ON THE BIEBERBACH CONJECTURE
AND HOLOMORPHIC DYNAMICS

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Abstract. In this note we prove that when \( P \) is a polynomial of degree \( d \) with connected Julia set and when \( z_0 \) belongs to the filled-in Julia set \( K(P) \), then \( |P'(z_0)| \leq d^2 \). We also show that equality is achieved if and only if \( K(P) \) is a segment of which one extremity is \( z_0 \). In that case, \( P \) is conjugate to a Tchebycheff polynomial or its opposite. The main tool in our proof is the Bieberbach conjecture proved by de Branges in 1984.

1. Introduction

Let us first recall two well-known dynamical results which are in the same vein as ours.

**Theorem 1.** Let \( P \) be a monic centered polynomial with connected Julia set. Then, for any \( z_0 \in K(P) \), we have \( |z_0| \leq 2 \) with equality if and only if \( K(P) \) is a segment of which one extremity is \( z_0 \).

**Proof.** Assume \( K \) is a compact connected subset of \( \mathbb{C} \) and \( \mathbb{C} \setminus K \) is conformally isomorphic to \( \mathbb{C} \setminus \mathbb{D} \). Let \( \phi : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus K \) be a conformal isomorphism with Laurent series expansion

\[
\phi(z) = b_1 z + b_0 + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \ldots .
\]

Then, the Gronwall Area Formula asserts that the area of \( K \) is equal to \( \pi \sum_{n \leq 1} n |b_n|^2 \). It follows that \( |b_1| \leq |b_{-1}| \), with equality if and only if \( K \) is a straight line segment. Moreover, when \( b_0 = 0 \) and \( z_0 \in K \), by considering the map \( \psi(w) = \sqrt{\phi(w^2)} - z_0 \), we get \( |z_0| \leq 2|b_1| \) with equality if and only if \( K \) is a straight line segment of which one extremity is \( z_0 \).

Then, observe that when \( P \) is a monic centered polynomial, the Böttcher coordinate \( \phi : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus K(P) \) has Laurent series expansion of the form

\[
\phi(z) = z + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \ldots .
\]

Indeed, \( b_1 = 1 \) because \( P \) is monic and \( b_0 = 0 \) because \( P \) is centered. Theorem 1 follows immediately. \( \square \)

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Theorem 2. Let $P$ be a polynomial of degree $d$ with connected Julia set. If $\alpha$ is a fixed point of $P$, then $|P'(\alpha)| \leq d^2$.

This is a weak version of an inequality due to Pommerenke [Po], Levine [L] and Yoccoz [Y] (see [H] or [Pe]). The idea of the proof goes back to Bers’s Inequality in the context of quasi-fuchsian groups. There, Bers proves that the length of a hyperbolic geodesic in $Q(X,Y)$ is bounded by the hyperbolic length of the corresponding geodesic on $X$ or $Y$ (see [B] Theorem 3 and [McM] Prop. 6.4). In [O] Sect. 5.1, Otal gives a proof of Bers’s Inequality based on Koebe’s One-Quarter Theorem. His proof is inspired by Ahlfors (see [A] Lemma 1).

In the present article, we present a generalization of those two theorems. We will use the Bieberbach conjecture proved by de Branges in 1984.

De Branges’s Theorem. Let $\phi : \mathbb{D} \to \mathbb{C}$ be a univalent mapping. If $\phi(z) = \sum_{n \geq 1} a_n z^n$, then for any $n \geq 1$, we have $|a_n/a_1| \leq n$. Besides, if $|a_k/a_1| = k$ for some integer $k > 1$, then $\phi$ is a rotation of the Koebe function, i.e., there exists a real $\theta$ such that

$$\phi(z) = \frac{z}{(1 - e^{i\theta}z)^2}.$$  

We obtain a result which does not only control the derivative of $P$ at its fixed points, but controls the derivative of $P$ at all the points in the Julia set. Our main observation is the following.

Lemma 1. Let $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a germ such that $0$ is a superattracting fixed point with local degree $k \geq 2$. Let $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a Böttcher coordinate, i.e., a germ which is univalent in a neighborhood of $0$ and which satisfies $\phi(z^k) = f(\phi(z))$ for $z$ sufficiently close to 0. If $\phi(z) = \sum_{n \geq 1} a_n z^n$, then

$$\text{res} \left( \frac{1}{f}, 0 \right) = k \frac{a_k}{a_1}.$$  

Remark. The result still holds if instead of germs one considers formal power series, but we are not aware of a formal proof.

We say that a polynomial $P$ is a Tchebycheff polynomial if $P(\cos z) = \cos(d\zeta)$, where $d$ is the degree of $P$. As a corollary of Lemma 1, we will show the following two theorems.

