

REDUCTION NUMBERS AND INITIAL IDEALS

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ABSTRACT. The reduction number $r(A)$ of a standard graded algebra A is the least integer k such that there exists a minimal reduction J of the homogeneous maximal ideal \mathbf{m} of A such that $J\mathbf{m}^k = \mathbf{m}^{k+1}$. Vasconcelos conjectured that $r(R/I) \leq r(R/\text{in}(I))$ where $\text{in}(I)$ is the initial ideal of an ideal I in a polynomial ring R with respect to a term order. The goal of this note is to prove the conjecture.

1. REDUCTION NUMBERS AND INITIAL IDEALS

Let K be an infinite field and let $A = \bigoplus_{i \in \mathbf{N}} A_i$ be a homogeneous K -algebra, that is, an algebra of the form R/I where $R = K[x_1, \dots, x_n]$ is a polynomial ring and I is a homogeneous ideal. The reduction number $r(A)$ of A is the least integer k such that there exists a minimal reduction J of the homogeneous maximal ideal \mathbf{m} of A such that $J\mathbf{m}^k = \mathbf{m}^{k+1}$. It is not difficult to see that $r(A)$ is the largest integer k such that the Hilbert function of A/J at k does not vanish; here J is the ideal of A generated by $d = \dim A$ generic linear forms.

Vasconcelos conjectured [10, Conjecture 7.2] that

$$r(R/I) \leq r(R/\text{in}_\tau(I))$$

where $\text{in}_\tau(I)$ is the initial ideal of I with respect to a term order τ . The conjecture has been proved by Bresinsky and Hoa [2] for the generic initial ideal $\text{Gin}_\tau(I)$, or, more generally, when $\text{in}_\tau(I)$ is Borel-fixed. Trung [8] showed that $r(R/I) = r(R/\text{Gin}_{\text{RL}}(I))$ where $\text{Gin}_{\text{RL}}(I)$ is the generic initial ideal of I with respect to the degree reverse lexicographic order RL (revlex for short).

The goal of this note is to prove the conjecture in general. After this paper was written we were informed that Trung [9] has independently solved the conjecture in general by a completely different method. What we prove is the following generalization of Vasconcelos' conjecture:

Theorem 1.1. *Let p be an integer, $0 \leq p \leq n$, and let $\text{in}_\tau(I)$ be the initial ideal of I with respect to a term order τ . Let J be an ideal generated by p generic linear forms. Then the Hilbert function of $R/I + J$ is \leq that of $R/\text{in}_\tau(I) + J$, that is,*

$$\dim_K [R/I + J]_j \leq \dim_K [R/\text{in}_\tau(I) + J]_j$$

for all $j \in \mathbf{N}$.

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Taking $p = \dim R/I$ and $j = r(R/\text{in}_\tau(I)) + 1$ one obtains $r(R/I) < j$ which implies Vasconcelos' conjecture. To prove Theorem 1.1 we need some preparation.

Lemma 1.2. *Let p be an integer, $0 \leq p \leq n$, and let J be an ideal generated by p generic linear forms. Then the Hilbert function of $R/I + J$ is equal to the Hilbert function of $R/\text{Gin}_{\text{RL}}(I) + H$ where $\text{Gin}_{\text{RL}}(I)$ is the revlex Gin of I and $H = (x_{n-p+1}, x_{n-p+2}, \dots, x_n)$.*

Proof. Set $J = (y_1, \dots, y_p)$. We take a matrix $g \in \text{GL}_n(K)$ such that the induced K -algebra graded homomorphism $g : R \rightarrow R$ maps y_i to x_{n-p+i} for $i = 1, \dots, p$. It follows that the Hilbert function of $R/I + J$ equals that of $R/g(I) + H$. Taking initial ideals does not change the Hilbert function and by the known properties of the revlex order one has $\text{in}_{\text{RL}}(g(I) + H) = \text{in}_{\text{RL}}(g(I)) + H$. But since the y_i are generic, g is generic as well. Then $\text{in}_{\text{RL}}(g(I)) = \text{Gin}_{\text{RL}}(I)$ and we are done. \square

We would now like to compare $\text{Gin}_{\text{RL}}(I)$ with $\text{Gin}_{\text{RL}}(\text{in}_\tau(I))$. To this end, let us introduce a piece of notation.

