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INVARIANTS OF SEMISIMPLE LIE ALGEBRAS ACTING ON ASSOCIATIVE ALGEBRAS

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ABSTRACT. If $\mathfrak{g}$ is a Lie algebra of derivations of an associative algebra R, then the subalgebra of invariants is the set $R^{\mathfrak{g}} = \{r \in R \mid \delta(r) = 0 \text{ for all } \delta \in \mathfrak{g}\}$. In this paper, we study the relationship between the structure of $R^{\mathfrak{g}}$ and the structure of R, where $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra over a field of characteristic zero acting finitely on R, when R is semiprime.

1. INTRODUCTION

Let R be a semiprime algebra over a field K acted on by a finite dimensional Lie K-algebra $\mathfrak{g}$. In this paper we show how the structure of the subalgebra of invariants $R^{\mathfrak{g}}$ is related to that of R, in the case when $\mathfrak{g}$ is semisimple and K has characteristic zero. We are interested mainly in the questions of whether $R^{\mathfrak{g}}$ is semiprime and whether the assumption that $R^{\mathfrak{g}}$ satisfies a polynomial identity implies that R also satisfies a polynomial identity. There are many papers in the literature analyzing similar problems in different situations (see Kharchenko's book [12]). For instance, in [13] and [14] the authors considered the case when R is a prime of a positive characteristic and $\mathfrak{g}$ is restricted with a quasi-Frobenius inner part. In [2] and [6] the invariants of reduced algebras and domains under actions of finite dimensional restricted Lie algebras were investigated.

Note that if we extend the context to semiprime algebras, then the restriction for $\mathfrak{g}$ to be semisimple is rather necessary. Indeed, let $R = M_2(K[x, y])$ be the algebra of 2×2-matrices over the noncommutative free algebra $K[x, y]$ and $\mathfrak{g}$ be the 2-dimensional abelian Lie algebra spanned by inner derivations induced by the elements $\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right)$. Then $R^{\mathfrak{g}}$ (the centralizer of these matrices in R) is the commutative algebra consisting of all matrices of the form $\left(\begin{smallmatrix} \alpha & b \\ 0 & \alpha \end{smallmatrix}\right)$, where $\alpha \in K$ and $b \in K[x, y]$. Clearly $R^{\mathfrak{g}}$ is not semiprime and R does not satisfy any polynomial identity. A particular case, when $\mathfrak{g} = sl_2(K)$, was considered in [9]. It was proved there that $R^{\mathfrak{g}}$ is not nilpotent provided R is semiprime.

We now introduce the definitions and terminology that we will use throughout the paper. As usual, when R is an algebra over a field K, by $\mathrm{End}_K(R)$ we denote the algebra of K-linear endomorphisms of R and by $\mathfrak{Der}_K(R)$ the Lie algebra of K-linear derivations of R. By $R^{(-)}$ we mean the Lie algebra R with respect to the ordinary commutator $[a, b] = ab - ba$ as the Lie bracket. If $a \in R$, then $\operatorname{ad} a$ denotes the

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inner derivation induced by a, that is, $\operatorname{ad} a(x) = ax - xa$. Given a Lie algebra $\mathfrak{g}$, we say that $\mathfrak{g}$ acts on R if there is a homomorphism of Lie algebras $\psi : \mathfrak{g} \longrightarrow \mathfrak{Der}_K(R)$. We say that $\mathfrak{g}$ acts finitely on R if the image of $\mathfrak{g}$ in $\operatorname{End}_K(R)$ generates a finite dimensional associative algebra. Since a homomorphism of Lie algebras $\psi : \mathfrak{g} \longrightarrow \mathfrak{Der}_K(R)$ induces an associative homomorphism $\overline{\psi}$ from the universal enveloping algebra $U(\mathfrak{g})$ to $\operatorname{End}_K(R)$, we can say that $\mathfrak{g}$ acts on R finitely of dimension N if and only if $\dim_K \overline{\psi}(U(\mathfrak{g})) = N < \infty$. In this case, for any $x \in \mathfrak{g}$ the derivation $\psi(x)$ is algebraic as a linear transformation of R. Conversely, if $\psi(x)$ is algebraic for any $x \in \mathfrak{g}$, then the Poincaré-Birkhoff-Witt theorem implies that $\dim_K \overline{\psi}(U(\mathfrak{g})) < \infty$.

If R is a semiprime K-algebra and Q is its symmetric Martindale quotient algebra, then the action of $\mathfrak{g}$ always extends uniquely to Q. Furthermore, if a derivation of R is algebraic, then its extension to Q satisfies the same polynomial. Therefore the hypothesis that $\mathfrak{g}$ acts finitely on R also extends to the action of $\mathfrak{g}$ on Q. Recall that Q is semiprime and its center C, known as the extended centroid of R, is von Neumann regular and selfinjective. We denote by $\max(C)$ the set of all maximal ideals of C and by $\mathcal{F}_R$ the set of all essential ideals of R.

Later in this paper, we will let $Q_{cl}(R)$ denote the classical ring of right quotients of the semiprime right Goldie ring R.

2. Split semisimple Lie algebras and their traces

Let $\mathfrak{g}$ be a split semisimple Lie algebra over a field K of characteristic zero and let $\mathfrak{h}$ be a splitting Cartan subalgebra of $\mathfrak{g}$. Recall that under the action of $\operatorname{ad}\mathfrak{h}$, $\mathfrak{g}$ decomposes into the direct sum of root spaces (that is, the eigenspaces of this action)

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0 = \mathfrak{h}$ and Φ, the set of roots, is $\{0 \neq \alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}$. If $\alpha \in \Phi$, then $\dim_K \mathfrak{g}_\alpha = 1$ and $\mathfrak{g}_{-\alpha} \neq 0$. Moreover, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, and Φ contains the sets Φ^+ (positive roots) and Φ^- (negative roots) such that $\Phi = \Phi^+ \cup \Phi^-$, $\Phi^- = -\Phi^+$. Then we have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-,$$

where $\mathfrak{n}^+ = \sum_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$, $\mathfrak{n}^- = \sum_{\alpha\in\Phi^-} \mathfrak{g}_\alpha$.

