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EQUIVALENT QUASI-NORMS ON LORENTZ SPACES

DAVID E. EDMUNDS AND BOHUMÍR OPIC

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Abstract. We give new characterizations of Lorentz spaces by means of certain quasi-norms which are shown to be equivalent to the classical ones.

1. Introduction and results

Lorentz spaces play an important role in many branches of mathematical analysis. The aim of this paper is to derive new formulae which provide equivalent quasi-norms on such spaces. These results are motivated by mapping properties of fractional maximal operators, Riesz potentials and the Hilbert transform. Our proofs are based on weighted norm inequalities for integral operators. To explain our results in more detail, we first need some notation.

Given two quasi-Banach spaces X and Y, we write $X = Y$ if X and Y are equal in the algebraic and the topological sense (their quasi-norms are equivalent). The symbol $X \hookrightarrow Y$ means that $X \subset Y$ and the natural embedding of X in Y is continuous.

We write $A \lesssim B$ (or $A \gtrsim B$) if $A \le cB$ (or $cA \ge B$) for some positive constant c independent of appropriate quantities involved in the expressions A and B, and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. Throughout the paper we use the abbreviation LHS($*$) (RHS($*$)) for the left (right) hand side of the relation ($*$). Moreover, we adopt the convention that $1/\infty = 0$.

Let $\Omega \subset \mathbb{R}^n$ be a measurable subset (with respect to n-dimensional Lebesgue measure), $|\Omega|$ its measure and χ_Ω its characteristic function. The symbol $\mathcal{M}(\Omega)$ is used to denote the family of all scalar-valued (real or complex) measurable functions on the set Ω. By $\mathcal{M}^+(\Omega)$ we mean the subset of $\mathcal{M}(\Omega)$ consisting of all non-negative functions on Ω. If $\Omega = (a, b) \subseteq \mathbb{R}$, we simply write $\mathcal{M}(a, b)$ and $\mathcal{M}^+(a, b)$ instead of $\mathcal{M}((a, b))$ and $\mathcal{M}^+((a, b))$.

Given $p, r \in (0, \infty]$, the Lorentz space $L^{p,r}(\Omega)$ is defined by (cf. [Lo1], [Lo2], [BS])

$$L^{p,r}(\Omega) = \{f \in \mathcal{M}(\Omega);\ \|f\|_{p,r} = \|f\|_{p,r;\Omega} < \infty\}, \tag{1.1}$$

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where

$$\|f\|_{p,r} := \|t^{\frac{1}{p}-\frac{1}{r}} f^*(t)\|_{r,(0,|\Omega|)}. \tag{1.2}$$

Here f^* stands for the non-increasing rearrangement of f given by

$$f^*(t) := \inf\{\lambda > 0;\ |\{x \in \Omega;\ |f(x)| > \lambda\}| \le t\}, \quad t \in (0,\infty),$$

and $\|\cdot\|_{r,(a,b)}$, $-\infty \le a < b \le \infty$, is the usual quasi-norm in the Lebesgue space $L^r(a,b)$.

The functional (1.2) is not always a norm, even when $p, r \ge 1$. Following A. P. Calderón [C], we replace f^* by its maximal function

$$f^{**}(t) := t^{-1} \int_0^t f^*(s)\, ds, \qquad t \in (0,\infty),$$

in (1.2) and define the Lorentz space $L^{(p,r)}(\Omega)$, $p, r \in (0,\infty]$, by

$$L^{(p,r)}(\Omega) = \{f \in \mathcal{M}(\Omega);\ \|f\|_{(p,r)} = \|f\|_{(p,r);\Omega} < \infty\}, \tag{1.3}$$

where

$$\|f\|_{(p,r)} := \|t^{\frac{1}{p}-\frac{1}{r}} f^{**}(t)\|_{r,(0,|\Omega|)}. \tag{1.4}$$

One can see that the functional (1.4) is a norm if $r \ge 1$. Moreover (cf., e.g., [OP, Theorem 3.8 (i)]),

$$L^{(p,r)}(\Omega) = L^{p,r}(\Omega) \qquad \text{if} \quad 1 < p \le \infty \text{ and } 0 < r \le \infty. \tag{1.5}$$

In general, $L^{(p,r)}(\Omega) \hookrightarrow L^{p,r}(\Omega)$.

