

## ON THE BIEBERBACH CONJECTURE AND HOLOMORPHIC DYNAMICS

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ABSTRACT. In this note we prove that when  $P$  is a polynomial of degree  $d$  with connected Julia set and when  $z_0$  belongs to the filled-in Julia set  $K(P)$ , then  $|P'(z_0)| \leq d^2$ . We also show that equality is achieved if and only if  $K(P)$  is a segment of which one extremity is  $z_0$ . In that case,  $P$  is conjugate to a Tchebycheff polynomial or its opposite. The main tool in our proof is the Bieberbach conjecture proved by de Branges in 1984.

### 1. INTRODUCTION

Let us first recall two well-known dynamical results which are in the same vein as ours.

**Theorem 1.** *Let  $P$  be a monic centered polynomial with connected Julia set. Then, for any  $z_0 \in K(P)$ , we have  $|z_0| \leq 2$  with equality if and only if  $K(P)$  is a segment of which one extremity is  $z_0$ .*

*Proof.* Assume  $K$  is a compact connected subset of  $\mathbb{C}$  and  $\mathbb{C} \setminus K$  is conformally isomorphic to  $\mathbb{C} \setminus \overline{\mathbb{D}}$ . Let  $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus K$  be a conformal isomorphism with Laurent series expansion

$$\phi(z) = b_1 z + b_0 + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \dots$$

Then, the Gronwall Area Formula asserts that the area of  $K$  is equal to  $\pi \sum_{n \leq -1} n |b_n|^2$ . It follows that  $|b_1| \leq |b_{-1}|$ , with equality if and only if  $K$  is a straight line segment. Moreover, when  $b_0 = 0$  and  $z_0 \in K$ , by considering the map  $\psi(w) = \sqrt{\phi(w^2) - z_0}$ , we get  $|z_0| \leq 2|b_1|$  with equality if and only if  $K$  is a straight line segment of which one extremity is  $z_0$ .

Then, observe that when  $P$  is a monic centered polynomial, the Böttcher coordinate  $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus K(P)$  has Laurent series expansion of the form

$$\phi(z) = z + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \dots$$

Indeed,  $b_1 = 1$  because  $P$  is monic and  $b_0 = 0$  because  $P$  is centered. Theorem 1 follows immediately.  $\square$

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**Theorem 2.** *Let  $P$  be a polynomial of degree  $d$  with connected Julia set. If  $\alpha$  is a fixed point of  $P$ , then  $|P'(\alpha)| \leq d^2$ .*

This is a weak version of an inequality due to Pommerenke [Po], Levine [L] and Yoccoz [Y] (see [H] or [Pe]). The idea of the proof goes back to Bers's Inequality in the context of quasi-fuchsian groups. There, Bers proves that the length of a hyperbolic geodesic in  $Q(X, Y)$  is bounded by the hyperbolic length of the corresponding geodesic on  $X$  or  $Y$  (see [B] Theorem 3 and [McM] Prop. 6.4). In [O] Sect. 5.1, Otal gives a proof of Bers's Inequality based on Koebe's One-Quarter Theorem. His proof is inspired by Ahlfors (see [A] Lemma 1).

In the present article, we present a generalization of those two theorems. We will use the Bieberbach conjecture proved by de Branges in 1984.

**De Branges's Theorem.** *Let  $\phi : \mathbb{D} \rightarrow \mathbb{C}$  be a univalent mapping. If  $\phi(z) = \sum_{n \geq 1} a_n z^n$ , then for any  $n \geq 1$ , we have  $|a_n/a_1| \leq n$ . Besides, if  $|a_k/a_1| = k$  for some integer  $k > 1$ , then  $\phi$  is a rotation of the Koebe function, i.e., there exists a real  $\theta$  such that*

$$\phi(z) = \frac{z}{(1 - e^{i\theta} z)^2}.$$

We obtain a result which does not only control the derivative of  $P$  at its fixed points, but controls the derivative of  $P$  at all the points in the Julia set. Our main observation is the following.

**Lemma 1.** *Let  $f : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  be a germ such that 0 is a superattracting fixed point with local degree  $k \geq 2$ . Let  $\phi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  be a Böttcher coordinate, i.e., a germ which is univalent in a neighborhood of 0 and which satisfies  $\phi(z^k) = f(\phi(z))$  for  $z$  sufficiently close to 0. If  $\phi(z) = \sum_{n \geq 1} a_n z^n$ , then*

$$\operatorname{res} \left( \frac{1}{f}, 0 \right) = k \frac{a_k}{a_1}.$$

*Remark.* The result still holds if instead of germs one considers formal power series, but we are not aware of a formal proof.

We say that a polynomial  $P$  is a Tchebycheff polynomial if  $P(\cos z) = \cos(dz)$ , where  $d$  is the degree of  $P$ . As a corollary of Lemma 1, we will show the following two theorems.