Suppose that a finite dimensional split semisimple Lie algebra $\mathfrak{g}$ acts finitely of dimension N on a semiprime K-algebra R. For any $\alpha \in \Phi$ let $\mathfrak{g}_\alpha = Kx_\alpha$. It is well known that the Lie subalgebra spanned by $x_\alpha, x_{-\alpha}$ and $h_\alpha = [x_\alpha, x_{-\alpha}]$ is isomorphic to $sl(2, K)$. Since every finite dimensional subspace of R is contained in a finite dimensional $\mathfrak{g}$-stable subspace, it follows from the description of finite dimensional representations of $sl(2, k)$ that $\psi(x_\alpha)$ is a nilpotent derivation for any $\alpha \in \Phi$. From the Engel-Jacobson theorem on weakly nil sets ([11, Theorem 11]) it follows that $\psi(\mathfrak{n}^+)$, $\psi(\mathfrak{n}^-)$ generate nilpotent subalgebras of $\operatorname{End}_K(R)$. In particular, $\psi(\mathfrak{n}^+)$ and $\psi(\mathfrak{n}^-)$ consist of nilpotent derivations of R. From [8, Corollary 8 and Remark 1], it follows that for any $\delta \in \psi(\mathfrak{n}^+) \cup \psi(\mathfrak{n}^-)$ there exists a unique nilpotent element $a_\delta \in Q$ such that $\delta = \operatorname{ad} a_\delta$. Since $\mathfrak{h} \subseteq [\mathfrak{n}^-, \mathfrak{n}^+]$, we see that $\mathfrak{g}$ acts by Q-inner derivations. We will show that the map $\delta \mapsto a_\delta$ induces a Lie algebra homomorphism from $\mathfrak{g}$ into $Q^{(-)}$. We begin with the following easy but very useful lemma.

Lemma 1. *If A is a finite dimensional algebra with unity over a field F of characteristic zero and $\lambda \in F^*, a \in A$ are such that $\lambda + a \in [A, A]$, then the element a is not nilpotent.*

Proof. Since A has unity, there is a natural embedding of A into $\mathrm{End}_F(A)$ via the map $x \mapsto x_l$, where x_l is the left multiplication by x acting on A. Thus A can be viewed as a subalgebra of the matrix algebra $M_n(F)$, where $n = \dim_F A$. Every element from $[A, A]$ has trace zero, so the trace $tr(a) = -n\lambda \neq 0$. Thus the element a cannot be nilpotent. □

Let $M \in \max(C)$. From [1, Theorem 1], it follows that the central localization Q_M of Q at M is a centrally closed prime algebra with the extended centroid C_M, equal to the localization of C at M. Let η_M denote the canonical homomorphism from Q into Q_M. Recall that $\eta_M(q) = 0$ if and only if q is annihilated by an element from $C \setminus M$. Note that if q is a nonzero element of Q, then $ann_C(q)$ is a proper ideal of C. Hence for any maximal ideal M containing $ann_C(q)$ we have $\eta_M(q) \neq 0$. It says that $\bigcap_M \ker \eta_M = 0$. We are now able to prove

Proposition 2. *There exists a homomorphism of Lie algebras $\theta : \mathfrak{g} \longrightarrow Q^{(-)}$ such that $\psi(x) = \mathrm{ad}\, \theta(x)$ for all $x \in \mathfrak{g}$.*

Proof. Let A be the associative C-subalgebra of Q generated by C and the elements $a_{\psi(x)}$, where $x \in \mathfrak{n}^+ \cup \mathfrak{n}^-$. By [2, Corollary 2.3], A is finitely generated as a C-module. It means that for any maximal ideal M of C, the algebra A_M is finite dimensional over the field C_M. If $x \in \mathfrak{g}_\alpha$, where $\alpha \in \Phi$, then we put $\theta(x) = a_{\psi(x)}$. If $x \in \mathfrak{g}_0 = \mathfrak{h}$, then clearly

$$x = \sum_{\alpha \in \Phi^+} \lambda_\alpha [x_\alpha, x_{-\alpha}],$$

where $\lambda_\alpha \in K$. Consequently

$$\begin{aligned}\psi(x) &= \sum_{\alpha \in \Phi^+} \lambda_\alpha [\psi(x_\alpha), \psi(x_{-\alpha})] = \sum_{\alpha \in \Phi^+} \lambda_\alpha [\mathrm{ad}\, \theta(x_\alpha), \mathrm{ad}\, \theta(x_{-\alpha})] \\ &= \sum_{\alpha \in \Phi^+} \lambda_\alpha \,\mathrm{ad}\, ([\theta(x_\alpha), \theta(x_{-\alpha})]) = \mathrm{ad}\, (\sum_{\alpha \in \Phi^+} \lambda_\alpha [\theta(x_\alpha), \theta(x_{-\alpha})]).\end{aligned}$$

In this case we put

$$\theta(x) = \sum_{\alpha \in \Phi^+} \lambda_\alpha [\theta(x_\alpha), \theta(x_{-\alpha})].$$

