REGULARIZATION OF $A_p$ WEIGHTS

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Abstract. We show how to approximate a given weight function in the class $A_p$ by weights that are bounded above by multiples of their infima in such a way that the $A_p$ constant is not increased. As an application, we show that the precise range of $p$ for which a given weight is in $A_p$ cannot be determined by extrapolating the $A_p$ constants.

1. Introduction

For $1 < p < \infty$, the class $A_p(\mathbb{R}^n)$ consists of those locally integrable weight functions $w$ that are positive almost everywhere and satisfy an a priori bound of the form

$$\left( \int_Q w(x) \, dx \right)^{1/p} \left( \int_Q w(x)^{1-p'} \, dx \right)^{1/p'} \leq A |Q|$$

for all cubes $Q$ in $\mathbb{R}^n$; here $p'$ is the Hölder conjugate of $p$, $A$ is a constant depending on $w$ and $|Q|$ is the Lebesgue measure of $Q$. The smallest value of $A$ for which (1.1) can hold is called $A_p(w)$, the $A_p$-constant for $w$. These classes were introduced by B. Muckenhoupt [5] in connection with weighted inequalities for the Hardy-Littlewood maximal function, and were soon seen to have a number of remarkable properties. Note that Hölder’s inequality shows

$$|Q| = \int_Q w(x)^{1/p} w(x)^{-1/p} \, dx \leq \left( \int_Q w(x) \, dx \right)^{1/p} \left( \int_Q w(x)^{1-p'} \, dx \right)^{1/p'} ;$$

for that reason a condition such as (1.1) is often referred to as a reverse Hölder inequality. A simple application of Hölder’s inequality shows that $A_q(w) \leq A_p(w)$ whenever $p < q < \infty$, and several delicate arguments have been devised to show that, when $w \in A_p$, $A_q(w)$ must always be finite for all $q$ in an open interval $(p_0, \infty)$ that includes $p$. These arguments give upper bounds for $p_0$ and for $A_q(w)$ in terms of $p, q$ and $A_p(w)$; see Muckenhoupt [5], Coifman and Fefferman [3], and Chung, Hunt, and Kurtz [2] as well as the related work by Gehring [4].

In [1], the authors showed how to derive Orlicz space bounds that substitute for $A_{p_0}(w)$ when $p_0$ is determined by such arguments, but left open the question of whether the best value of $p_0$ could be obtained by such methods. That is, must $A_{p_0}(w) = \infty$ for some weight having the given $A_p$-constant? Our results here do

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not answer that question, but do show that we can never determine $w \notin A_{p_{0}}$ by considering only values of $A_{p}(w)$ for finitely many $p > p_{0}$, each having $A_{p}(w) < \infty$.

Our investigations here develop a regularization principle that can be useful for establishing a priori bounds for classes of the form $\{w \in A_{p} : A_{p}(w) \leq M\}$. Given a locally integrable weight function $w$, we show how to construct a family of weights $\{w_{t} : t > 0\}$ such that each $w_{t}$ and its reciprocal is uniformly bounded, with $A_{p}(w_{t}) \leq A_{p}(w)$ and $A_{p}(w) = \lim_{t \to 0+} A_{p}(w_{t})$. By working with $w_{t}$ in place of $w$, we can improve some arguments by guaranteeing that all the integrals involved are finite.

2. Regularization of $A_{p}$ weights

Given a locally integrable function $w$ on $\mathbb{R}^{n}$ that is positive almost everywhere, define

$$A_{p}(w) = \sup_{Q} \frac{1}{|Q|} \left( \int_{Q} w \right)^{1/p} \left( \int_{Q} w^{1-p'} \right)^{1/p'}$$

so that $w \in A_{p}$ if and only if $A_{p}(w) < \infty$. While $\int_{Q} w^{1-p'}$ need not be finite for all cubes, $\int_{Q} w$ is always positive so this supremum is well-defined, either as positive and real or as $\infty$. For $t > 0$ we define

