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REGULARIZATION OF A_p WEIGHTS

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Abstract. We show how to approximate a given weight function in the class A_p by weights that are bounded above by multiples of their infima in such a way that the A_p constant is not increased. As an application, we show that the precise range of p for which a given weight is in A_p cannot be determined by extrapolating the A_p constants.

1. Introduction

For $1 < p < \infty$, the class $A_p(\mathbf{R}^n)$ consists of those locally integrable weight functions w that are positive almost everywhere and satisfy an *a priori* bound of the form

$$\left(\int_Q w(x)\ dx\right)^{1/p}\left(\int_Q w(x)^{1-p'}\,dx\right)^{1/p'} \le A\,|Q| \tag{1.1}$$

for all cubes Q in $\mathbf{R}^n$; here p' is the Hölder conjugate of p, A is a constant depending on w and $|Q|$ is the Lebesgue measure of Q. The smallest value of A for which (1.1) can hold is called $A_p(w)$, the A_p-constant for w. These classes were introduced by B. Muckenhoupt [5] in connection with weighted inequalities for the Hardy-Littlewood maximal function, and were soon seen to have a number of remarkable properties. Note that Hölder's inequality shows

$$|Q| = \int_Q w(x)^{1/p}\,w(x)^{-1/p}\,dx \le \left(\int_Q w(x)\ dx\right)^{1/p}\left(\int_Q w(x)^{1-p'}\,dx\right)^{1/p'};$$

for that reason a condition such as (1.1) is often referred to as a reverse Hölder inequality. A simple application of Hölder's inequality shows that $A_q(w) \le A_p(w)$ whenever $p < q < \infty$, and several delicate arguments have been devised to show that, when $w \in A_p$, $A_q(w)$ must always be finite for all q in an open interval (p_0, ∞) that includes p. These arguments give upper bounds for p_0 and for $A_q(w)$ in terms of p, q and $A_p(w)$; see Muckenhoupt [5], Coifman and Fefferman [3], and Chung, Hunt, and Kurtz [2] as well as the related work by Gehring [4].

In [1], the authors showed how to derive Orlicz space bounds that substitute for $A_{p_0}(w)$ when p_0 is determined by such arguments, but left open the question of whether the best value of p_0 could be obtained by such methods. That is, must $A_{p_0}(w) = \infty$ for some weight having the given A_p-constant? Our results here do

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not answer that question, but do show that we can never determine $w \notin A_{p_0}$ by considering only values of $A_p(w)$ for finitely many $p > p_0$, each having $A_p(w) < \infty$.

Our investigations here develop a regularization principle that can be useful for establishing *a priori* bounds for classes of the form $\{w \in A_p : A_p(w) \le M\}$. Given a locally integrable weight function w, we show how to construct a family of weights $\{w_t : t > 0\}$ such that each w_t and its reciprocal is uniformly bounded, with $A_p(w_t) \le A_p(w)$ and $A_p(w) = \lim_{t\to 0+} A_p(w_t)$. By working with w_t in place of w, we can improve some arguments by guaranteeing that all the integrals involved are finite.

2. Regularization of A_p weights

Given a locally integrable function w on $\mathbf{R}^n$ that is positive almost everywhere, define

$$A_p(w) = \sup_Q \frac{1}{|Q|} \left(\int_Q w \right)^{1/p} \left(\int_Q w^{1-p'} \right)^{1/p'}$$

so that $w \in A_p$ if and only if $A_p(w) < \infty$. While $\int_Q w^{1-p'}$ need not be finite for all cubes, $\int_Q w$ is always positive so this supremum is well-defined, either as positive and real or as ∞. For $t > 0$ we define

$$w_t(x) = \frac{1}{t + 1/(t + w(x))} = \frac{t + w(x)}{t^2 + tw(x) + 1};$$

the first form gives us $w_t(x) = 1/t$ when $w(x) = \infty$. Then $\lim_{t\to 0+} w_t(x) = w(x)$ pointwise, and $t/\left(t^2+1\right) \le w_t(x) \le 1/t$. Note that both w_t and $1/w_t$ are in Muckenhoupt's class A_1, since that class consists of all weights for which the average over each cube is dominated by a fixed multiple of the essential infimum over the same cube. For $1 < p < \infty$ we easily calculate

$$A_p(w_t) \le \left(\frac{1}{t}\right)^{1/p} \left[\left(\frac{t}{t^2+1} \right)^{-p'/p} \right]^{1/p'} = \left(\frac{t^2+1}{t^2} \right)^{1/p}$$

and find exactly the same bound for $A_p(1/w_t)$. Of course, these are only crude bounds; the theorem below shows that $A_p(w_t)$ mimics the behavior of $A_p(w)$ for small t.

