphi : X \rightarrow Y$, X a irreducible smooth variety, is a *rational resolution* of Y if:

$$\begin{aligned} \varphi_*\mathcal{O}_X &= \mathcal{O}_Y, & R^i\varphi_*(\mathcal{O}_X) &= 0 & i > 0, \\ \varphi_*\omega_X &= \omega_Y, & R^i\varphi_*(\omega_X) &= 0 & i > 0. \end{aligned}$$

Here ω_X is the canonical sheaf of X , and $\omega_Y = i_*\omega_{Y_{reg}}$, where $i : Y_{reg} \hookrightarrow Y$ is the inclusion of the regular locus. If there exists a rational resolution for Y , then Y is normal and Cohen-Macaulay.

Definition 3. Assume that $\text{char } k = p > 0$. If X is a scheme over k , we denote F to be the absolute Frobenius morphism. We say that X is *Frobenius split* if the induced map $F : \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$ is split, i.e. there exists a map $\sigma \in \text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ such that $\sigma \circ F$ is the identity. We call σ a (*Frobenius*) *splitting*. If $Y \subset X$ is a closed subscheme with associated ideal sheaf \mathcal{I}_Y , we say that σ is compatible with Y if $\sigma(F_*\mathcal{I}_Y) \subset \mathcal{I}_Y$.

In order to find a Frobenius splitting for a given variety, we will use the following criterion:

Proposition 1 ([12],[11]). *Let X be a complete smooth variety over a field k of characteristic $p > 0$. Then $\text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \cong F_*H^0(X, \omega_X^{1-p})$. In particular, a section $\sigma \in H^0(X, \omega_X^{1-p})$ induces a Frobenius splitting if and only if for some point*

$q \in X$ and some local system of coordinates $x = (x_1, \dots, x_N)$ at q , the coefficient of $(x/dx)^{p-1}$ in the expansion of σ at q is non-zero. \square

First of all, we extend some known results about resolutions of singularities of G -embeddings to arbitrary characteristics.

Proposition-Definition ([4],[18],[15]). Let G be a reductive group and Z be its center. Then there exists a maximal simple projective embedding of $G_{ad} = G/Z \cong (G_{ad} \times G_{ad})/\Delta(G_{ad})$. We call this embedding the *wonderful embedding* of G_{ad} (or G) and denote it as \mathbf{X} .

Let us summarize the properties of the wonderful embedding to be used in our work:

The wonderful embedding is a smooth variety.

If $C = C(G)$ notes the Weyl chamber of G associated to a Borel subgroup B , then the colored cone associated to the wonderful embedding is

$$(\mathcal{C}(\mathbf{X}), \mathcal{F}(\mathbf{X})) = (\mathcal{V}(G_{ad}), \emptyset) = -C(G_{ad}).$$

We denote X_i to be the irreducible component (divisor) of $\mathbf{X} \setminus G_{ad}$ that has associated valuation $-\omega_i$. The unique closed orbit of \mathbf{X} is $Y = \bigcap_{i=1}^l X_i \cong (G/B) \times (G/B^-)$.

Moreover, there exists a morphism $\gamma : \mathbb{A}^l \rightarrow \mathbf{X}$, where \mathbb{A}^l is the affine space of dimension l , and an open neighborhood $\mathcal{U} \subset \mathbf{X}$ of $1 \in G_{ad}$ meeting Y , such that the map

$$U \times U^- \times \mathbb{A}^l \ni (u, u', a) \mapsto (u, u') \cdot \gamma(a) \in \mathcal{U}$$

is an isomorphism. The morphism γ verifies that for all $(t, s) \in T_0 \times T_0$, we have $\gamma^{-1}((t, s) \cdot 1) = (t^{\alpha_1} s^{-\alpha_1}, \dots, t^{\alpha_l} s^{-\alpha_l})$ (recall that $\alpha_i, i = 1, \dots, l$, are the simple roots of G_0).