$$w_{t}(x) = \frac{1}{t + 1/(t + w(x))} = \frac{t + w(x)}{t^{2} + tw(x) + 1},$$

the first form gives us $w_{t}(x) = 1/t$ when $w(x) = \infty$. Then $\lim_{t \to 0+} w_{t}(x) = w(x)$ pointwise, and $t/(t^{2} + 1) \leq w_{t}(x) \leq 1/t$. Note that both $w_{t}$ and $1/w_{t}$ are in Muckenhoupt’s class $A_{1}$, since that class consists of all weights for which the average over each cube is dominated by a fixed multiple of the essential infimum over the same cube. For $1 < p < \infty$ we easily calculate

$$A_{p}(w_{t}) \leq \left( \frac{1}{t} \right)^{1/p} \left[ \left( \frac{t}{t^{2} + 1} \right)^{-p'/p} \right]^{1/p'} = \left( \frac{t^{2} + 1}{t^{2}} \right)^{1/p}$$

and find exactly the same bound for $A_{p}(1/w_{t})$. Of course, these are only crude bounds; the theorem below shows that $A_{p}(w_{t})$ mimics the behavior of $A_{p}(w)$ for small $t$.

**Theorem 1.** For each a.e. positive locally integrable function $w$ on $\mathbb{R}^{n}$ and for $1 < p < \infty$, $A_{p}(w_{t}) \leq A_{p}(w)$ on $(0, \infty)$, with $A_{p}(w) = \lim_{t \to 0+} A_{p}(w_{t})$.

**Proof.** For $s, t \geq 0$ call $u(x) = s + w(x)$ and $v(x) = tu(x) + 1$, and then define

$$F(s, t) = \left( \int_{Q} \frac{u}{v} \right)^{p'-1} \int_{Q} \left( \frac{u}{v} \right)^{1-p'}$$

with $Q$ a fixed but arbitrary cube in $\mathbb{R}^{n}$. We begin by showing that the partial derivatives $\frac{\partial F}{\partial s}(s, 0)$ and $\frac{\partial F}{\partial t}(s, t)$ are never positive. The bound for $\frac{\partial F}{\partial s}$ was done earlier by the authors in [1]; we reprove it here with a slightly different argument. Since

$$F(s, 0) = \left( \int_{Q} (s + w) \right)^{p'-1} \int_{Q} (s + w)^{1-p'},$$
for $s > 0$ we can differentiate with respect to $s$ to find
\[
\frac{\partial F}{\partial s}(s, 0) = (p' - 1) \left( \int_Q (s + w) \right)^{p' - 2} |Q| \int_Q (s + w)^{1 - p'} + (1 - p') \left( \int_Q (s + w) \right)^{p' - 1} \int_Q (s + w)^{1 - p'}
\]
\[
= (p' - 1) \left( \int_Q u \right)^{p' - 2} \left[ |Q| \int_Q u^{1 - p'} - \left( \int_Q u \right) \int_Q u^{-p'} \right].
\]

We use a simple corollary of Hölder’s inequality to show that this is never positive: for $u > 0$ and $0 \leq \theta \leq 1$ we have
\[
\left( \int u^a + b(1 - \theta) \right) \left( \int u^a + b(1 - \theta) \right) \leq \left( \int u^a \right) \left( \int u^b \right).
\]

Note $|Q| = \int_Q u^0$, and calling $\theta = 1/(p' + 1)$ lets us write
\[
0 = 1 - \theta - p'\theta \quad \text{and} \quad 1 - p' = \theta - p'(1 - \theta).
\]

Thus
\[
|Q| \int_Q u^{1 - p'} \leq \left( \int_Q u \right) \int_Q u^{-p'} < \infty,
\]
proving that $\frac{\partial F}{\partial s}(s, 0) \leq 0$ for each $s > 0$.

Obviously $\lim_{s \to 0} \int_Q (s + w) = \int_Q w$, and the monotone convergence theorem shows $\lim_{s \to 0} \int_Q (s + w)^{1 - p'} = \int_Q w^{1 - p'}$ whether this last quantity is finite or infinite, so we have established that $F(s, 0) \leq F(0, 0)$.