Let $V \subset R_i$ be a subspace of forms of degree i and dimension d . Then $\bigwedge^d V$ is a subspace of dimension 1 of $\bigwedge^d R_i$. We identify in the following $\bigwedge^d V$ with any non-zero element contained in it. Fix a term order $<_\sigma$ in R . An exterior monomial is an element of the form $m_1 \wedge \dots \wedge m_d$ where the m_j are distinct monomials of R_i . An exterior monomial $m_1 \wedge \dots \wedge m_d$ is σ -standard if $m_1 >_\sigma \dots >_\sigma m_d$. Note that $\bigwedge^d R_i$ has a basis consisting of the σ -standard exterior monomials. We order the σ -standard exterior monomials lexicographically:

$$m_1 \wedge \dots \wedge m_d >_\sigma n_1 \wedge \dots \wedge n_d$$

if $m_i >_\sigma n_i$ for the smallest index i such that $m_i \neq n_i$. Then one defines the initial (exterior) monomial with respect to σ of any element f in the exterior space $\bigwedge^d R_i$ and the initial subspace of any subspace of $\bigwedge^d R_i$. By construction one has that $\text{in}_\sigma(V) = \langle m_1, \dots, m_d \rangle$ and $m_1 >_\sigma \dots >_\sigma m_d$ if and only if $\text{in}_\sigma(\bigwedge^d V) = m_1 \wedge \dots \wedge m_d$. For an element $F \in \bigwedge^d R_i$ we define its σ -support $\text{Support}_\sigma(F)$ to be the set of the σ -standard exterior monomials which appear with a non-zero coefficient in F . Note that any exterior monomial n is equal (up to sign) to a σ -standard exterior monomial. Note also that given an element $F \in \bigwedge^d R_i$ and two term orders σ and τ , the τ -support of F is obtained by taking the τ -standard form of the elements in $\text{Support}_\sigma(F)$. One has:

Lemma 1.3. *Let σ be a term order. Let $m = m_1 \wedge \dots \wedge m_d$ be a σ -standard exterior monomial, and let $q = q_1 \wedge \dots \wedge q_d$ be an exterior monomial with $q_i \leq_\sigma m_i$ for $i = 1, \dots, d$. Let $n = n_1 \wedge \dots \wedge n_d$ be the σ -standard exterior monomial corresponding to q . Then $n_i \leq_\sigma m_i$ for $i = 1, \dots, d$.*

Proof. Since n is obtained from q by a sequence of transposition exchanging q_j with q_{j+1} whenever $q_j <_\sigma q_{j+1}$, it suffices to check that the property $q_i \leq_\sigma m_i$ for all i is preserved by any such transposition. This is clear since $m_j >_\sigma m_{j+1} \geq_\sigma q_{j+1}$ and $m_{j+1} \geq_\sigma q_{j+1} >_\sigma q_j$. \square

Lemma 1.4. *Let σ be a term order. Let V be a subspace of R_i of dimension d , and let $\text{in}_\sigma(\bigwedge^d V) = m_1 \wedge \dots \wedge m_d$. For every $n_1 \wedge \dots \wedge n_d \in \text{Support}_\sigma(\bigwedge^d V)$ one has $m_i \geq_\sigma n_i$ for $i = 1, \dots, d$.*

Proof. Let f_1, \dots, f_d be elements in V such that $\text{in}(f_i) = m_i$. Then $\bigwedge^d V = f_1 \wedge \dots \wedge f_d$. For $i = 1, \dots, d$ let q_i be a monomial in f_i . It suffices to show that the σ -standard exterior monomial corresponding to $q_1 \wedge \dots \wedge q_d$ satisfies the desired property. This follows from Lemma 1.3 since $q_i \leq_\sigma m_i$. \square