Finally, a G -embedding has no colors if and only if it equivariantly dominates \mathbf{X} ([8]).

Lemma 1. *Let M be a complete G -embedding, without colors, and let $Y \subset M$ be a closed $(G \times G)$ -orbit. Then $Y \cong G/B \times G/B^-$; in particular, Y is separable.*

Proof. Consider the canonical $(G \times G)$ -equivariant morphism $\pi : M \rightarrow \mathbf{X}$; then π is dominant, and as M is complete, $Y \subset \pi^{-1}(G/B \times G/B^-)$. On the other hand, it is known that Y is isomorphic to $(G \times G)/S$, S a subgroup scheme. It follows that $S \subset B \times B^-$. As Y is projective, $S = B \times B^-$, and the Lemma is proved. \square

We recall that if M is a G -embedding, then $B \times B^-$ has a finite number of orbits in M . In particular, if $\mathcal{O} \subset M$ is a $(G \times G)$ -orbit, then there exists a $(B \times B^-)$ -orbit open in \mathcal{O} .

Proposition 2. *Let M be a normal G -embedding, without colors, and let \mathcal{E} be its associated fan. Then there exists an open $(B \times B^-)$ -stable subset \mathcal{W} of M isomorphic to $U \times U^- \times A$, where A is the toric T -variety associated to \mathcal{E} , such that \mathcal{W} intersects every $(G \times G)$ -orbit in its open $(B \times B^-)$ -orbit.*

Proof. Let $\pi : M \rightarrow \mathbf{X}$ be the canonical $(G \times G)$ -equivariant morphism and let $\mathcal{U} \subset \mathbf{X}$ be the open subset isomorphic to $U \times U^- \times \mathbb{A}^l$ constructed for the wonderful embedding. As $\mathcal{U} \cap G_{ad} = UT_{G_{ad}}U^-$ it follows that $\pi^{-1}(\mathcal{U}) \cap G = UTU^-$ and that $\mathcal{W} = \pi^{-1}(\mathcal{U}) \cong U \times U^- \times A$, where $A = \pi^{-1}(\overline{T_{G_{ad}}} \cap \mathcal{U}) = \pi^{-1}(\mathbb{A}^l)$. It is clear

that A is a normal T -variety containing T as an open dense set; that is, A is a toric T -variety.

Recall that $\mathbf{X} \setminus \mathcal{U} = \bigcup_{i=1}^l \overline{D}_i^{\mathbf{X}}$, where as before \overline{D}_i is the color associated to the simple root α_i and $\overline{\cdot}^{\mathbf{X}}$ means that we consider the closure in \mathbf{X} . As $\pi(\overline{D}_i^M) \subset \overline{D}_i^{\mathbf{X}}$, it follows that $\mathcal{W} \subset M \setminus \bigcup_{i=1}^l \overline{D}_i^M$. Consider the $(G \times G)$ -stable open subset $M_{\mathcal{C}} \subset M$ associated to a cone $\mathcal{C} \in \mathcal{E}$ ([8]). Then $M_{\mathcal{C}}$ is a G -embedding, and

$$\mathcal{W} \cap M_{\mathcal{C}} = U \times U^- \times (A \cap M_{\mathcal{C}}) \subset M_{\mathcal{C}} \setminus \bigcup_{i=1}^l \overline{D}_i^{M_{\mathcal{C}}}.$$

As M is without colors, $M_{\mathcal{C}} \setminus \bigcup_{i=1}^l \overline{D}_i^{M_{\mathcal{C}}}$ is the affine open $(B \times B^-)$ -stable subset constructed in [8, §3]. Thus, $\pi(M_{\mathcal{C}} \setminus \bigcup_{i=1}^l \overline{D}_i^{M_{\mathcal{C}}}) \subset \mathcal{U}$. It follows that $\mathcal{W} = M_{\mathcal{C}} \setminus \bigcup_{i=1}^l \overline{D}_i^{M_{\mathcal{C}}}$, so the intersection of \mathcal{W} with a $(G \times G)$ -orbit \mathcal{O} of M is the unique open $(B \times B^-)$ -orbit of \mathcal{O} . Furthermore, as $M_{\mathcal{C}} \cap \mathcal{U} = U \times U^- \times (A \cap M_{\mathcal{C}})$, it follows that $A_{\mathcal{C}} = A \cap M_{\mathcal{C}}$ is an affine T -toric variety.