Theorem 1. *For each a.e. positive locally integrable function w on $\mathbf{R}^n$ and for $1 < p < \infty$, $A_p(w_t) \le A_p(w)$ on $(0, \infty)$, with $A_p(w) = \lim_{t\to 0+} A_p(w_t)$.*

Proof. For $s, t \ge 0$ call $u(x) = s + w(x)$ and $v(x) = tu(x) + 1$, and then define

$$F(s,t) = \left(\int_Q \frac{u}{v} \right)^{p'-1} \int_Q \left(\frac{u}{v} \right)^{1-p'}$$

with Q a fixed but arbitrary cube in $\mathbf{R}^n$. We begin by showing that the partial derivatives $\frac{\partial F}{\partial s}(s,0)$ and $\frac{\partial F}{\partial t}(s,t)$ are never positive. The bound for $\frac{\partial F}{\partial s}$ was done earlier by the authors in [1]; we reprove it here with a slightly different argument.

Since

$$F(s,0) = \left(\int_Q (s+w) \right)^{p'-1} \int_Q (s+w)^{1-p'},$$

for $s>0$ we can differentiate with respect to s to find

$$\begin{aligned}\frac{\partial F}{\partial s}(s,0) &= (p'-1)\left(\int_Q (s+w)\right)^{p'-2}|Q|\int_Q (s+w)^{1-p'} \\ &\quad + (1-p')\left(\int_Q (s+w)\right)^{p'-1}\int_Q (s+w)^{-p'} \\ &= (p'-1)\left(\int_Q u\right)^{p'-2}\left[|Q|\int_Q u^{1-p'} - \left(\int_Q u\right)\int_Q u^{-p'}\right].\end{aligned}$$

We use a simple corollary of Hölder's inequality to show that this is never positive: for $u>0$ and $0\le\theta\le 1$ we have

$$\left(\int u^{a(1-\theta)+b\theta}\right)\left(\int u^{a\theta+b(1-\theta)}\right)\le\left(\int u^a\right)\left(\int u^b\right).$$

Note $|Q|=\int_Q u^0$, and calling $\theta=1/(p'+1)$ lets us write

$$0=1-\theta-p'\theta \quad\text{and}\quad 1-p'=\theta-p'(1-\theta).$$

Thus

$$|Q|\int_Q u^{1-p'}\le\left(\int_Q u\right)\int_Q u^{-p'}<\infty,$$

proving that $\frac{\partial F}{\partial s}(s,0)\le 0$ for each $s>0$.

Obviously $\lim_{s\to 0+}\int_Q (s+w)=\int_Q w$, and the monotone convergence theorem shows $\lim_{s\to 0+}\int_Q (s+w)^{1-p'}=\int_Q w^{1-p'}$ whether this last quantity is finite or infinite, so we have established that $F(s,0)\le F(0,0)$.

Next we fix $s>0$ and differentiate with respect to t. Since

$$\frac{\partial}{\partial t}\left(\frac{u}{v}\right)=\frac{\partial}{\partial t}\left(\frac{u}{tu+1}\right)=-\frac{u^2}{v^2},$$

we find

$$\frac{\partial F}{\partial t}(s,t)=(p'-1)\left(\int_Q\frac{u}{v}\right)^{p'-2}\left[\left(\int_Q\frac{u}{v}\right)\int_Q\left(\frac{u}{v}\right)^{2-p'}-\left(\int_Q\frac{u^2}{v^2}\right)\int_Q\left(\frac{u}{v}\right)^{1-p'}\right].$$

Again calling $\theta=1/(p'+1)$, we may write

$$1=2(1-\theta)+\theta(1-p') \quad\text{and}\quad 2-p'=2\theta+(1-\theta)(1-p')$$

and then show that $\frac{\partial F}{\partial t}(s,t)\le 0$ as above; note all powers of u/v are locally integrable.