The equality (1.5) is a consequence of the fact that $f^* \le f^{**}$ and that the Hardy-Littlewood maximal operator M is bounded on the space $L^{p,r}(\Omega)$ when $1 < p \le \infty$ and $0 < r \le \infty$. If the role of the Hardy-Littlewood maximal operator M is played by a fractional maximal operator, we arrive at the following result.

Theorem 1.1. *Let $0 < r \le \infty$ and either $0 < q \le p \le \infty$ or $0 < p < q < \infty$. Then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{(p,r);\Omega} \approx \|t^{\frac{1}{q}-\frac{1}{r}} \sup_{\tau \in (t,|\Omega|)} \tau^{\frac{1}{p}-\frac{1}{q}} f^{**}(\tau)\|_{r,(0,|\Omega|)}. \tag{1.6}$$

Corollaries. (i) *Let $0 < p, r < \infty$. Then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{(p,r);\Omega} \approx \|\sup_{\tau \in (t,|\Omega|)} \tau^{\frac{1}{p}-\frac{1}{r}} f^{**}(\tau)\|_{r,(0,|\Omega|)}.$$

(ii) *Let $0 < p \le \infty$ and $0 < q < \infty$. Then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{(p,p);\Omega} \approx \|t^{\frac{1}{q}-\frac{1}{p}} \sup_{\tau \in (t,|\Omega|)} \tau^{\frac{1}{p}-\frac{1}{q}} f^{**}(\tau)\|_{p,(0,|\Omega|)}.$$

(iii) *Let $0 < r \le \infty$ and either $0 < q \le p \le \infty$ or $0 < p < q < \infty$. If, in addition, $p > 1$, then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{p,r;\Omega} \approx \|t^{\frac{1}{q}-\frac{1}{r}} \sup_{\tau \in (t,|\Omega|)} \tau^{\frac{1}{p}-\frac{1}{q}} f^{**}(\tau)\|_{r,(0,|\Omega|)}.$$

The main part of Theorem 1.1 concerns the case when $0 < p < q < \infty$ and $1/p < 1+1/q$ (cf. the proof of Theorem 1.1 in Section 2). To explain the idea which is behind (1.6), assume in addition that $p > 1$ and $\Omega = \mathbb{R}^n$. Putting $\gamma = n(1/p-1/q)$, we see that $\gamma \in (0, n)$. Let M_γ be the fractional maximal operator given by

$$(M_\gamma f)(x) = \sup_{Q \ni x} |Q|^{\frac{\gamma}{n}-1} \int_Q |f(y)|\, dy, \quad f \in \mathcal{M}(\mathbb{R}^n), \quad x \in \mathbb{R}^n,$$

where the supremum is extended over all the cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes. By [CKOP, Theorem 1.1],

$$(M_\gamma f)^*(t) \lesssim \sup_{\tau \in (t,\infty)} \tau^{\frac{\gamma}{n}} f^{**}(\tau), \quad t \in (0,\infty), \tag{1.7}$$

for all $f \in \mathcal{M}(\mathbb{R}^n)$ and this estimate is sharp (in the sense that for any $f \in \mathcal{M}^+(\mathbb{R}^n)$ which is radially non-increasing – notation $f \in \mathcal{M}^+_r(\mathbb{R}^n; \downarrow)$ – the symbol $\lesssim$ can be replaced by $\gtrsim$ in (1.7)). Now, given the space $L^{q,r}(\mathbb{R}^n) =: Y$ with $r \in (0,\infty]$, put $\bar{Y} = L^{q,r}((0,\infty))$. Then one can show that the space X,

$$X := \{f \in \mathcal{M}(\mathbb{R}^n);\ \|f\|_X < \infty\}, \tag{1.8}$$

where

$$\|f\|_X := \| \sup_{\tau \in (t,\infty)} \tau^{\frac{\gamma}{n}} f^{**}(\tau)\|_{\bar{Y}},$$

is the largest rearrangement-invariant space which is mapped by M_γ into Y. On the other hand, Theorem 1.1 asserts that $X = L^{(p,r)}(\mathbb{R}^n)$.