**Theorem 3.** *Assume  $P$  is a polynomial of degree  $d$  with connected Julia set. Then, for any  $z_0 \in K(P)$ , we have  $|P'(z_0)| \leq d^2$  with equality if and only if  $K(P)$  is a segment, one extremity of which is  $z_0$ . In that case,  $P$  is conjugate to a Tchebycheff polynomial or to its opposite.*

**Theorem 4.** *Assume  $P$  is a polynomial of degree  $d$  with disconnected Julia set. Let  $g_P : \mathbb{C} \rightarrow \mathbb{R}^+$  be the Green's function of  $K(P)$  and set*

$$G(P) = \max_{\{\omega | P'(\omega)=0\}} g_P(\omega).$$

*Then, for any  $z_0 \in \mathbb{C}$  with  $g_P(z_0) \leq G(P)$ , we have  $|P'(z_0)| < d^2 e^{(d-1)G(P)}$ .*

*Remark.* This inequality always holds for points in  $K(P)$ .

2. PROOFS OF THE RESULTS

*Proof of Lemma 1.* Let  $\gamma_1$  be a small circle around 0 and let  $\gamma_2$  be its image by  $\phi$ . Then,

$$\operatorname{res}\left(\frac{1}{f}, 0\right) = \int_{\gamma_2} \frac{dw}{f(w)} \underset{w=\phi(z)}{=} \int_{\gamma_1} \frac{\phi'(z)}{f(\phi(z))} dz = \int_{\gamma_1} \frac{\phi'(z)}{\phi(z^k)} dz = \operatorname{res}\left(\frac{\phi'(z)}{\phi(z^k)}, 0\right).$$

Since  $\phi(z) = \sum_{n \geq 1} a_n z^n$ , we have

$$\frac{\phi'(z)}{\phi(z^k)} = \frac{a_1 + 2a_2z + \dots + ka_kz^{k-1} + \mathcal{O}(|z|^k)}{a_1z^k(1 + \mathcal{O}(|z|^k))} = \frac{1}{z^k} + \frac{2a_2}{a_1} \frac{1}{z^{k-1}} + \dots + \frac{ka_k}{a_1} \frac{1}{z} + \mathcal{O}(1).$$

Therefore

$$\operatorname{res}\left(\frac{1}{f}, 0\right) = \operatorname{res}\left(\frac{\phi'(z)}{\phi(z^k)}, 0\right) = \frac{ka_k}{a_1}.$$

□

*Proof of Theorem 3.* First, observe that when  $P$  is conjugate to a Tchebycheff polynomial of degree  $d$  (or its opposite),  $K(P)$  is a segment and the derivative at an extremity is  $\pm d^2$ . The proof is not difficult and left to the reader.

Next, assume  $P$  is a polynomial of degree  $d$  with connected Julia set and  $z_0$  belongs to the filled-in Julia set  $K(P)$ . Let  $\Omega$  be the simply connected sub-domain of  $\mathbb{P}^1$  defined by

$$\Omega = \left\{ w \in \mathbb{P}^1 \mid z_0 + \frac{1}{w} \in \mathbb{P}^1 \setminus K(P) \right\}.$$

Since  $z_0 \in K(P)$ , we see that  $\Omega \subset \mathbb{C}$ , and since  $P$  has a superattracting fixed point with local degree  $d$  at infinity, the rational map  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  defined by

$$f(w) = \frac{1}{P(z_0 + 1/w) - z_0}$$

has a superattracting fixed point at 0 with local degree  $d$ . Any Böttcher coordinate of  $f$  extends to a univalent mapping  $\phi : \mathbb{D} \rightarrow \Omega$  and Lemma 1 asserts that writing  $\phi(z) = \sum_{n \geq 1} a_n z^n$ , we get

$$d \frac{a_d}{a_1} = \operatorname{res}\left(\frac{1}{f}, 0\right).$$

Since  $P(z) = b_0 + b_1(z - z_0) + \dots + b_d(z - z_0)^d$ , we see that

$$\frac{1}{f(w)} = P(z_0 + 1/w) - z_0 = b_0 - z_0 + \frac{b_1}{w} + \dots + \frac{b_d}{w^d}.$$

Therefore,  $\operatorname{res}(1/f, 0) = b_1 = P'(z_0)$ . It now follows from de Branges's Theorem that

$$|P'(z_0)| = \left| d \frac{a_d}{a_1} \right| \leq d^2,$$

with equality if and only if  $\phi$  is a rotation of the Koebe function. In that case,  $\Omega$  is a slit plane, and thus  $K(P)$  is a segment of which one extremity is  $z_0$ .