Recall that $\mathcal{C}_{\mathbb{A}^l}$, the cone associated to \mathbb{A}^l as a toric variety, is $-C = \mathcal{V}(G)$. If a 1-PS λ of T has a limit in $A_{\mathcal{C}}$, then its image by π has a limit in \mathbb{A}^l ; thus $\lambda \in \mathcal{V}(G)$. On the other hand, the 1-PS belonging to $\mathcal{V}(G)$ whose limit exists in $M_{\mathcal{C}}$ are exactly those of $\mathcal{V}(M_{\mathcal{C}}) = \mathcal{C} \cap \mathcal{V}(G)$ ([14]). It follows that $\mathcal{C}_{A_{\mathcal{C}}} = \mathcal{C}$, and thus that $\mathcal{E}_A = \mathcal{E}$. □

Corollary 1. *X be a normal G -embedding, without colors, with associated fan $\{\mathcal{C}_i, i \in I\}$. Suppose that \mathcal{C}_i is generated by a part of a basis of $\Xi(T)$ for all $i \in I$. Then X is smooth.*

Proof. Consider the affine chart $\mathcal{W} \cong U \times U^- \times A$ of X constructed in Proposition 2. The fan associated to A as a T -toric variety is $\{\mathcal{C}_i\}_{i \in I}$, so A is smooth ([5]), and it follows that X is smooth. □

Proposition 3. *Let X be a normal G -embedding. Then there exist a smooth G -embedding \tilde{X} , without colors, and a proper birational $(G \times G)$ -equivariant morphism $\varphi : \tilde{X} \rightarrow X$.*

Proof. Let $\mathcal{E} = \{(\mathcal{C}_i(X), \mathcal{F}_i(X))\}$ be the colored fan associated to X , and consider the fan \mathcal{E}' obtained by considering the cones $\mathcal{C}_i(X) \cap \mathcal{V}(G)$ and their faces. Take a subdivision of \mathcal{E}' into a fan \mathcal{E}'' (necessarily without colors) such that each cone \mathcal{C} of \mathcal{E}'' is generated by part of a basis of $\Xi(T)$; this subdivision always exists; cf. the toric case ([5]). We claim that the G -embedding \tilde{X} associated to \mathcal{E}'' verifies the required properties.

Indeed, the subdivision above induces a proper birational equivariant morphism $\varphi : \tilde{X} \rightarrow X$ ([8]), and it follows from Corollary 1 that \tilde{X} is smooth. □

Remark. As the resolution of singularities $\varphi : \tilde{X} \rightarrow X$ constructed above is a birational proper morphism and that X is normal, then $\varphi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$. It follows that if \tilde{X} is Frobenius split, then so is X ([12], [11]).

Proposition 4. *Let X be a normal G -embedding, without colors. Then a canonical divisor is given by $-\sum_{j=1}^r X_j - \sum_{i=1}^l 2\overline{D}_i$, where $X_j, j = 1, \dots, r$, are the $(G \times G)$ -stable divisors of X and $D_i, i = 1, \dots, l$, are the colors.*

Proof. We identify the G -invariant sections of $\mathcal{T}G$, the tangent bundle of G , with $\text{Lie}(G)$, the Lie algebra associated to G . Consider a basis $\gamma_1, \dots, \gamma_h$ of the U -left invariant sections of $\mathcal{T}U$, the tangent bundle over U , and consider a basis $\gamma'_1, \dots, \gamma'_h$ of the U^- -right invariant sections of $\mathcal{T}U^-$, such that the γ_i (resp. γ'_i) correspond to the positive (resp. negative) root vectors of $\text{Lie}(G)$. We choose subscripts in such a way that if γ_i corresponds to α , α a positive root, then γ'_i corresponds to $-\alpha$. Let $\delta_1, \dots, \delta_n$ be a basis of the $(T \times T)$ -invariant sections of the tangent bundle of T .