There is the following counterpart of Theorem 1.1.

Theorem 1.2. *Let $0 < r \le \infty$, $0 < p \le \infty$ and $0 > q > -\infty$. Then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{(p,r);\Omega} \approx \|t^{\frac{1}{q}-\frac{1}{r}} \sup_{\tau \in (0,t)} \tau^{\frac{1}{p}-\frac{1}{q}} f^{**}(\tau)\|_{r,(0,|\Omega|)}. \tag{1.9}$$

Corollaries. (i) *Let $0 < p \le \infty$, and $0 > q > -\infty$. Then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{(p,p);\Omega} \approx \|t^{\frac{1}{q}-\frac{1}{p}} \sup_{\tau \in (0,t)} \tau^{\frac{1}{p}-\frac{1}{q}} f^{**}(\tau)\|_{p,(0,|\Omega|)}.$$

(ii) *Let $0 < r \le \infty$, $1 < p \le \infty$ and $0 > q > -\infty$. Then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{p,r;\Omega} \approx \|t^{\frac{1}{q}-\frac{1}{r}} \sup_{\tau \in (0,t)} \tau^{\frac{1}{p}-\frac{1}{q}} f^{**}(\tau)\|_{r,(0,|\Omega|)}.$$

Checking the proof of the inequality RHS(1.9) $\lesssim$ LHS(1.9) (see the proof of Theorem 1.2 in Section 2) and using the estimate

$$t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t) \lesssim \Big(\int_0^t \sigma^{\frac{1}{p}-\frac{1}{q}-1}\, d\sigma\Big) f^{**}(t) \le \int_0^t \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{**}(\sigma)\, d\sigma$$

for all $f \in \mathcal{M}(\Omega)$ and every $t \in (0,\infty)$, one can see that the following assertion is true.

Theorem 1.3. *Let $0 < r \le \infty$, $0 < p \le \infty$ and $0 > q > -\infty$. Then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{(p,r);\Omega} \approx \Big\| t^{\frac{1}{q}-\frac{1}{r}} \int_0^t \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{**}(\sigma)\, d\sigma \Big\|_{r,(0,|\Omega|)}. \tag{1.10}$$

Corollary. *Let $0 < r \le \infty$ and $1 < p \le \infty$. Then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{(p,r);\Omega} \approx \Big\| t^{\frac{1}{p}-\frac{1}{r}} \Big(\frac{1}{t} \int_0^t f^{**}(\sigma)\, d\sigma \Big) \Big\|_{r,(0,|\Omega|)}$$

(*compare with* (1.5)).

Similarly, it follows from the proof of Theorem 1.1 (cf. Section 2) that, for all $f \in \mathcal{M}(\Omega)$,

$$\|f\|_{(p,r);\Omega} \approx \Big\| t^{\frac{1}{q}-\frac{1}{r}} \int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{**}(\sigma)\, d\sigma \Big\|_{r,(0,|\Omega|)}$$

provided that

$$0 < r \le \infty, \quad 0 < p < q < \infty \quad \text{and} \quad 1/p < 1 + 1/q. \tag{1.11}$$

The next theorem shows that the restriction (1.11) can be relaxed.

Theorem 1.4. *Let $0 < r \le \infty$, $0 < p \le \infty$ and $0 < q < \infty$. Assume that either $1/p < 1 + 1/q$ or $|\Omega| = \infty$. Then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{(p,r);\Omega} \approx \Big\| t^{\frac{1}{q}-\frac{1}{r}} \int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{**}(\sigma)\, d\sigma \Big\|_{r,(0,|\Omega|)}. \tag{1.12}$$

Corollaries. (i) *Let $0 < p, r < \infty$ and let either $1/p < 1 + 1/q$ or $|\Omega| = \infty$. Then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{(p,r);\Omega} \approx \Big\| \int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{r}-1} f^{**}(\sigma)\, d\sigma \Big\|_{r,(0,|\Omega|)}.$$

(ii) *Let $0 < r \le \infty$ and $0 < p < \infty$. Then, for all $f \in \mathcal{M}(\Omega)$,*

$$\|f\|_{(p,r);\Omega} \approx \Big\| t^{\frac{1}{p}-\frac{1}{r}} \int_t^\infty \sigma^{-1} f^{**}(\sigma)\, d\sigma \Big\|_{r,(0,|\Omega|)}. \tag{1.13}$$

In particular,

$$\|f\|_{p,p;\Omega} \approx \Big\| \int_t^\infty \sigma^{-1} f^{**}(\sigma)\, d\sigma \Big\|_{p,(0,|\Omega|)} \tag{1.14}$$

provided that $1 < p < \infty$.