Theorem 3. Assume $P$ is a polynomial of degree $d$ with connected Julia set. Then, for any $z_0 \in K(P)$, we have $|P'(z_0)| \leq d^2$ with equality if and only if $K(P)$ is a segment, one extremity of which is $z_0$. In that case, $P$ is conjugate to a Tchebycheff polynomial or to its opposite.

Theorem 4. Assume $P$ is a polynomial of degree $d$ with disconnected Julia set. Let $g_P : \mathbb{C} \to \mathbb{R}^+$ be the Green’s function of $K(P)$ and set

$$G(P) = \max_{\{\omega \mid P(\omega) = 0\}} g_P(\omega).$$  

Then, for any $z_0 \in \mathbb{C}$ with $g_P(z_0) \leq G(P)$, we have $|P'(z_0)| < d^2 e^{(d-1)G(P)}$.

Remark. This inequality always holds for points in $K(P)$.
2. Proofs of the results

Proof of Lemma [4] Let \( \gamma_1 \) be a small circle around 0 and let \( \gamma_2 \) be its image by \( \phi \). Then,

\[
\text{res} \left( \frac{1}{f}, 0 \right) = \int_{\gamma_2} \frac{dw}{f(w)} \equiv \int_{\gamma_1} \frac{\phi'(z)}{\phi(z)^k} dz = \int_{\gamma_1} \frac{\phi'(z)}{\phi(z)^k} dz = \text{res} \left( \frac{\phi'(z)}{\phi(z)^k}, 0 \right).
\]

Since \( \phi(z) = \sum_{n \geq 1} a_n z^n \), we have

\[
\frac{\phi'(z)}{\phi(z)^k} = \frac{a_1 + 2a_2 z + \ldots + ka_k z^{k-1} + O(|z|^k)}{a_1 z^k (1 + O(|z|^k))} = \frac{1}{z^k} + \frac{2a_2}{a_1} \frac{1}{z^{k-1}} + \ldots + \frac{ka_k}{a_1} \frac{1}{z} + O(1).
\]

Therefore

\[
\text{res} \left( \frac{1}{f}, 0 \right) = \text{res} \left( \frac{\phi'(z)}{\phi(z)^k}, 0 \right) = \frac{ka_k}{a_1}.
\]

\( \blacksquare \)

Proof of Theorem [5] First, observe that when \( P \) is conjugate to a Tchebycheff polynomial of degree \( d \) (or its opposite), \( K(P) \) is a segment and the derivative at an extremity is \( \pm d^2 \). The proof is not difficult and left to the reader.

Next, assume \( P \) is a polynomial of degree \( d \) with connected Julia set and \( z_0 \) belongs to the filled-in Julia set \( K(P) \). Let \( \Omega \) be the simply connected sub-domain of \( \mathbb{P}^1 \) defined by

\[
\Omega = \left\{ w \in \mathbb{P}^1 \mid z_0 + \frac{1}{w} \in \mathbb{P}^1 \setminus K(P) \right\}.
\]

Since \( z_0 \in K(P) \), we see that \( \Omega \subset \mathbb{C} \), and since \( P \) has a superattracting fixed point with local degree \( d \) at infinity, the rational map \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) defined by

\[
f(w) = \frac{1}{P(z_0 + 1/w) - z_0}
\]

has a superattracting fixed point at 0 with local degree \( d \). Any Böttcher coordinate of \( f \) extends to a univalent mapping \( \phi : \mathbb{D} \to \Omega \) and Lemma [4] asserts that writing \( \phi(z) = \sum_{n \geq 1} a_n z^n \), we get

\[
d \frac{a_d}{a_1} = \text{res} \left( \frac{1}{f}, 0 \right).
\]

Since \( P(z) = b_0 + b_1(z - z_0) + \ldots + b_d(z - z_0)^d \), we see that

\[
\frac{1}{f(w)} = P(z_0 + 1/w) - z_0 = b_0 - z_0 + \frac{b_1}{w} + \ldots + \frac{b_d}{w^d}.
\]

Therefore, \( \text{res} \left( 1/f, 0 \right) = b_1 = P'(z_0) \). It now follows from de Branges’s Theorem that

\[
|P'(z_0)| = \left| d \frac{a_d}{a_1} \right| \leq d^2,
\]

with equality if and only if \( \phi \) is a rotation of the Koebe function. In that case, \( \Omega \) is a slit plane, and thus \( K(P) \) is a segment of which one extremity is \( z_0 \).