As $BB^- = UTU^-$, we obtain a $(B \times B^-)$ -invariant section $0 \neq \sigma \in H^0(X, \omega_X^{-1})$, given by $\sigma = \gamma_1 \wedge \dots \wedge \gamma_h \wedge \delta_1 \wedge \dots \wedge \delta_n \wedge \gamma'_1 \wedge \dots \wedge \gamma'_h \in H^0(G, \bigwedge^{max} \mathcal{T}X_{reg})$. Indeed, by construction $\sigma \neq 0$ over the open $(B \times B^-)$ -orbit BB^- . Let us calculate $\text{div}(\sigma)$; as σ is $(B \times B^-)$ -invariant, we have

$$\text{div}(\sigma) = \sum_{j=1}^r a_j X_j + \sum_{i=1}^l b_i \overline{D}_i$$

for some integers a_j, b_i . In order to calculate the order of vanishing of σ at X_j , consider the open chart $U \times U^- \times A \hookrightarrow X$ constructed in Proposition 2. It is clear that the vanishing order of σ at the divisor X_j is given by the vanishing order of the restriction to A of $\delta_1 \wedge \dots \wedge \delta_s$, which is a section of $\bigwedge^{max} \mathcal{T}A_{reg}$. We are reduced to the toric case, and the vanishing order of σ at a divisor of the boundary of A is 1, so $\text{div}(\sigma) = \sum X_j + \sum b_i \overline{D}_i$ ([5]). Indeed, if ν_k is the valuation associated to X_k , consider $\nu_{i_1}, \dots, \nu_{i_s}$ such that $\nu_j, \nu_{i_1}, \dots, \nu_{i_s}$ generate a cone of maximal dimension belonging to \mathcal{E}_A . Then in local coordinates the equation of σ is $x_j x_{i_1} \dots x_{i_s}$, so σ has vanishing order 1 at $\{x_j = 0\}$.

As $X \setminus \bigcup X_j = G$, in order to calculate the coefficients b_i , $i = 1, \dots, l$, we can restrict ourselves to G . Recall that the $(B \times B^-)$ -stable irreducible divisors in G are the colors $D_i = \overline{Bs_{\alpha_i} B^-} = \overline{Us_{\alpha_i} TU^-}$, α_i a simple root. Consider $(\ker \alpha_i)^\circ$, the identity irreducible component of the kernel of the character α_i , and put $G_i = Z_G((\ker \alpha_i)^\circ)$. Let P_i (resp. Q_i) be the minimal parabolic containing B (resp. B^-) associated to α_i (resp. $-\alpha_i$). From the properties of the Bruhat decomposition we get $Bs_{\alpha_i} B^- = P_i Q_i = R_u(P_i) G_i R_u(Q_i)$, where $R_u(H)$ notes the unipotent radical of a group H . Recall that $R_u(P_i)$ (resp. $R_u(Q_i)$) is a closed subgroup of U (resp. U^-). Moreover, $\{\gamma_j, \gamma_j \neq \gamma_i\}$ (resp. $\{\gamma'_j, \gamma'_j \neq \gamma'_i\}$) is a basis of the $R_u(P_i)$ -left (resp. $R_u(Q_i)$ -right) invariant sections of $\mathcal{T}R_u(P_i)$ (resp. $\mathcal{T}R_u(Q_i)$).

From the discussion above it is clear that we can restrict ourselves to G_i , and then to its semisimple part. This one is isomorphic to $\text{Sl}(2)$ or $\text{PSl}(2)$, and these cases are easily checked. Finally, observe that as the coefficients a_j, b_i are greater than or equal to zero, it follows that the section σ belongs to $H^0(X, \omega_X^{-1})$. \square

Notation. We denote \tilde{G} to be the simply connected cover of G .