Next we fix $s > 0$ and differentiate with respect to $t$. Since
\[
\frac{\partial}{\partial t} \left( \frac{u}{v} \right) = \frac{\partial}{\partial t} \left( \frac{u}{tu + 1} \right) = -\frac{u^2}{v^2},
\]
we find
\[
\frac{\partial F}{\partial t}(s, t) = (p' - 1) \left( \int_Q \frac{u}{v} \right)^{p' - 2} \left[ \left( \int_Q \frac{u}{v} \right) \int_Q \left( \frac{u}{v} \right)^{2 - p'} - \left( \int_Q \frac{u^2}{v^2} \right) \int_Q \left( \frac{u}{v} \right)^{1 - p'} \right].
\]

Again calling $\theta = 1/(p' + 1)$, we may write
\[
1 = 2(1 - \theta) + \theta (1 - p') \quad \text{and} \quad 2 - p' = 2\theta + (1 - \theta)(1 - p')
\]
and then show that $\frac{\partial F}{\partial t}(s, t) \leq 0$ as above; note all powers of $u/v$ are locally integrable.

As for $F(s, t)$ itself, we have
\[
\lim_{t \to 0^+} \int_Q u/v = \int_Q u \quad \text{and} \quad \lim_{t \to 0^+} \int_Q (u/v)^{1 - p'} = \int_Q u^{1 - p'}
\]
by monotone convergence in the first case and by uniform convergence in the second, so that $\lim_{t \to 0^+} F(s, t) = F(s, 0) \leq F(0, 0)$. Hence we have $F(s, t) \leq F(0, 0)$ for all $s, t > 0$. In particular, this holds true for $s = t$, and after taking $p'$ roots we obtain
\[
\left( \int_Q w^2 \right)^{1/p} \left( \int_Q \left( w^{1 - p'} \right)^{1/p'} \right) \leq \left( \int_Q w \right)^{1/p} \left( \int_Q \left( w^{1 - p'} \right)^{1/p'} \right).
\]
for all $t > 0$, all nonnegative locally integrable $w$, and all cubes $Q$ in $\mathbb{R}^n$. Consequently, $A_p(w_t) \leq A_p(w)$ for each nonnegative locally integrable $w$ on $\mathbb{R}^n$, and this is valid as long as $t > 0$ and $1 < p < \infty$.

We conclude the proof by appealing to Fatou’s lemma. Since $\int_Q w$ and $\int_Q w^{1-p'}$ are always positive or infinite, for each cube $Q$ we have

$$
\frac{1}{|Q|} \left( \int_Q w \right)^{1/p} \left( \int_Q w^{1-p'} \right)^{1/p'} \leq \frac{1}{|Q|} \left( \liminf_{t \to 0+} \int_Q w_t \right)^{1/p} \left( \liminf_{t \to 0+} \int_Q w_t^{1-p'} \right)^{1/p'}
$$

$$
\leq \liminf_{t \to 0+} \frac{1}{|Q|} \left( \int_Q w_t \right)^{1/p} \left( \int_Q w_t^{1-p'} \right)^{1/p'}
$$

$$
\leq \liminf_{t \to 0+} A_p(w_t) \leq \limsup_{t \to 0+} A_p(w_t) \leq A_p(w).
$$

Taking the supremum over all cubes completes the proof. 

3. Concluding remarks

Given any weight $w$ that satisfies $A_{p_1}(w) > C_1$ and $A_{p_2}(w) \leq C_2$, for sufficiently small $t$ the weight $w_t$ will satisfy exactly the same inequalities, and $w_t$ is in every $A_p$ class. Consequently, no such system of inequalities can be used to show that the given weight $w$ fails to belong to some weight class.

Our theorem fills a gap in some of the proofs that $\{p > 1 : w \in A_p\}$ is open. For example, the $A_p$ condition and a Calderón-Zygmund covering argument can be used to show

$$
\int_Q w^{1-q} \leq C(p, q, A_p(w)) \int_Q w^{1-q} + |Q|^q (\int_Q w^{1-p'})^{(1-q)/p}
$$

for $1 < q < p$, with $C(p, q, A_p(w)) < 1$ when $q$ is close enough to $p$. Then subtraction gives a bound for $\int_Q w^{1-q}$ that leads to a bound for $A_q(w)$ in terms of $A_p(w)$, but to justify the subtraction $\int_Q w^{1-q}$ needs to be finite. Such arguments are always valid when $w$ is replaced by $w_t$, and then the bound for $A_q(w)$ in terms of $A_p(\omega)$ can be obtained by letting $t \to 0$.

References


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