Just as we have explained the idea behind Theorem 1.1, so we can clarify the idea hidden behind Theorem 1.4. To this end suppose that (1.11) holds, $\gamma := n(1/p - 1/q)$, $p > 1$ and $\Omega = \mathbb{R}^n$. Define the Riesz potential I_γ by

$$(I_\gamma f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\, dy, \quad x \in \mathbb{R}^n.$$

It is well known that, for all $t \in (0, \infty)$,

$$(I_\gamma f)^*(t) \lesssim t^{\frac{1}{p}-\frac{1}{q}-1} \int_0^t f^*(\sigma)\, d\sigma + \int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^*(\sigma)\, d\sigma \tag{1.15}$$

and that this estimate is sharp (in the same sense as (1.7); cf. [S, (1.20)] or [EGO, Lemma 3.4]). Rewriting RHS(1.15) by Fubini's theorem, we arrive at

$$(I_\gamma f)^*(t) \lesssim \int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{**}(\sigma)\, d\sigma \quad \text{for all } t \in (0, \infty).$$

Now, given the space $L^{q,r}(\mathbb{R}^n) =: Y$ with $r \in (0,\infty]$, put $\bar{Y} = L^{q,r}((0,\infty))$. Then one can show that the space X from (1.8), where

$$\|f\|_X := \Big\| \int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{**}(\sigma)\, d\sigma \Big\|_{\bar{Y}},$$

is the largest rearrangement-invariant space which is mapped by I_γ into Y. On the other hand, Theorem 1.4 asserts that $X = L^{(p,r)}(\mathbb{R}^n)$.

Since (cf. Theorem 4.7 and Proposition 4.10 in Chapter 3 of [BS])

$$\int_t^\infty \sigma^{-1} f^{**}(\sigma)\, d\sigma = t^{-1} \int_0^t f^*(\sigma)\, d\sigma + \int_t^\infty \sigma^{-1} f^*(\sigma)\, d\sigma$$

gives the sharp estimate of $(Hf)^*(t)$, $t \in (0,\infty)$, where H is the Hilbert transform, defined by

$$(Hf)(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy, \quad x \in \mathbb{R},$$

one can similarly explain the idea behind formula (1.14).

If $1 < p < q < \infty$ and $\gamma := n\ (1/p - 1/q) < n$, then, by the result of B. Muckenhoupt and R. L. Wheeden (cf. [MW] or [AH, Theorem 3.6.1]),

$$\|M_\gamma f\|_{q;\mathbb{R}^n} \approx \|I_\gamma f\|_{q;\mathbb{R}^n} \quad \text{for all non-negative } f \in L^p(\mathbb{R}^n) \tag{1.16}$$

(here $\|\cdot\|_{q;\mathbb{R}^n}$ stands for the usual $L^q(\mathbb{R}^n)$-norm). Together with the fact that estimates (1.7) and (1.15) are sharp for all $f \in \mathcal{M}_r^+(\mathbb{R}^n;\downarrow)$, Theorems 1.1 and 1.4 imply that there is the following variant of (1.16) involving quasi-norms of Lorentz spaces.

Corollary. *Let $0 < r \le \infty$, $1 < p < q < \infty$ and $\gamma := n\ (1/p - 1/q) < n$. Then, for all $f \in \mathcal{M}_r^+(\mathbb{R}^n;\downarrow)$,*

$$\|M_\gamma f\|_{q,r;\mathbb{R}^n} \approx \|I_\gamma f\|_{q,r;\mathbb{R}^n} \approx \|f\|_{p,r;\mathbb{R}^n}\,. \tag{1.17}$$

Similarly, one can prove:

Corollary. *Let $0 < r \le \infty$ and $1 < p < \infty$. Then, for all $f \in \mathcal{M}_r^+(\mathbb{R};\downarrow)$,*

$$\|Hf\|_{p,r;\mathbb{R}} \approx \|f\|_{p,r;\mathbb{R}}\,.$$

Remarks. (i) The estimate (1.17) motivates us to look for an extension of [AH, Theorem 3.6.1] to the scale of Lorentz spaces. It reads as follows:

Let $0 < r \le \infty$, $1 < p < q < \infty$ and $\gamma := n\ (1/p - 1/q) < n$. Then, for all positive Radon measures μ on $\mathbb{R}^n$,

$$\|M_\gamma \mu\|_{q,r;\mathbb{R}^n} \approx \|I_\gamma \mu\|_{q,r;\mathbb{R}^n}. \tag{1.18}$$

Indeed, the pointwise estimate (cf. [AH, p. 72])

$$(M_\gamma \mu)(x) \lesssim (I_\gamma \mu)(x), \quad x \in \mathbb{R}^n,$$

shows that LHS(1.18) $\lesssim$ RHS(1.18). To prove the converse inequality, one applies the fact that, for all $g \in \mathcal{M}(\mathbb{R}^n)$,

$$\|g\|_{q,r;\mathbb{R}^n} = \|t^{\frac{1}{q}-\frac{1}{r}} g^*(t)\|_{r,(0,\infty)} \approx \|\lambda^{1-\frac{1}{r}}\ |\{x \in \mathbb{R}^n;\ |g(x)| > \lambda\}|^{\frac{1}{q}}\ \|_{r,(0,\infty)}$$

and the good λ inequality (3.6.1) from [AH].

(ii) It follows from the proofs of Theorems 1.1–1.4 that the constants of equivalence in the relations (1.6), (1.9), (1.10) and (1.12) depend only on p, q and r.

(iii) One can easily extend Theorems 1.1–1.4 to the case when the Lorentz space $L^{(p,r)}(\Omega)$ is replaced by the Lorentz-Zygmund space $L^{(p,r)}(\log L)^{\alpha}(\Omega)$ ($p, r \in (0,\infty]$, $\alpha \in \mathbb{R}$) defined by (cf. [BR])

$$L^{(p,r)}(\log L)^{\alpha}(\Omega) = \{f \in \mathcal{M}(\Omega);\ \|f\|_{(p,r),\alpha} = \|f\|_{(p,r),\alpha;\Omega} < \infty\},$$

where

$$\|f\|_{(p,r),\alpha} := \|t^{\frac{1}{p}-\frac{1}{r}}\ell^{\alpha}(t)f^{**}(t)\|_{r,(0,|\Omega|)}$$

and $\ell(t) := 1 + |\log t|$, $t \in (0,\infty)$.

(iv) Theorems 1.1–1.4 remain true if the role of the measure space (Ω, dx) is played by a totally σ-finite measure space (R,μ) with a non-atomic measure μ.

(v) Theorems 1.1–1.4 continue to hold if $\|f\|_{(p,r);\Omega}$ and f^{**} are replaced by $\|f\|_{p,r;\Omega}$ and f^{*}, respectively. This is a consequence of more general results proved in [O].

2. Proof of Theorems 1.1, 1.2 and 1.4

Throughout this section by $\mathcal{M}^{+}(a,b;\downarrow)$, $(a,b) \subseteq \mathbb{R}$, we mean the collection of all $f \in \mathcal{M}^{+}(a,b)$ which are non-increasing on (a,b).

To prove Theorems 1.1 and 1.2, we shall use the following result.

Lemma 2.1 (cf. [La, Theorem 2.2])**.** *Let $0 < P \le Q \le 1$. Suppose that $w, v \in \mathcal{M}^{+}(0,\infty)$ and $k \in \mathcal{M}^{+}((0,\infty)\times(0,\infty))$. Then there is $C \in [0,\infty)$ such that the inequality*

$$\Big\|w(t)\int_0^{\infty} k(t,\sigma)g(\sigma)\,d\sigma\Big\|_{Q,(0,\infty)} \le C\,\|v(t)g(t)\|_{P,(0,\infty)} \tag{2.1}$$

holds for all $g \in \mathcal{M}^{+}(0,\infty;\downarrow)$ if and only if

$$\Big\|w(t)\int_0^{\rho} k(t,\sigma)\,d\sigma\Big\|_{Q,(0,\infty)} \le C\,\|v(t)\|_{P,(0,\rho)} \quad \textit{for all } \rho \in (0,\infty). \tag{2.2}$$