We need to show that θ is well defined. To this end, suppose that

$$\sum_{\alpha \in \Phi^+} \lambda_\alpha [x_\alpha, x_{-\alpha}] = \sum_{\alpha \in \Phi^+} \lambda'_\alpha [x_\alpha, x_{-\alpha}].$$

Then $\psi(x) = \mathrm{ad}\, (\sum_{\alpha \in \Phi^+} \lambda'_\alpha [\theta(x_\alpha), \theta(x_{-\alpha})])$ and hence there exists $\lambda \in C$ such that $\sum_{\alpha \in \Phi^+} (\lambda_\alpha - \lambda'_\alpha)[\theta(x_\alpha), \theta(x_{-\alpha}]) = \lambda$. In particular, $\lambda \in [A, A]$. By Lemma 1, we obtain that $\eta_M(\lambda) = 0$ for any $M \in \max(C)$. Since $\bigcap_M \ker \eta_M = 0$, $\lambda = 0$ and

$$\sum_{\alpha \in \Phi^+} \lambda_\alpha [\theta(x_\alpha), \theta(x_{-\alpha})] = \sum_{\alpha \in \Phi^+} \lambda'_\alpha [\theta(x_\alpha), \theta(x_{-\alpha})].$$

In order to show that θ is a homomorphism of Lie algebras, let $x, y \in \mathfrak{g}$ be homogeneous with respect to Cartan's decomposition of $\mathfrak{g}$. From the definition of θ it follows that

$$[\theta(x), \theta(y)] = \theta([x, y]) + \lambda$$

for some $\lambda \in C$. Note that if $[x, y] \in \mathfrak{h}$, then $\theta([x, y]) \in [A, A]$ and if $[x, y] \in \mathfrak{n}^+ \cup \mathfrak{n}^-$, then $\theta([x, y])$ is a nilpotent element of A. In both cases we get that $\lambda + a \in [A, A]$ for some nilpotent element $a \in A$. Applying Lemma 1 to the localization A_M at any $M \in \max(C)$, we obtain $\eta_M(\lambda) = 0$, and consequently $\lambda = 0$. □

Given a maximal ideal M of C, we let $\mathfrak{g}_M$ denote the Lie algebra $C_M \otimes \mathfrak{g}$. Clearly $\mathfrak{g}_M$ is a finite dimensional and semisimple C_M-algebra. The action of $\mathfrak{g}$ on Q is inner, so any $\delta \in \psi(\mathfrak{g})$ is C-linear. Hence we have the induced action of $\mathfrak{g}_M$ on the prime C_M-algebra Q_M. Moreover, it is clear that the dimension of this action over C_M does not exceed N. We denote the induced homomorphism of Lie algebras from $\mathfrak{g}_M$ into $A_M^{(-)}$ by θ_M, where A is the associative C-subalgebra of Q generated by C and $\theta(\mathfrak{g})$.

Proposition 3. *The algebra A_M is semisimple and* $\dim_{C_M} A_M \le N$.

Proof. The homomorphism of Lie algebras $\theta_M : \mathfrak{g}_M \longrightarrow A_M^{(-)}$ induces a homomorphism of associative algebras with unity $\overline{\theta}_M : U(\mathfrak{g}_M) \longrightarrow A_M$. Since A_M is generated by C_M and $\theta_M(\mathfrak{g}_M)$, $\overline{\theta}_M$ is onto. Thus $A_M \simeq U(\mathfrak{g}_M)/I$ for some ideal of finite codimension. By Weyl's theorem ([10, Section III, Theorem 8]), the left $U(\mathfrak{g}_M)$-module $U(\mathfrak{g}_M)/I$ is completely reducible. In particular, $U(\mathfrak{g}_M)/I$ is completely reducible as a left $U(\mathfrak{g}_M)/I$-module. Hence A_M is a semisimple finite dimensional C_M-algebra.

It remains to prove that $\dim_{C_M} A_M \le N$. We let $\psi_M : \mathfrak{g}_M \longrightarrow \mathfrak{Der}(Q_M)$ denote the action of $\mathfrak{g}_M$ on Q_M and $\overline{\psi}_M : U(\mathfrak{g}_M) \longrightarrow \mathrm{End}_{C_M}(Q_M)$ the induced associative homomorphism. Let $\{x_1, \dots, x_n\}$ be a basis of $\mathfrak{g}_M$ over C_M. We identify any x_i with its image in the universal enveloping algebra $U(\mathfrak{g}_M)$ and we put $\theta_M(x_i) = t_i$. Consider the natural filtration $\{U_n(\mathfrak{g}_M)\}_{n \ge 0}$ of $U(\mathfrak{g}_M)$. Given $a \in A_M$ we let $\deg(a)$ denote the minimal number n such that $U_n(\mathfrak{g}_M) \cap \overline{\theta}_M^{-1}(a) \ne \emptyset$. Let $\mathcal{B}$ be a linear basis of A_M with minimal sum $\sum_{b \in \mathcal{B}} \deg(b)$. Without loss of generality we may assume that $1 \in \mathcal{B}$. For any $b \in \mathcal{B}$ let $f_b = f_b(x_1, \dots, x_n) \in U_{\deg b}(\mathfrak{g}_M)$ be such that $\overline{\theta}_M(f_b) = b$. We claim that the subset $\{\overline{\psi}_M(f_b) \mid b \in \mathcal{B}\}$ of $\overline{\psi}_M(U(\mathfrak{g}_M))$ is linearly independent over C_M. Suppose there exists $\alpha_b \in C_M$ such that $\sum_{b \in \mathcal{B}} \alpha_b \overline{\psi}_M(f_b) = 0$ and not all α_b are equal zero. It says that $\sum_{b \in \mathcal{B}} \alpha_b f_b(\operatorname{ad} t_1, \dots, \operatorname{ad} t_n) = 0$ in $\mathfrak{Der}(Q_M)$. Hence Q_M satisfies the identity

$$\sum_{b \in \mathcal{B}} \alpha_b f_b(\operatorname{ad} t_1, \dots, \operatorname{ad} t_n)(X) = 0.$$