We must now show that  $P$  is conjugate to a Tchebycheff polynomial or to its opposite. Knowing that  $K(P)$  is a segment, this is classical. Conjugating  $P$  with an affine map, we may assume that  $K(P) = [-1, 1]$ . We define  $\psi : \mathbb{P}^1 \setminus \overline{\mathbb{D}} \rightarrow \mathbb{P}^1 \setminus [-1, 1]$  to be the conformal representation

$$\psi(z) = \frac{1}{2} \left( z + \frac{1}{z} \right).$$

The conformal representation  $\psi^{-1} : \mathbb{P}^1 \setminus [-1, 1] \rightarrow \mathbb{P}^1 \setminus \overline{\mathbb{D}}$  conjugates the proper mapping  $P : \mathbb{P}^1 \setminus [-1, 1] \rightarrow \mathbb{P}^1 \setminus [-1, 1]$  to a proper mapping from  $\mathbb{P}^1 \setminus \overline{\mathbb{D}}$  to itself, having a superattracting fixed point of degree  $d$  at infinity. This mapping is necessarily of the form  $z \mapsto \lambda z^d$ , with  $|\lambda| = 1$ .

Since  $K(P)$  is totally invariant, the polynomial  $P$  necessarily maps the set  $\{-1, 1\}$  into itself. Besides,  $\psi^{-1}(z)$  tends to  $\pm 1$  as  $z$  tends to  $\pm 1$ . Therefore, the map  $z \mapsto \lambda z^d$  maps the set  $\{-1, 1\}$  into itself. This shows that  $\lambda = \pm 1$ . Hence,

$$(\forall z \in \mathbb{P}^1 \setminus \overline{\mathbb{D}}) \quad P(\psi(z)) = \psi(\pm z^d).$$

As  $z \rightarrow e^{i\theta} \in S^1$ , we get

$$P(\cos \theta) = P\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) = \pm \frac{e^{id\theta} + e^{-id\theta}}{2} = \pm \cos(d\theta).$$

□

*Proof of Theorem 4.* We will mimick the previous proof. We assume that  $g_P(z_0) \leq G(P)$  and we set

$$\Omega = \left\{ w \in \mathbb{P}^1 \mid g_P\left(z_0 + \frac{1}{w}\right) > G(P) \right\}.$$

We define  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  by  $f(w) = 1/(P(z_0 + 1/w) - z_0)$ . Then the Böttcher coordinate of  $f$  at 0 extends to a univalent mapping  $\phi$  between the disk centered at 0 with radius  $e^{-G(P)}$  and the domain  $\Omega \subset \mathbb{C}$ . Since the mapping

$$z \mapsto \phi(e^{-G(P)}z) = \sum_{n \geq 1} a_n e^{-nG(P)} z^n$$

is univalent in the unit disk, de Branges's Theorem only allows us to conclude that

$$|P'(z_0)| = d \left| \frac{a_d}{a_1} \right| = d e^{(d-1)G(P)} \left| \frac{a_d e^{-dG(P)}}{a_1 e^{-G(P)}} \right| < d^2 e^{(d-1)G(P)}.$$

The inequality is strict because the complement of  $\Omega$  has non-empty interior, and therefore,  $\Omega$  cannot be a slit plane. □

### 3. APPLICATION

A possible application of Theorem 3 is the following.

**Corollary 1.** *Let  $d \geq 3$  be an integer, and  $a = (a_2, \dots, a_{d-1})$  be a point in  $\mathbb{C}^{d-2}$ . Then, the Julia set of the polynomial  $P_a(z) = d^2z + a_2z^2 + \dots + a_{d-1}z^{d-1} + z^d$  is connected if and only if  $P_a$  is conjugate to a Tchebycheff polynomial.*

*Proof.* On the one hand, if  $P_a$  is conjugate to a Tchebycheff polynomial, its Julia set is a segment and therefore it is connected. On the other hand, observe that 0 is a fixed point with multiplier  $d^2$ . Therefore, Theorem 3 shows that if  $J(P_a)$  is connected, then  $P_a$  is conjugate to a Tchebycheff polynomial or its opposite and 0 is an extremity of  $K(P_a)$ . Since 0 is fixed,  $P_a$  may always be conjugate to a Tchebycheff polynomial. □

Every polynomial of degree  $d$  having a fixed point with multiplier  $d^2$  is conjugate to a polynomial  $P_a$ . The family  $(P_a)_{a \in \mathbb{C}^{d-2}}$  is a co-dimension 1 algebraic sub-variety of the space of polynomials up to conjugacy. The set of polynomials  $P_a$  which are conjugate to a Tchebycheff polynomial is finite but not empty. Therefore, for

each degree  $d \geq 3$ , we produce an example of co-dimension 1 algebraic family of polynomials for which the connectivity locus is non-empty and discrete.

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