Proof of Theorem 1.1. (i) If $0 < q \le p \le \infty$, then the function $\tau \to \tau^{\frac{1}{p}-\frac{1}{q}}$ is non-increasing on $(0,\infty)$. Hence,

$$\sup_{\tau\in(t,|\Omega|)} \tau^{\frac{1}{p}-\frac{1}{q}} f^{**}(\tau) = t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t) \quad \text{for all } t \in (0,|\Omega|),$$

which implies that LHS(1.6) = RHS(1.6).

(ii) Assume now that $0 < p < q < \infty$ and $1/p < 1 + 1/q$. The inequality RHS(1.6) $\gtrsim$ LHS(1.6) is obvious. To prove the converse one, we observe that, for all $t \in (0,|\Omega|)$,

$$\begin{aligned} \sup_{\tau\in(t,|\Omega|)} \tau^{\frac{1}{p}-\frac{1}{q}} f^{**}(\tau) &\approx \sup_{\tau\in(t,|\Omega|)} \Big(\int_{\tau}^{\infty} s^{\frac{1}{p}-\frac{1}{q}-2}\,ds\Big)\Big(\int_0^{\tau} f^{*}(\sigma)\,d\sigma\Big) \\ &\le \int_t^{\infty} s^{\frac{1}{p}-\frac{1}{q}-2}\Big(\int_0^{s} f^{*}(\sigma)\,d\sigma\Big)\,ds \end{aligned} \tag{2.3}$$

and, by Fubini's theorem,

$$\text{RHS(2.3)} \approx t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t) + \int_t^{\infty} \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{*}(\sigma)\,d\sigma, \tag{2.4}$$

which yields

$$\text{RHS(1.6)} \lesssim \text{LHS(1.6)} + V(f), \tag{2.5}$$

where

$$V(f) := \left\| t^{\frac{1}{q}-\frac{1}{r}} \int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^*(\sigma)\, d\sigma \right\|_{r,(0,\infty)}.$$

If $r \in [1,\infty]$, we apply the Hardy inequality (cf. [OK])

$$\left\| t^{\frac{1}{q}-\frac{1}{r}} \int_t^\infty g(\sigma)\, d\sigma \right\|_{r,(0,\infty)} \le C\, \| t^{\frac{1}{q}+1-\frac{1}{r}} g(t) \|_{r,(0,\infty)},$$

which holds for all $g \in \mathcal{M}^+(0,\infty)$ (with a positive constant C independent of g), to get, for all $f \in \mathcal{M}(\Omega)$,

$$V(f) \lesssim \| t^{\frac{1}{p}-\frac{1}{r}} f^*(t) \|_{r,(0,\infty)} \le \mathrm{LHS}(1.6). \tag{2.6}$$

If $r \in (0,1)$, we put $P = Q = r$,

$$k(t,\sigma) = \chi_{(t,\infty)}(\sigma)\, \sigma^{\frac{1}{p}-\frac{1}{q}-1}, \quad w(t) = t^{\frac{1}{q}-\frac{1}{r}}, \quad v(t) = t^{\frac{1}{p}-\frac{1}{r}} \quad \text{if } \ t,\sigma \in (0,\infty),$$

and apply Lemma 2.1 to arrive at the estimate (2.6). Combining (2.5) and (2.6), we obtain the desired inequality RHS(1.6) $\lesssim$ LHS(1.6).