Let us consider the "monomial" $\omega = (\operatorname{ad} t_1)^{k_1} \cdots (\operatorname{ad} t_n)^{k_n}(X)$. Applying the formula $(\operatorname{ad} a)^k(X) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} a^i X a^{k-i}$ to ω and expressing any coefficient lying on the right side of X as a linear combination of elements of $\mathcal{B}$ we can decompose ω into the sum of $(t_1^{k_1} \cdots t_n^{k_n} + g_\omega)X$ and terms of the form $h_b(t_1, \dots, t_n)Xb$, where g_ω, h_b are polynomials in $t_1, \dots, t_n$ of degree smaller than $k_1 + \cdots + k_n$. Thus we obtain the generalized identity for Q_M

$$(\sum_{b\in\mathcal{B}}\alpha_b f_b(t_1,\ldots,t_n)+g(t_1,\ldots,t_n))X+\sum_{b\in\mathcal{B}\setminus\{1\}}g_b(t_1,\ldots,t_n)Xb=0,$$

where g is a polynomial in $t_1,\ldots,t_n$ of degree smaller than $\max\{\deg(b)\mid\alpha_b\neq 0\}$. Since Q_M is centrally closed, from the theorem of Martindale ([16, Theorem 2]), it follows that $\sum_{b\in\mathcal{B}}\alpha_b f_b(t_1,\ldots,t_n)+g(t_1,\ldots,t_n)=0$. Let $b_0\in\mathcal{B}$ be such that $\alpha_{b_0}\neq 0$ and $\deg(b_0)=\max\{\deg(b)\mid\alpha_b\neq 0\}$. Note that $\tilde{b}=\sum_{b\in\mathcal{B}}\alpha_b b\neq 0$ and $\deg(\tilde{b})\leq\deg(b_0)$. Replacing b_0 by $\tilde{b}$ we obtain a basis $\widetilde{\mathcal{B}}$ of A_M such that $\sum_{b\in\widetilde{\mathcal{B}}}\deg(b)\leq\sum_{b\in\mathcal{B}}\deg(b)$. By the definition of $\mathcal{B}$ it now follows that $\deg(\tilde{b})=\deg(b_0)$. On the other hand $\tilde{b}=-g(t_1,\ldots,t_n)$. Thus $\deg(\tilde{b})=\deg(g(t_1,\ldots,t_n))<\max\{\deg(b)\mid\alpha_b\neq 0\}=\deg(b_0)$. The obtained contradiction shows that

$$\dim_{C_M}A_M\leq\dim_{C_M}\overline{\psi}_M(U(\mathfrak{g}_M)).$$

Since $\dim_{C_M}\overline{\psi}_M(U(\mathfrak{g}_M))\leq\dim_K\overline{\psi}(U(\mathfrak{g}))=N$, we obtain $\dim_{C_M}A_M\leq N$, concluding the proof. □

We will now briefly describe a method of construction of a trace function, which allows us to produce nontrivial invariants. A similar construction was used in [2], where actions of Lie algebras on reduced rings were considered.

Since the extended centroid C is selfinjective and A is finitely generated as a C-module, there exists a finite subset $\mathcal{A}$ of A such that $A=\bigoplus_{a\in\mathcal{A}}Ca$. If M is a maximal ideal of C, then all nonzero elements among $\eta_M(a)$ form a linear basis of A_M over C_M.

By [15, Example 16.59], the algebra A_M is symmetric. It says that there exists a nonsingular, symmetric and bilinear form $\varphi_M\colon A_M\times A_M\longrightarrow C_M$ with the associativity property $\varphi_M(xy,z)=\varphi_M(x,yz)$ for all $x,y,z\in A_M$. Let $\mathcal{B}_M\subseteq A_M$ be a fixed basis of A_M over C_M and let $\mathcal{B}_M^\star=\{x^\star\mid x\in\mathcal{B}_M\}$ be the dual basis with respect to the form φ_M. Then for any $x,y\in\mathcal{B}_M$, $\varphi_M(x,x^\star)=1$ and $\varphi_M(x,y^\star)=0$ if $x\neq y$. For any $a\in\mathcal{A}$ and $b\in\mathcal{B}_M$ there exist $\lambda_c^{a,b},\mu_c^{a,b}\in C_M$ such that

$$\eta_M(a)b^\star=\sum_{c^\star\in\mathcal{B}_M^\star}\lambda_c^{a,b}c^\star\qquad\text{and}\qquad b\eta_M(a)=\sum_{c\in\mathcal{B}_M}\mu_c^{a,b}c.$$

Note that $\lambda_c^{a,b}=\varphi_M(c,\eta_M(a)b^\star)=\varphi_M(c\eta_M(a),b^\star)=\mu_b^{a,c}$. Thus

$$b\eta_M(a)=\sum_{c\in\mathcal{B}_M}\lambda_b^{a,c}c.$$

Let $\xi_M\in C\setminus M$ be such that for any $a\in\mathcal{A}$, $b,c\in\mathcal{B}_M$ there exist $\bar{b},\bar{b}^\star\in A$, $\gamma_c^{a,b}\in C$ satisfying $\xi_M b^\star=\eta_M(\bar{b}^\star),\xi_M b=\eta_M(\bar{b}),\xi_M\lambda_c^{a,b}=\eta_M(\gamma_c^{a,b})$. The above formulas can be rewritten in the form

$$\eta_M(\xi_M a\bar{b}^\star-\sum_{c^\star\in\mathcal{B}_M^\star}\gamma_c^{a,b}\bar{c}^\star)=0\qquad\text{and}\qquad\eta_M(\xi_M\bar{b}a-\sum_{c\in\mathcal{B}_M}\gamma_b^{a,c}\bar{c})=0.$$