The crucial fact is the following:

Lemma 1.5. *Let V be a d -dimensional subspace of R_i . Let σ and τ be term orders. Set $W = \text{in}_\tau(V)$. Let $g \in \text{GL}_n(K)$ be a generic matrix acting as a K -algebra graded homomorphism on R . Then*

$$\text{Support}_\sigma(g(\bigwedge^d W)) \subseteq \text{Support}_\sigma(g(\bigwedge^d V)).$$

Proof. Let $W = \langle m_1, \dots, m_d \rangle$ and let f_1, \dots, f_d in V so that $\text{in}_\tau(f_i) = m_i$. Set $F = f_1 \wedge \dots \wedge f_d$ and $M = m_1 \wedge \dots \wedge m_d$. We have to show that $\text{Support}_\sigma(g(M)) \subseteq \text{Support}_\sigma(g(F))$. The matrix g acts on R by, say, $g(x_i) = \sum_j g_{ij} x_j$. We give to g_{ij} a multidegree: $\text{deg}(g_{ij}) = e_i \in \mathbf{Z}^n$. In the following $\log(m)$ denotes the exponent of a monomial m . For any monomial m of R_i we have that $g(m)$ is a sum of monomials of R_i whose coefficients are polynomials of degree $\log(m)$ in the g_{ij} . Similarly, if $n = n_1 \wedge \dots \wedge n_d$ is an exterior monomial, then $g(n)$ is a sum of σ -standard exterior monomials whose coefficients are polynomials in the g_{ij} of degree $\log(n_1 \dots n_d)$. Now assume $n = n_1 \wedge \dots \wedge n_d$ is a σ -standard exterior monomial in the σ -support of $g(M)$. If n arises in the expansion of $g(Q)$ where $Q = q_1 \wedge \dots \wedge q_d$ for monomials q_i in the support of f_i , then the coefficient of n in $g(Q)$ is a polynomial of degree $\log(q_1 \dots q_d)$ in the g_{ij} . If at least one of the q_i , say q_j , is $<_\tau m_i$, then $q_1 \dots q_d <_\tau m_1 \dots m_d$. In particular $q_1 \dots q_d \neq m_1 \dots m_d$. It follows that the coefficients of n in $g(M)$ and in $g(Q)$ are polynomials in the g_{ij} of different degree. Therefore the coefficient of n in $g(F)$ is a multi-homogeneous polynomial in the g_{ij} and one of its homogeneous components is exactly the coefficient of n in $g(M)$. This suffices to show that, for a generic g , the element n is in the σ -support of $g(F)$. \square

Corollary 1.6. *Let V be a d -dimensional subspace of R_i . Let τ and σ be term orders. Let $\text{Gin}_\sigma(V) = \langle m_1, \dots, m_d \rangle$ and $\text{Gin}_\sigma(\text{in}_\tau(V)) = \langle n_1, \dots, n_d \rangle$ with $m_i >_\sigma m_{i+1}$ and $n_i >_\sigma n_{i+1}$ for all $i = 1, \dots, d - 1$. Then $m_i \geq_\sigma n_i$ for all $i = 1, \dots, d$.*

Proof. Set $W = \text{in}_\tau(V)$, $m = m_1 \wedge \dots \wedge m_d$ and $n = n_1 \wedge \dots \wedge n_d$. By construction $\text{in}_\sigma(g(\bigwedge^d W)) = n$ for a generic matrix g . By virtue of Lemma 1.5, $n \in \text{Support}_\sigma(g(\bigwedge^d V))$ and by construction $\text{in}_\sigma(g(\bigwedge^d V)) = m$. It follows from Lemma 1.4 that $n_i \leq_\sigma m_i$ for all $i = 1, \dots, d$. \square

We are ready to prove Theorem 1.1:

Proof of Theorem 1.1. Set $H = (x_{n-p+1}, x_{n-p+2}, \dots, x_n)$. By virtue of Lemma 1.2, it is enough to show that the Hilbert function of $R/\text{Gin}_{\text{RL}}(I) + H$ is less than or equal to that of $R/\text{Gin}_{\text{RL}}(\text{in}_\tau(I)) + H$. Fix an integer j and set

$$a = \dim [R/\text{Gin}_{\text{RL}}(I) + H]_j \quad \text{and} \quad b = \dim [R/\text{Gin}_{\text{RL}}(\text{in}_\tau(I)) + H]_j.$$

We must show that $a \leq b$. Let V be the component of degree j of I . Set $d = \dim V$, $\text{Gin}_{\text{RL}}(V) = \langle m_1, \dots, m_d \rangle$, $\text{Gin}_{\text{RL}}(\text{in}_\tau(V)) = \langle n_1, \dots, n_d \rangle$ and assume that $m_i >_{\text{RL}}$

m_{i+1} and $n_i >_{\text{RL}} n_{i+1}$. For a monomial m we set $\max(m) = \max\{i : x_i \text{ divides } m\}$. By construction we have

$$b - a = |\{k : \max(m_k) \leq n - p\}| - |\{k : \max(n_k) \leq n - p\}|.$$

By Corollary 1.6 we know that $m_i \geq_{\text{RL}} n_i$ for all i . This implies that $\max(m_i) \leq \max(n_i)$ for all i . Hence $\{k : \max(m_k) \leq n - p\} \supseteq \{k : \max(n_k) \leq n - p\}$ and we are done. \square

Remark 1.7. One easily checks that the proof of Theorem 1.1 also works if one takes the initial ideal with respect to a positive weight function ω . In particular Vasconcelos' conjecture holds in this situation, too.

Remark 1.8. With the notation of Theorem 1.1, one could ask whether there is a relation between the graded Betti numbers of $R/I + J$ and those of $R/\text{in}_\tau(I) + J$. The known properties of the initial ideal imply that the former are smaller than the latter for instance when $p \leq \text{depth } R/\text{in}_\tau(I)$. But this relation does not hold in general. This is because, as we know, the Hilbert function of $R/I + J$ is less than or equal to that of $R/\text{in}_\tau(I) + J$ and hence the number of generators in low degrees of $I + J$ tends to be larger than that of $\text{in}_\tau(I) + J$. For instance, taking $I = (x^2 + yz, xy, xz)$ and τ to be the lex order, then $\text{in}_\tau(I) = (x^2, xy, xz, yz^2, y^2z)$ and for a general linear form L the ideal $I + (L)$ has 3 minimal generators in degree 2, while $\text{in}_\tau(I) + (L)$ has only 2 minimal generators in degree 2.

Remark 1.9. Recall that the analytic spread $\ell(I)$ of an ideal I is the Krull dimension of the fiber ring $\bigoplus_{i=0}^{\infty} I^i / \mathfrak{m} I^i$. One can ask whether there is a relation between the analytic spread I and that of $\text{in}_\tau(I)$. There are examples where $\ell(I) > \ell(\text{in}_\tau(I))$ and others where $\ell(I) < \ell(\text{in}_\tau(I))$. As for the former, take for instance the ideal I of the 2-minors of a generic 3×3 symmetric matrix and τ a diagonal term order (i.e. the initial term of a minor is the product of the elements of the main diagonal). One has $\ell(I) = 6$ and $\ell(\text{in}_\tau(I)) = 5$. On the other hand, if I is the ideal generated by 2 generic quadrics in 3 variables and τ is the lex order, then $\ell(I) = 2$ and $\ell(\text{in}_\tau(I)) = 3$.