(iii) Finally, let $0 < p < q < \infty$ and $1/p \ge 1 + 1/q$. Then $p \in (0,1)$ and, by [OP, Lemmas 3.5 (ii) and 3.15],

$$L^{(p,r)}(\Omega) = \{0\} \ \text{ if } \ |\Omega| = \infty, \quad L^{(p,r)}(\Omega) = L^1(\Omega) \ \text{ if } \ |\Omega| < \infty.$$

Consequently, if $f \in \mathcal{M}(\Omega)$, $f \ne 0$ a.e. in Ω, then

$$\begin{cases} \|f\|_{(p,r);\Omega} = \infty & \text{if} \quad |\Omega| = \infty, \\ \|f\|_{(p,r);\Omega} \approx \|f\|_{1,1;\Omega} & \text{if} \quad |\Omega| < \infty. \end{cases} \tag{2.7}$$

On the other hand, since the function $\tau \to \tau^{\frac{1}{p}-\frac{1}{q}-1}$ is non-decreasing on $(0,\infty)$, we obtain

$$\sup_{\tau \in (t,|\Omega|)} \tau^{\frac{1}{p}-\frac{1}{q}} f^{**}(\tau) = \Big(\lim_{\tau \to |\Omega|} \tau^{\frac{1}{p}-\frac{1}{q}-1} \Big) \int_0^{|\Omega|} f^*(\sigma)\, d\sigma, \quad t \in (0,|\Omega|).$$

This easily yields that

$$\begin{cases} \mathrm{RHS}(1.6) = \infty & \text{if} \quad |\Omega| = \infty, \\ \mathrm{RHS}(1.6) \approx \|f\|_{1,1;\Omega} & \text{if} \quad |\Omega| < \infty. \end{cases} \tag{2.8}$$

Comparing (2.7) and (2.8), we see that (1.6) is again satisfied. □

Remark. There is another proof of Theorem 1.1 if $0 < p < q < \infty$ and $1/p < 1 + 1/q$. Indeed, put $\Delta = 1/p - 1/q$ and define the operators $\mathcal{T}_\Delta$, $\mathcal{S}_\Delta$ and $\mathcal{R}_\Delta$ on $\mathcal{M}^+(0,|\Omega|;\downarrow)$ by

$$\begin{aligned} (\mathcal{T}_\Delta \varphi)(t) &= \sup_{\tau \in (t,|\Omega|)} \tau^\Delta \varphi^{**}(\tau), \\ (\mathcal{S}_\Delta \varphi)(t) &= t^\Delta \varphi^{**}(t), \\ (\mathcal{R}_\Delta \varphi)(t) &= \sup_{\tau \in (t,|\Omega|)} \tau^\Delta \varphi(\tau). \end{aligned} \tag{2.9}$$

First, one can show that (cf. [CKOP] or [EO])

$$\mathcal{T}_\Delta \approx \mathcal{S}_\Delta + \mathcal{R}_\Delta \quad \text{on} \quad \mathcal{M}^+(0,|\Omega|;\downarrow).$$

Hence, for all $f \in \mathcal{M}(\Omega)$,

$$\begin{aligned} \mathrm{RHS}(1.6) &= \|t^{\frac{1}{q}-\frac{1}{r}}(\mathcal{T}_\Delta f^*)(t)\|_{r,(0,|\Omega|)} \\ &\approx \|t^{\frac{1}{q}-\frac{1}{r}}(\mathcal{S}_\Delta f^*)(t)\|_{r,(0,|\Omega|)} + \|t^{\frac{1}{q}-\frac{1}{r}}(\mathcal{R}_\Delta f^*)(t)\|_{r,(0,|\Omega|)}\,. \end{aligned}$$

On using the equality $\Delta = 1/p - 1/q$ and (2.9), we see that

$$\|t^{\frac{1}{q}-\frac{1}{r}}(\mathcal{S}_\Delta f^*)(t)\|_{r,(0,|\Omega|)} = \|f\|_{(p,r);\Omega}.$$

Moreover, since one can prove that (cf. the sufficiency part of the proof of [CKOP, Lemma 3.1] or [EO, Lemma 4.5])

$$\|t^{\frac{1}{q}-\frac{1}{r}}(\mathcal{R}_\Delta f^*)\|_{r,(0,|\Omega|)} \lesssim \|t^{\frac{1}{p}-\frac{1}{r}} f^*(t)\|_{r,(0,|\Omega|)}$$

for all $f \in \mathcal{M}(\Omega)$, (1.6) follows.