Thus we can find $\gamma\in C\setminus M$ such that

$$\gamma\xi_M a\bar{b}^\star=\sum_{c^\star\in\mathcal{B}_M^\star}\gamma\gamma_c^{a,b}\bar{c}^\star\qquad\text{and}\qquad\gamma\xi_M\bar{b}a=\sum_{c\in\mathcal{B}_M}\gamma\gamma_b^{a,c}\bar{c}.$$

Putting $\mathcal{E} = \{\gamma\bar{b} \mid b \in \mathcal{B}_M\}$ and $\mathcal{E}^\star = \{\gamma\bar{b}^\star \mid b^\star \in \mathcal{B}_M^\star\}$ we see that $\mathcal{E}_M = \eta_M(\mathcal{E})$, $\mathcal{E}_M^\star = \eta_M(\mathcal{E}^\star)$ are linear bases for A_M over C_M and for any $a \in \mathcal{A}$, $f \in \mathcal{E}$

$$\xi_M a f^\star = \sum_{e^\star \in \mathcal{E}^\star} \gamma_e^{a,f} e^\star \quad \text{and} \quad \xi_M f a = \sum_{e \in \mathcal{E}} \gamma_f^{a,e} e.$$

Consider the map $t_M\colon Q \longrightarrow Q$ given by $t_M(x) = \sum_{e\in\mathcal{E}} \xi_M e^\star x e$. The above formulas imply that for any $a \in \mathcal{A}$

$$\begin{aligned} a t_M(x) - t_M(x) a &= \sum_{f\in\mathcal{E}} (\xi_M a f^\star) x f - \sum_{f^\star\in\mathcal{E}^\star} f^\star x (\xi_M f a) \\ &= \sum_{f\in\mathcal{E}} \Big(\sum_{e^\star\in\mathcal{E}^\star} \gamma_e^{a,f} e^\star\Big) x f - \sum_{f^\star\in\mathcal{E}^\star} f^\star x \Big(\sum_{e\in\mathcal{E}} \gamma_f^{a,e} e\Big) \\ &= \sum_{e,f\in\mathcal{E}} \gamma_e^{a,f} e^\star x f - \sum_{e,f\in\mathcal{E}} \gamma_f^{a,e} f^\star x e = 0. \end{aligned}$$

This gives us that t_M maps Q into $Q^{\mathfrak{g}}$. Clearly, the action of $\mathfrak{g}$ on Q determines a unique action of $\mathfrak{g}$ on Q_M. It is easy to see that $(Q_M)^{\mathfrak{g}} = (Q^{\mathfrak{g}})_M$. Indeed, if $\xi^{-1} q \in (Q_M)^{\mathfrak{g}}$, then for any $\delta \in \psi(\mathfrak{g})$ there exists $m_\delta \in C \setminus M$ such that $m_\delta \delta(q) = 0$. Therefore, if $\mathcal{D}$ is a basis of $\psi(\mathfrak{g})$ and $m = \prod_{\delta\in\mathcal{D}} m_\delta$, then $m \in C \setminus M$ and $\delta(mq) = m\delta(q) = 0$ for all $\delta \in \psi(\mathfrak{g})$. As a result, $mq \in Q^{\mathfrak{g}}$ and $\xi^{-1} q = (m\xi)^{-1}(mq) \in (Q^{\mathfrak{g}})_M$. Thus $(Q_M)^{\mathfrak{g}} \subseteq (Q^{\mathfrak{g}})_M$. The reverse inclusion is clear. Therefore t_M induces the map $T_M\colon Q_M \longrightarrow (Q_M)^{\mathfrak{g}}$ given by $T_M(x) = \sum_{e\in\mathcal{E}} \eta_M(e^\star) x \eta_M(e)$. We can summarize the above considerations in the following proposition.

Proposition 4. *If M is a maximal ideal of C, then there exist $\xi_M \in C \setminus M$ and finite subsets $\mathcal{E}$, $\mathcal{E}^\star = \{e^\star \mid e \in \mathcal{E}\}$ of nonzero elements of A such that:*

(1) *the sets $\eta_M(\mathcal{E})$ and $\eta_M(\mathcal{E}^\star)$ form linear bases of A_M over C_M,*
(2) *the functions $t_M(x) = \sum_{e\in\mathcal{E}} \xi_M e^\star x e$ and $T_M(x) = \sum_{e\in\mathcal{E}} \eta_M(e^\star) x \eta_M(e)$ map Q and Q_M into $Q^{\mathfrak{g}}$ and $(Q_M)^{\mathfrak{g}}$, respectively.*

3. Main result

In this part we study how the structure of $R^{\mathfrak{g}}$ is related to that of R. In particular, we prove that $R^{\mathfrak{g}}$ is semiprime provided R is semiprime, that the PI property can be lifted from $R^{\mathfrak{g}}$ to R, and that the classical rings of quotients of $R^{\mathfrak{g}}$ and R are closely connected. We obtain these by reducing the questions to the case of centralizers of finite dimensional semisimple algebras. Therefore we will frequently refer to works of Montgomery [17] and Montgomery-Smith [18] on centralizers of separable algebras.