Let $\lambda \in \Xi_+(\tilde{T})$ be a dominant weight of \tilde{G} . We denote $L(\lambda) \rightarrow G/B$ and $L(-\lambda) \rightarrow G/B^-$ to be the line bundles associated to the weights λ (with respect to B) and $-\lambda$ (with respect to B^-) respectively ([7]), and write $H^0(G/B, \lambda) = H^0(G/B, L(\lambda))$. The external tensor product $L(\lambda) \boxtimes L(-\lambda) \rightarrow (G/B) \times (G/B^-)$ is denoted $\mathcal{L}(\lambda)$.

Given a dominant weight λ , there exists a unique line bundle \mathcal{L} over \mathbf{X} such that its restriction to the unique closed orbit Y is $\mathcal{L}(\lambda)$ ([15]); we denote it as $\mathcal{L}_{\mathbf{X}}(\lambda)$.

Definition 4. Let k be of characteristic $p > 0$, and let X be a G -variety. We say that a splitting $\sigma \in H^0(X, \omega_X^{1-p})$ is canonical if

- i) $t \cdot \sigma = \sigma$ for all $t \in T$,
- ii) for every simple root α there exists $\sigma_{\alpha,r} \in H^0(X, \omega_X^{1-p})$, $r = 1, \dots, p - 1$, such that $x_\alpha(t) \cdot \sigma = \sum_{r=0}^{p-1} t^r \cdot \sigma_{\alpha,r}$.

In order to construct a canonical splitting for an arbitrary normal G -embedding, we first recall the construction of a splitting for the wonderful embedding ([2]), and show that it is a canonical one.

Definition 5. If $w \in W$ is an element of the Weyl group, we define $\mathbf{X}(w)$, the large Schubert variety, as the closure in \mathbf{X} of the double class $BwB \subset G$.

Notation. From now on, we assume that $\text{char } k = p > 0$.

Lemma 2 ([2]). *The wonderful embedding \mathbf{X} is canonically split compatibly with all $(G \times G)$ -orbit closures and all subvarieties $\mathbf{X}(w)$ and $(w_0, w_0)\mathbf{X}(w)$ (obvious action), for $w \in W$ and for $w_0 \in W$ the element of maximal length.*

Proof. In [2] is constructed a splitting $\sigma \in H^0(\mathbf{X}, \omega_{\mathbf{X}}^{1-p})$ verifying all but the first assertion. We recall the construction given there in order to prove that σ is a canonical splitting.

Consider the Steinberg module $St = H^0(G/B, (p - 1)\rho)$. Using Frobenius reciprocity and selfduality of St we obtain a $(\tilde{G} \times \tilde{G})$ -homomorphism

$$f : St \boxtimes St \rightarrow H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}((p - 1)\rho)),$$

such that the composition

$$\text{res}_Y \circ f : St \boxtimes St \rightarrow H^0(Y, \mathcal{L}((p - 1)\rho))$$

is an isomorphism, where Y denotes as usual the unique closed orbit of X .

Consider the $(\tilde{G} \times \tilde{G})$ -morphism

$$\begin{aligned} f^2 : (St \boxtimes St)^{\otimes 2} &\rightarrow H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(2(p - 1)\rho)), \\ x_1 \boxtimes y_1 \otimes x_2 \boxtimes y_2 &\mapsto f(x_1 \boxtimes y_1)f(x_2 \boxtimes y_2). \end{aligned}$$

The composition $\text{res}_Y \circ f^2 : (St \boxtimes St)^{\otimes 2} \rightarrow H^0(Y, \mathcal{L}((p - 1)\rho))$ is surjective. As $\text{Hom}(F_*\mathcal{O}_Y, \mathcal{O}_Y)$ is isomorphic to $H^0(Y, \mathcal{L}((p - 1)\rho))$, we obtain an unique homomorphism (up to a constant)

$$\varphi : (St \boxtimes St)^{\otimes 2} \rightarrow \text{Hom}(F_*\mathcal{O}_Y, \mathcal{O}_Y).$$

Let v_+ (resp. v_-) be a highest (resp. lowest) weight vector in $H^0(G/B, \rho)$. Then $a = (v_+ \boxtimes v_+)^{p-1}$ and $b = (v_- \boxtimes v_-)^{p-1}$ are in $St \boxtimes St$ and are such that $\varphi(a \otimes b)$ is a splitting of Y .