2. REDUCTION NUMBERS AND LEX-SEGMENT IDEALS

A monomial ideal L of R is said to be a Lex-segment if whenever m is a monomial in L and n is a monomial with $\deg(n) = \deg(m)$ and $n > m$ with respect to the lexicographic order, then one has that $n \in L$. Given a homogeneous ideal I there is a unique Lex-segment ideal I^{Lex} such that the Hilbert function of I^{Lex} is equal to that of I . It is well-known that I^{Lex} is "extremal" with respect to many invariants in the class of the ideals with a given Hilbert function (e.g. absolute Betti numbers Bigatti [1], Hulett [4] and Pardue [6], relative Betti numbers Iyengar and Pardue [5], local cohomology Sbarra [7], etc.). Therefore it is natural to ask whether the same also holds for the reduction number, i.e. whether $r(R/I) \leq r(R/I^{\text{Lex}})$ holds in general. We have:

Proposition 2.1. *If K has characteristic 0, then*

$$r(R/I) \leq r(R/I^{\text{Lex}})$$

holds for every homogeneous ideal I of $K[x_1, \dots, x_n]$.

Proof. Let I be a homogeneous ideal of $R = K[x_1, \dots, x_n]$ and set $d = \dim R/I$ and $J = \text{Gin}_{\text{RL}}(I)$. It is known that J is Borel-fixed, that is, fixed under the action of the group of the upper-triangular matrix. In characteristic 0 this is equivalent to saying that J is strongly stable, that is, if $x_i m$ is a monomial in J and $j < i$, then $x_j m$ is in J as well. From this and from Lemma 1.2, it follows immediately that if $\text{char } K = 0$, then $r(R/I)$ is equal to the least integer k such that x_{n-d}^{k+1} is in J (this has also been observed in [8]). Then the desired inequality is a consequence of the following fact:

Claim. Let V and L be sets of monomials of degree k with the same cardinality such that V is strongly stable and L is a Lex-segment. If $x_i^k \in L$ for some i , then $x_i^k \in V$.

To prove the claim one observes that since L contains x_i^k and is a Lex-segment, then L also contains the set, say A , of all the monomials of degree k which are divisible by some x_j with $j < i$. Therefore $|L| \geq |A| + 1$. Since $|V| = |L|$ it follows that V contains a monomial m which is not in A . In other words, V contains a monomial supported only on the variables x_i, x_{i+1}, \dots, x_n . Since V is stable, then V also contains x_i^k . \square

We believe that the inequality of Proposition 2.1 is also true if the characteristic of the base field is finite. Pardue developed in [6] a characteristic-free strategy for proving that the Lex-segment ideal is extremal with respect to a certain invariant, say $\alpha(R/I)$. Roughly speaking, it says that if α does not decrease by taking initial ideals and also does not decrease by performing a certain deformation process, called polarization, then one has $\alpha(R/I) \leq \alpha(R/I^{\text{Lex}})$ for all the ideals I . For the definition of polarization of a monomial ideal we refer the reader to the paper of Pardue [6]. Unfortunately one cannot use Pardue's argument to prove the above inequality. This is because the reduction number can decrease under polarizations. For example, let $R = K[x_1, \dots, x_4]$ and

$$I = (x_4^2, x_1 x_3^3, x_3^3 x_4, x_2^3 x_4, x_2 x_3^3, x_2^3 x_3, x_1^2 x_3^2, x_1^4, x_1 x_2^2 x_4, x_2^4)$$

and J its polarization; one can check that $r(R/I) = 4$ and $r(R/J) = 3$. In this case $r(R/I^{\text{Lex}}) = 5$.

Let us also note that the above ideal can be used to construct an example of a standard graded algebra A and a non-zero divisor z of degree 1 such that $r(A) < r(A/zA)$. To this end it suffices to take $S = K[x_1, \dots, x_5]$, and

$$I_1 = (x_4 x_5, x_1 x_3^3, x_3^3 x_4, x_2^3 x_4, x_2 x_3^3, x_2^3 x_3, x_1^2 x_3^2, x_1^4, x_1 x_2^2 x_4, x_2^4).$$

In other words, I_1 is the polarization of the ideal I above with respect to the variable x_4 . Set $A = S/I_1$ and $z = x_4 - x_5$. Then z is a non-zero divisor of A and $r(A) = 3$ and $r(A/zA) = 4$.

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