Theorem 5. *Let R be a semiprime algebra over a field K of characteristic zero on which a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ acts finitely of dimension N. Then:*

(1) *the subalgebra of invariants $R^{\mathfrak{g}}$ is semiprime and $I \cap R^{\mathfrak{g}} \neq 0$ for any nonzero ideal I of R,*
(2) *if $R^{\mathfrak{g}}$ satisfies a polynomial identity of degree d, then R satisfies the standard polynomial identity of degree dN,*
(3) *if $R^{\mathfrak{g}}$ is a simple ring with unity, then R is a simple ring with unity,*
(4) *if $t \in R^{\mathfrak{g}}$ is regular in $R^{\mathfrak{g}}$, then t is regular in R,*
(5) *$R^{\mathfrak{g}}$ is right Artinian if and only if R is right Artinian,*

(6) *$R^{\mathfrak{g}}$ is right Goldie if and only if R is right Goldie. Furthermore, in the case when both R and $R^{\mathfrak{g}}$ are right Goldie, $Q_{cl}(R) = RT^{-1}$, where T is the set of all regular elements of $R^{\mathfrak{g}}$, and $Q_{cl}(R)^{\mathfrak{g}} = Q_{cl}(R^{\mathfrak{g}})$.*

Proof. Since for any field extension $K \subseteq F$ the F-algebra $R \otimes_K F$ is semiprime, $\mathfrak{g} \otimes_K F$ is semisimple and $(R \otimes_K F)^{\mathfrak{g} \otimes_K F} = R^{\mathfrak{g}} \otimes_K F$, without loss of generality we may assume that $\mathfrak{g}$ is a split semisimple Lie algebra. Indeed, it suffices to consider a finite field extension $K \subseteq F$ containing all characteristic roots of every $\operatorname{ad} h$, $h \in \mathfrak{h}$. Then properties from (1), (2), (4), (5) and the first part of (6) are preserved by $R \otimes_K F$ and $R^{\mathfrak{g}} \otimes_K F$.

First, we prove that $Q^{\mathfrak{g}}$ is semiprime. According to Proposition 2, let $\theta \colon \mathfrak{g} \longrightarrow Q$ be the homomorphism of Lie algebras satisfying $\psi(x) = \operatorname{ad} \theta(x)$, and let A be the subring of Q generated by $\theta(\mathfrak{g})$ and the extended centroid C. We know that for any $M \in \max(C)$ the localization A_M is finite dimensional over C_M and semisimple. By the result of Montgomery on centralizers of separable algebras ([17, Theorem 3.3]), we obtain that $Q^{\mathfrak{g}}_M = C_{Q_M}(A_M)$ is semiprime. If I is an ideal of $Q^{\mathfrak{g}}$, then there exists an ideal $M \in \max(C)$ such that $\eta_M(I) \neq 0$. Note that $\eta_M(I)C_M$ is a nonzero ideal of $Q^{\mathfrak{g}}_M$ and $(\eta_M(I)C_M)^2 = \eta_M(I^2)C_M$. Now it is clear that $Q^{\mathfrak{g}}$ must be semiprime.

We will make use of the following fact:

(*) if $M \in \max(C)$, $0 \neq q \in Q$, $I \in \mathcal{F}_R$ are such that either $qt_M(I) = 0$ or $t_M(I)q = 0$, then $\eta_M(q) = 0$.

By [3, Theorem 6.4.1], the generalized polynomial identity $\sum_{\eta_M \in \mathcal{E}} q\xi_M e^\star X e = 0$ on I can be lifted from I to $Q(I) = Q(R) = Q$. Thus $\sum_{e \in \mathcal{E}} \eta_M(q)\eta_M(e^\star)X\eta_M(e) = 0$ is the generalized polynomial identity on a centrally closed prime ring Q_M. We are now in a position to apply a theorem of Martindale ([16, Theorem 2]). It gives us that for any $e^\star \in \mathcal{E}^\star$, $\eta_M(q)\eta_M(e^\star) = 0$. Since $\eta_M(\mathcal{E}^\star)$ is a basis of A_M and A_M contains unity of Q_M, we obtain that $\eta_M(q) = 0$. The case when $t_M(I)q = 0$ is analogous. This finishes the proof of (*).

From the basic properties of Q it follows that for any $M \in \max(C)$ and $I \in \mathcal{F}_R$ one can choose $I(M) \in \mathcal{F}_R$ such that $t_M(I(M)) \subseteq I$. By (*), for $q = 1$, the map $t_M \colon I(M) \longrightarrow I^{\mathfrak{g}}$ is nonzero. Suppose that $a \in R^{\mathfrak{g}}$ is such that $aR^{\mathfrak{g}}a = 0$. Then for any $M \in \max(C)$, $at_M(I(M))a = 0$. Applying the result on lifting generalized polynomial identities again, we obtain $at_M(Q)a = 0$. Since $at_M(Q)$ is a right ideal of $Q^{\mathfrak{g}}$ and $Q^{\mathfrak{g}}$ is semiprime, $at_M(Q) = 0$. From (*) it follows that $\eta_M(a) = 0$. Thus $a = 0$ and $R^{\mathfrak{g}}$ is semiprime.

If I is a nonzero ideal of R, then by [5, Proposition 1.12], I contains a nonzero $\mathfrak{g}$-stable ideal J. Since J is semiprime, we can apply the above to conclude that $0 \neq J^{\mathfrak{g}} \subseteq R^{\mathfrak{g}} \cap I$.