By the proof of [16, Prop. 4.3.17], $a \otimes b$ verifies the conditions i) and ii) of the definition of canonical splitting. As f^2 is $(\tilde{G} \times \tilde{G})$ -equivariant, it follows that $f^2(a \otimes b)$, and thus $\tau = \varphi(a \otimes b)$ satisfy the conditions, too. Write $x_\alpha(t) \cdot \tau = \sum_{r=0}^{p-1} t^r \cdot \tau_{\alpha,r}$, with $\tau_{\alpha,r} \in H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(2(p - 1)\rho))$.

Consider the $(\tilde{G} \times \tilde{G})$ -invariant sections σ_i of $\mathcal{L}_X(\alpha_i) = \mathcal{O}_X(-X_i)$, $i = 1, \dots, l$; then the section

$$\sigma = \tau \prod \sigma_i^{p-1}$$

is the splitting we search for. Indeed, it is a Frobenius splitting ([2]), and if $t \in T \times T$, then

$$t \cdot \sigma = t \cdot \tau \prod_{i=1}^l (t \cdot \sigma_i)^{p-1} = \tau \prod \sigma_i^{p-1}$$

because σ_i is $(G \times G)$ -invariant for $i = 1, \dots, l$. Moreover, if α is a simple root of $G \times G$, then

$$x_\alpha(t) \cdot \sigma = \left(\sum_{r=0}^{p-1} t^r \cdot \tau_{\alpha,r} \right) \prod \sigma_i^{p-1} = \sum_{r=0}^{p-1} t^r \cdot \left(\tau_{\alpha,r} \prod \sigma_i^{p-1} \right),$$

and the proposition is proved. □

Theorem 2. *Any normal G -embedding is canonically Frobenius split.*

Proof. Let M be a normal G -embedding. Consider the resolution of singularities $\varphi : \widetilde{M} \rightarrow M$ constructed in Proposition 3. A (canonically) Frobenius splitting for \widetilde{M} induces a (canonical) Frobenius splitting for M , so we can assume that M is smooth, without colors. If we consider a smooth completion $M \hookrightarrow \overline{M}$ of M , then if \overline{M} is (canonical) Frobenius split so is M , thus we suppose M without colors, smooth and complete.

By Proposition 4, a canonical divisor for M is given by $\sum_{i=1}^l -2\overline{D}_i - \sum_{j=1}^r X_j$, where \overline{D}_i is the closure of the color associated to the simple root α_i , $i = 1, \dots, l$, and X_j , $j = 1, \dots, r$, are the $(G \times G)$ -stable irreducible divisors.

Let $Y \cong G/B \times G/B^-$ be a closed orbit of M . Then Y is Frobenius split ([11],[15],[2]), and $\omega_Y \cong \mathcal{L}(-2\rho)$, where ρ is the half sum of the positive roots. As M is smooth, from the proof of Proposition 3 we get that a cone of maximal dimension of the colored fan associated to M must be generated by a basis of $\Xi(T)$. We change subscripts of the $(G \times G)$ -stable divisors X_j , $j = 1, \dots, r$, in such a way that Y is a transversal intersection of the divisors X_j , $j = 1, \dots, n$, where $n = \dim T$.