As for $F(s,t)$ itself, we have

$$\lim_{t\to 0+}\int_Q u/v=\int_Q u \quad\text{and}\quad \lim_{t\to 0+}\int_Q (u/v)^{1-p'}=\int_Q u^{1-p'}$$

by monotone convergence in the first case and by uniform convergence in the second, so that $\lim_{t\to 0+}F(s,t)=F(s,0)\le F(0,0)$. Hence we have $F(s,t)\le F(0,0)$ for all $s,t>0$. In particular, this holds true for $s=t$, and after taking p' roots we obtain

$$\left(\int_Q w_t\right)^{1/p}\left(\int_Q w_t^{1-p'}\right)^{1/p'}\le\left(\int_Q w\right)^{1/p}\left(\int_Q w^{1-p'}\right)^{1/p'}$$

for all $t > 0$, all nonnegative locally integrable w, and all cubes Q in $\mathbf{R}^n$. Consequently, $A_p(w_t) \le A_p(w)$ for each nonnegative locally integrable w on $\mathbf{R}^n$, and this is valid as long as $t > 0$ and $1 < p < \infty$.

We conclude the proof by appealing to Fatou's lemma. Since $\int_Q w$ and $\int_Q w^{1-p'}$ are always positive or infinite, for each cube Q we have

$$\begin{aligned}\frac{1}{|Q|}\left(\int_Q w\right)^{1/p}\left(\int_Q w^{1-p'}\right)^{1/p'} &\le \frac{1}{|Q|}\left(\liminf_{t\to 0+}\int_Q w_t\right)^{1/p}\left(\liminf_{t\to 0+}\int_Q w_t^{1-p'}\right)^{1/p'}\\ &\le \liminf_{t\to 0+}\frac{1}{|Q|}\left(\int_Q w_t\right)^{1/p}\left(\int_Q w_t^{1-p'}\right)^{1/p'}\\ &\le \liminf_{t\to 0+} A_p(w_t) \le \limsup_{t\to 0+} A_p(w_t) \le A_p(w).\end{aligned}$$

Taking the supremum over all cubes completes the proof. □

3. Concluding remarks

Given any weight w that satisfies $A_{p_1}(w) > C_1$ and $A_{p_2}(w) \le C_2$, for sufficiently small t the weight w_t will satisfy exactly the same inequalities, and w_t is in every A_p class. Consequently, no such system of inequalities can be used to show that the given weight w fails to belong to some weight class.

Our theorem fills a gap in some of the proofs that $\{p > 1 : w \in A_p\}$ is open. For example, the A_p condition and a Calderón-Zygmund covering argument can be used to show

$$\int_Q w^{1-q'} \le C(p,q,A_p(w))\int_Q w^{1-q'} + |Q|^{q'(1-p/q)}\left(\int_Q w^{1-p'}\right)^{(p/q)q'/p}$$

for $1 < q < p$, with $C(p,q,A_p(w)) < 1$ when q is close enough to p. Then subtraction gives a bound for $\int_Q w^{1-q'}$ that leads to a bound for $A_q(w)$ in terms of $A_p(w)$, but to justify the subtraction $\int_Q w^{1-q'}$ needs to be finite. Such arguments are always valid when w is replaced by w_t, and then the bound for $A_q(w)$ in terms of $A_p(\omega)$ can be obtained by letting $t \to 0$.

References

[1] R. J. Bagby and B. Masaedeh, *Optimal Extrapolation of Weight Conditions*, Panamerican Mathematical Journal 7 (1997), Number 1, 55-63. MR **97k:**42038

[2] H.-M. Chung, R. A. Hunt, and D. S. Kurtz, *The Hardy-Littlewood maximal function on L(p,q) spaces with weights*, Indiana Univ. Math. J. 31 (1982), 109-120. MR **83b:**42021

[3] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241-250. MR **50:**10670

[4] F. W. Gehring, *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. 130 (1973), 265-277. MR **53:**5861

[5] B. Muckenhoupt, *Weighted inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207-226. MR **45:**2461

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