Proof of Theorem 1.2. Since

$$\sup_{\tau\in(0,t)} \tau^{\frac{1}{p}-\frac{1}{q}} f^{**}(\tau) \geq t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t) \quad \text{for all } t \in (0,\infty),$$

it follows that $\mathrm{RHS}(1.9) \geq \mathrm{LHS}(1.9)$. Moreover, the estimate

$$\sup_{\tau\in(0,t)} \tau^{\frac{1}{p}-\frac{1}{q}} f^{**}(\tau) \approx \sup_{\tau\in(0,t)} f^{**}(\tau) \int_0^\tau \sigma^{\frac{1}{p}-\frac{1}{q}-1} d\sigma \leq \int_0^t \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{**}(\sigma)\, d\sigma$$

implies that

$$\mathrm{RHS}(1.9) \lesssim \left\| t^{\frac{1}{q}-\frac{1}{r}} \int_0^t \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{**}(\sigma)\, d\sigma \right\|_{r,(0,|\Omega|)}. \tag{2.10}$$

If $r \in [1,\infty]$, we apply the Hardy inequality (cf. [OK])

$$\left\| t^{\frac{1}{q}-\frac{1}{r}} \int_0^t g(\sigma)\, d\sigma \right\|_{r,(0,|\Omega|)} \leq C\, \|t^{\frac{1}{q}+1-\frac{1}{r}} g(t)\|_{r,(0,|\Omega|)},$$

which holds on $\mathcal{M}^+(0,|\Omega|)$ (with a positive constant C independent of g and $|\Omega|$), to get, for all $f \in \mathcal{M}(\Omega)$,

$$\mathrm{RHS}(2.10) \lesssim \|t^{\frac{1}{p}-\frac{1}{r}} f^{**}(t)\|_{r,(0,|\Omega|)} = \mathrm{LHS}(1.9). \tag{2.11}$$

If $r \in (0,1)$, we put $P = Q = r$ and

$$k(t,\sigma) = \chi_{(0,t)}(\sigma)\, \sigma^{\frac{1}{p}-\frac{1}{q}-1}, \quad w(t) = t^{\frac{1}{q}-\frac{1}{r}} \chi_{(0,|\Omega|)}(t), \quad v(t) = t^{\frac{1}{p}-\frac{1}{r}} \chi_{(0,|\Omega|)}(t)$$

when $t, \sigma \in (0,\infty)$. Then (2.11) is a consequence of Lemma 2.1. Combining (2.10) and (2.11), we arrive at the estimate $\mathrm{RHS}(1.9) \lesssim \mathrm{LHS}(1.9)$ and Theorem 1.2 is proved. □

Proof of Theorem 1.4. Since for all $f \in \mathcal{M}(\Omega)$ and every $t \in (0,\infty)$,

$$t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t) \lesssim \Big(\int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-2}\, d\sigma \Big) \int_0^t f^*(s)\, ds \le \int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{**}(\sigma)\, d\sigma,$$

we see that LHS(1.12) $\lesssim$ RHS(1.12).

Now, we are going to verify the reverse estimate. By Fubini's theorem, for all $f \in \mathcal{M}(\Omega)$ and every $t \in (0,\infty)$,

$$\int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{**}(\sigma)\, d\sigma = \int_0^t f^*(\tau) \Big(\int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-2}\, d\sigma \Big) d\tau + \int_t^\infty f^*(\tau) \Big(\int_\tau^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-2}\, d\sigma \Big) d\tau. \tag{2.12}$$

Therefore, if $1/p < 1 + 1/q$ and $t \in (0,\infty)$,

$$\int_t^\infty \sigma^{\frac{1}{p}-\frac{1}{q}-1} f^{**}(\sigma)\, d\sigma = \text{RHS}(2.4),$$

and, proceeding just as in the proof of Theorem 1.1, we arrive at the desired estimate.

Assume now that $1/p \ge 1 + 1/q$ and $|\Omega| = \infty$. Then (cf. (2.7)) LHS(1.12)$=\infty$ for all $f \in \mathcal{M}(\Omega)$, $f \neq 0$ a.e. in Ω. Since (2.12) yields that RHS(1.12) $= \infty$ for all such f, we see that LHS(1.12) = RHS(1.12). □

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Centre for Mathematical Analysis and Its Applications, University of Sussex, Falmer, Brighton BN1 9QH, England

E-mail address: d.e.edmunds@sussex.ac.uk

Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic

E-mail address: opic@math.cas.cz