For (2) we first prove that $Q^{\mathfrak{g}}$ can be viewed as a subring of $Q(R^{\mathfrak{g}})$. From (*) it follows that for $I \in \mathcal{F}_R$, $I^{\mathfrak{g}}$ has a zero left and right annihilator in Q. In particular, $I^{\mathfrak{g}}$ is an essential ideal of $R^{\mathfrak{g}}$. If $0 \neq q \in Q^{\mathfrak{g}}$ and $I \in \mathcal{F}_R$ is such that $qI \cup Iq \subseteq R$, then clearly $0 \neq qI^{\mathfrak{g}} \subseteq R^{\mathfrak{g}}$ and $0 \neq I^{\mathfrak{g}}q \subseteq R^{\mathfrak{g}}$. Finally, suppose that J is an essential ideal of $R^{\mathfrak{g}}$ and $q \in Q^{\mathfrak{g}}$ is such that either $qJ = 0$ or $Jq = 0$. Take $I \in \mathcal{F}_R$ such that $qI \cup Iq \subseteq R$. If $qJ = 0$, then $(I^{\mathfrak{g}}q)J = I^{\mathfrak{g}}(qJ) = 0$ and hence $I^{\mathfrak{g}}q = 0$. The above implies that $q = 0$. A similar argument shows that $Jq = 0$ implies $q = 0$. Therefore $Q^{\mathfrak{g}}$ can be treated as a subring of $Q(R^{\mathfrak{g}})$. Suppose that $R^{\mathfrak{g}}$ satisfies a polynomial identity of degree d. Since $R^{\mathfrak{g}}$ is semiprime, it satisfies the standard identity s_d. Then by [3, Theorem 6.4.1], s_d is satisfied by $Q(R^{\mathfrak{g}})$, and hence also by $Q^{\mathfrak{g}}$. This forces that for any $M \in \max(C)$ the C_M-algebra $(Q^{\mathfrak{g}})_M = (Q_M)^{\mathfrak{g}} = C_{Q_M}(A_M)$ satisfies s_d. Applying the result of Montgomery and Smith on centralizers of separable algebras [18], we conclude that Q_M satisfies the standard identity s_{dN}. Since it holds for any $M \in \max(C)$, the ring Q satisfies s_{dN}.

For (3), note that if I is a nonzero ideal of R, then by (1) $I^{\mathfrak{g}}$ is a nonzero ideal of $R^{\mathfrak{g}}$. Hence $I^{\mathfrak{g}} = R^{\mathfrak{g}}$ and $1 \in I^{\mathfrak{g}}$. Therefore $I = R$ and R must be a simple ring.

(4) Let $t \in R^{\mathfrak{g}}$ be regular in $R^{\mathfrak{g}}$. Since $R^{\mathfrak{g}} \subseteq Q^{\mathfrak{g}} \subseteq Q(R^{\mathfrak{g}})$ and t is regular in $Q(R^{\mathfrak{g}})$, t must be regular in $Q^{\mathfrak{g}}$. Suppose that $tx = 0$ for some $x \in R$. Take any $M \in \max\ (C)$. Then $\eta_M(t)$ is regular in $(Q^{\mathfrak{g}})_M = (Q_M)^{\mathfrak{g}}$. Since $(Q_M)^{\mathfrak{g}} = C_{Q_M}(A_M)$ for some simple finite dimensional C_M-algebra, by [17, Theorem 5.1], the element $\eta_M(t)$ is regular in Q_M, so $\eta_M(x) = 0$. This holds for any maximal ideal of C, so $x = 0$. Consequently t is regular in Q.

(5) If $R^{\mathfrak{g}}$ is left Artinian, then $Q(R^{\mathfrak{g}}) = R^{\mathfrak{g}}$. Since $Q(R)^{\mathfrak{g}} \subseteq Q(R^{\mathfrak{g}})$, we obtain $Q(R)^{\mathfrak{g}} = R^{\mathfrak{g}}$. In particular, $C \subseteq R$, so R has unity. From (*) it follows that $\mathcal{F}_R = \{R\}$. By (1) we now obtain that R is a finite direct sum of minimal ideals which are certainly $\mathfrak{g}$-stable. Hence it suffices to consider the case when R is a simple algebra with unity and $R^{\mathfrak{g}}$ is semisimple Artinian. Then the algebra A (defined in Proposition 2) is semisimple and finite dimensional over the center of R. From [17, Corollary 4.1], it follows that R must be semisimple Artinian. Conversely, if R is semisimple Artinian, then the result is a direct consequence of [17, Theorem 4.2].

(6) Suppose R is right Goldie. Then the classical ring of right quotients $Q_{cl}(R)$ and the symmetric Martindale ring of quotients Q coincide and hence Q is semisimple Artinian. By (4), $Q^{\mathfrak{g}}$ is semisimple Artinian. Since $R^{\mathfrak{g}} \subseteq Q^{\mathfrak{g}} \subseteq Q(R^{\mathfrak{g}})$, we obtain $Q^{\mathfrak{g}} = Q(R^{\mathfrak{g}})$. It says that $R^{\mathfrak{g}}$ is a right order in a semisimple Artinian ring $Q(R^{\mathfrak{g}})$. Therefore $R^{\mathfrak{g}}$ is right Goldie.

Conversely, suppose that $R^{\mathfrak{g}}$ is right Goldie. From (1) it follows that R does not contain infinite direct sums of nonzero ideals. Thus there exist $\mathfrak{g}$-stable ideals $R_1, \dots, R_k$ of R such that $R_1 \oplus \dots \oplus R_k \in \mathcal{F}_R$ and any R_i is a prime ring. Clearly, it suffices to show that any R_i is right Goldie. On the other hand, note that $R_i^{\mathfrak{g}}$ is right Goldie as a two-sided ideal of $R^{\mathfrak{g}}$, so without loss of generality we may assume that R is prime. Since $R^{\mathfrak{g}} \subseteq Q^{\mathfrak{g}} \subseteq Q(R^{\mathfrak{g}})$ and $Q(R^{\mathfrak{g}})$ is semisimple Artinian, $Q^{\mathfrak{g}}$ is right Goldie. Moreover $Q^{\mathfrak{g}} = C_Q(A)$, where A is a semisimple finite dimensional C-algebra (Proposition 3), so by the result of Cohen [7], Q is right Goldie. It immediately implies that Q is semisimple Artinian and hence R is right Goldie.