We must now show that \( P \) is conjugate to a Tchebycheff polynomial or to its opposite. Knowing that \( K(P) \) is a segment, this is classical. Conjugating \( P \) with an affine map, we may assume that \( K(P) = [-1, 1] \). We define \( \psi : \mathbb{P}^1 \setminus \mathbb{D} \to \mathbb{P}^1 \setminus [-1, 1] \) to be the conformal representation

\[
\psi(z) = \frac{1}{2} \left( z + \frac{1}{z} \right).
\]
The conformal representation \( \psi^{-1} : \mathbb{P}^1 \setminus [-1, 1] \rightarrow \mathbb{P}^1 \setminus \overline{\mathbb{D}} \) conjugates the proper mapping \( P : \mathbb{P}^1 \setminus [-1, 1] \rightarrow \mathbb{P}^1 \setminus [-1, 1] \) to a proper mapping from \( \mathbb{P}^1 \setminus \overline{\mathbb{D}} \) to itself, having a superattracting fixed point of degree \( d \) at infinity. This mapping is necessarily of the form \( z \mapsto \lambda z^d \), with \( |\lambda| = 1 \).

Since \( K(P) \) is totally invariant, the polynomial \( P \) necessarily maps the set \( \{-1, 1\} \) into itself. Besides, \( \psi^{-1}(z) \) tends to \( \pm 1 \) as \( z \) tends to \( \pm 1 \). Therefore, the map \( z \mapsto \lambda z^d \) maps the set \( \{-1, 1\} \) into itself. This shows that \( \lambda = \pm 1 \). Hence, \( P \) is conjugate to a Tchebycheff polynomial.

As \( z \rightarrow e^{i\theta} \in S^1 \), we get
\[
P(\cos \theta) = P \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right) = \pm \frac{e^{i\theta} + e^{-i\theta}}{2} = \pm \cos(d\theta).
\]

\( \square \)

Proof of Theorem 4. We will mimic the previous proof. We assume that \( g_P(z_0) \leq G(P) \) and we set
\[
\Omega = \left\{ w \in \mathbb{P}^1 \mid g_P \left( z_0 + \frac{1}{w} \right) > G(P) \right\}.
\]
We define \( f : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) by \( f(w) = 1/(P(z_0 + 1/w) - z_0) \). Then the Böttcher coordinate of \( f \) at 0 extends to a univalent mapping \( \phi \) between the disk centered at 0 with radius \( e^{-G(P)} \) and the domain \( \Omega \subset \mathbb{C} \). Since the mapping
\[
z \mapsto \phi(e^{-G(P)} z) = \sum_{n \geq 1} a_n e^{-nG(P)} z^n
\]
is univalent in the unit disk, de Branges’s Theorem only allows us to conclude that
\[
|P'(z_0)| = \left| \frac{a_d}{a_1} \right| = de^{(d-1)G(P)} \left| \frac{a_d e^{-dG(P)}}{a_1 e^{-G(P)}} \right| < d^2 e^{(d-1)G(P)}.
\]
The inequality is strict because the complement of \( \Omega \) has non-empty interior, and therefore, \( \Omega \) cannot be a slit plane.

\( \square \)

3. Application

A possible application of Theorem 3 is the following.

**Corollary 1.** Let \( d \geq 3 \) be an integer, and \( a = (a_2, \ldots, a_d) \) be a point in \( \mathbb{C}^{d-2} \). Then, the Julia set of the polynomial \( P_a(z) = d^2 z + a_2 z^2 + \ldots + a_d z^d + 1 \) is connected if and only if \( P_a \) is conjugate to a Tchebycheff polynomial.

Proof. On the one hand, if \( P_a \) is conjugate to a Tchebycheff polynomial, its Julia set is a segment and therefore it is connected. On the other hand, observe that 0 is a fixed point with multiplier \( d^2 \). Therefore, Theorem 3 shows that if \( J(P_a) \) is connected, then \( P_a \) is conjugate to a Tchebycheff polynomial or its opposite and 0 is an extremity of \( K(P_a) \). Since 0 is fixed, \( P_a \) may always be conjugate to a Tchebycheff polynomial.

Every polynomial of degree \( d \) having a fixed point with multiplier \( d^2 \) is conjugate to a polynomial \( P_a \). The family \( (P_a)_{a \in \mathbb{C}^{d-2}} \) is a co-dimension 1 algebraic sub-variety of the space of polynomials up to conjugacy. The set of polynomials \( P_a \) which are conjugate to a Tchebycheff polynomial is finite but not empty. Therefore, for
each degree \( d \geq 3 \), we produce an example of co-dimension 1 algebraic family of polynomials for which the connectivity locus is non-empty and discrete.

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References


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