Consider $\varphi : M \rightarrow \mathbf{X}$, the canonical $(G \times G)$ -equivariant morphism. As M and \mathbf{X} are without colors, $\varphi^{-1}(\bigcup \overline{D}_i^{\mathbf{X}}) = \bigcup \overline{D}_i^M$. It follows that $\varphi^* \mathcal{O}(\sum \overline{D}_i^{\mathbf{X}}) = \mathcal{O}(\sum \overline{D}_i^M)$. By the projection formula, we have

$$\begin{aligned} \varphi_* \mathcal{O}(2(p-1) \sum \overline{D}_i^M) &= \varphi_* \varphi^* \mathcal{O}(2(p-1) \sum \overline{D}_i^{\mathbf{X}}) \\ &= \varphi_* (\mathcal{O}_M) \otimes_{\mathcal{O}_{\mathbf{X}}} \mathcal{O}(2(p-1) \sum \overline{D}_i^{\mathbf{X}}). \end{aligned}$$

So, if the section σ is as in the proof of Lemma 2, we can think of σ as belonging to $H^0(\mathbf{X}, \varphi_* \mathcal{O}(2(p-1) \sum \overline{D}_i^{\mathbf{X}})) = H^0(M, \mathcal{O}(2(p-1) \sum \overline{D}_i^M))$.

Take $\tilde{\sigma} = \sigma \prod_{j=1}^r \sigma_j^{p-1}$, σ_j a $(G \times G)$ -invariant section for $\mathcal{O}(X_j)$; then $\tilde{\sigma} \in H^0(M, \omega_M^{1-p})$. We take as coordinates in a neighbourhood of the point x_0 the system $y_1, \dots, y_m, s_1 = \sigma_1, \dots, s_n = \sigma_n$. It follows that the expansion in these coordinates of σ has a non-zero coefficient at the point x_0 for the term $(ys/dyds)^{p-1}$, so M is Frobenius split ([11]).

Moreover, it is clear from the construction that the section $\tilde{\sigma}$ constructed gives a canonical splitting (using the same calculations as in Lemma 2). □

Corollary 2. *In arbitrary characteristic, any normal G -embedding has a rational resolution. In particular, it is a Cohen-Macaulay variety.*

Proof. Suppose $\text{char } k = p > 0$, and let $\pi : \tilde{X} \rightarrow X$ be the desingularisation of a normal G -embedding X constructed in Proposition 3. It is clear that the exceptional locus of π is included set theoretically in the union $\bigcup_{j=1}^r X_j = X \setminus G$ of the $(G \times G)$ -stable irreducible divisors. The divisor associated to the section $\sigma \in H^0(X, \omega_X^{1-p})$ constructed in the proof of Theorem 2 contains this union, so we are in conditions to apply the results of [1, §3], and deduce that X has a rational resolution. The case $\text{char } k = 0$ is obtained by reduction mod p ([1]). \square

4. GOOD FILTRATIONS

Let M be a (normal, irreducible) reductive monoid with unit group G , such that the center of G is one-dimensional. In [3], it is proved that in that case the algebra of regular functions of M has a good filtration for the induced action of $G \times G$. We extend this result to arbitrary (normal, irreducible) reductive monoids, as a consequence of the existence of a canonical Frobenius splitting for M .

Definition 6. Let G be a reductive group; a rational G -module V has a *good filtration* if there exists a filtration

$$(0) = V_0 \subset V_1 \subset \dots$$

such that $\bigcup V_i = V$ and for all i there exists $\lambda_i \in \Xi^+(T)$ such that $V_{i+1}/V_i \cong H^0(G/B, \lambda_i)$, the highest weight module of greatest weight λ_i .

Corollary 3. *Let M be a reductive monoid. Then $k[M]$, algebra of regular functions of M , has a good filtration for the action of $G \times G$ induced by the action on M .*

Proof. As M admits a canonical Frobenius splitting, it follows from [10, Thm. 6.2] that the $(G \times G)$ -module $H^0(M, \mathcal{L})$ has a good filtration for all $(G \times G)$ -equivariant line bundles \mathcal{L} . In particular $k[M] = H^0(M, \mathcal{O}_M)$ has a good filtration. \square

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