It remains to show that $Q_{cl}(R) = RT^{-1}$, where T is the set of all regular elements of $R^{\mathfrak{g}}$. We claim that any essential right ideal of R intersects T nontrivially. Since R is right nonsingular, any essential right ideal J of R contains an essential $\mathfrak{g}$-stable right ideal J_*. Indeed, in [4, Lemma 4.2], and in [19], it is shown that if δ is a derivation and J is an essential right ideal of a ring whose right singular ideal is zero, then the right ideal $J \cap \delta^{-1}(J) = \{r \in J \mid \delta(r) \in J\}$ is also essential. Hence, if $\mathcal{D}$ is a basis of $\psi(\mathfrak{g})$, then applying the above many times one obtains that $J_* = \bigcap_{\delta \in \mathcal{D},\, 0 \le j \le N} \delta^{-j}(J)$ is essential and $\mathfrak{g}$-stable. Since R is semiprime Goldie, J_* is semiprime as a ring and we may consider the induced action of $\mathfrak{g}$ on J_*. Moreover it is clear that J_* and R have the same Goldie ranks, so J_* is right Goldie. The above implies that $J_*^{\mathfrak{g}}$ is semiprime right Goldie. In particular, $J_*^{\mathfrak{g}}$ contains a regular element t (regular in $J_*^{\mathfrak{g}}$). By (4), t is regular in J_*. If $0 \neq r \in R$, then there exists $r' \in R$ such that $0 \neq rr' \in J_*$. Hence, the regularity of t in J_* implies the right regularity of t in R. Since R is semiprime Goldie, t must also be left regular in R, so $t \in T$. Consequently, T is a right Ore set in R and we may consider the localization RT^{-1}. Since $(RT^{-1})^{\mathfrak{g}} = R^{\mathfrak{g}}T^{-1}$ is right Artinian, from (5) it follows that RT^{-1} is right Artinian. Hence $Q_{cl}(R) = RT^{-1}$ and it is clear now that $Q_{cl}(R)^{\mathfrak{g}} = R^{\mathfrak{g}}T^{-1} = Q_{cl}(R^{\mathfrak{g}})$. □

References

1. K.I. Beidar, *Rings of quotients of semiprime rings*, Vestnik Moscow. Univers. Math. (1978), 36–43.
2. K.I. Beidar and P. Grzeszczuk, *Actions of Lie algebras on rings without nilpotent elements*, Algebra Colloquium, **2** (1995), 105–116. MR **96f:**16043
3. K. I. Beidar, W. S. Martindale III, A. V. Mikhalev, *Rings with generalized identities*, Marcel Dekker, New York, 1996. MR **97g:**16035
4. J. Bergen and S. Montgomery, *Smash products and outer derivations*, Israel J. Math., **53** (1986), 321–345. MR **87i:**16065
5. J. Bergen, *Constants of Lie algebra actions*, J. Algebra, **114** (1988), 452–465. MR **89c:**16048
6. J. Bergen, *Invariants of domains under the actions of restricted Lie algebras*, J. Algebra, **177** (1995), 115–131. MR **97c:**16045
7. M. Cohen, *Goldie centralizers of separable algebras*, Michigan Math. J., **23** (1976), 185–191. MR **54:**346
8. P. Grzeszczuk, *On nilpotent derivations of semiprime rings*, J. Algebra, **149** (1992), 313–321. MR **93h:**16068
9. P. Grzeszczuk, *Constants of algebraic derivations*, Comm. in Algebra, **21** (1993), 1857–1868. MR **94f:**16057
10. N. Jacobson, *Lie algebras*, Dover Publications Inc., New York, 1979. MR **80k:**17001
11. I. Kaplansky, *Lie algebras and locally compact groups*, Lectures in Mathematics, University of Chicago Press, Chicago, 1971. MR **43:**2145
12. V.K. Kharchenko, *Automorphisms and derivations of associative rings*, Kluwer Academic Publishers, Dordrecht, vol. 69, 1991. MR **93i:**16048
13. V.K. Kharchenko, *On derivations of prime rings of positive characteristic*, Algebra i Logika, **35** (1996), 88–104. (English transl. Algebra and Logic, **35** (1996), 49–58.) MR **97j:**16052
14. V.K. Kharchenko, J. Keller and S. Rodriguez-Romo, *Prime rings with PI rings of constants*, Israel J. Math., **86** (1996), 357–377. MR **97k:**16052
15. T.Y. Lam, *Lectures on Modules and Rings*, Springer Verlag, New York, 1998. MR **99i:**16001
16. W.S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra, **12** (1969), 576–584. MR **39:**257
17. S. Montgomery, *Centralizers of separable algebras*, Michigan Math. J., **22** (1975), 15–24. MR **52:**5721
18. S. Montgomery and M. Smith, *Algebras with a separable subalgebra whose centralizer satisfies a polynomial identity*, Comm. in Algebra, **3** (1975), 151–168. MR **51:**10386
19. A.Z. Popov, *Derivations of prime rings*, Algebra i Logika, **22** (1983), 79–92. MR **85h:**16043

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