

FINITENESS OF REPRESENTATION DIMENSION

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ABSTRACT. We will show that any module over an artin algebra is a direct summand of some module whose endomorphism ring is quasi-hereditary. As a conclusion, any artin algebra has a finite representation dimension.

M. Auslander introduced a concept of representation dimension of artin algebras in [A], which was a trial to give a reasonable way of measuring homologically how far an artin algebra is from being of finite representation type ([X1], [FGR]). His methods given there have been effectively applied not only for the representation theory of artin algebras [ARS], but also for the theory of quasi-hereditary algebras of Cline-Parshall-Scott [CPS] by Dlab-Ringel in [DR2]. Unfortunately, much seems to be unknown about representation dimension itself. In particular, Reiten asked in 1998 whether any artin algebra has a finite representation dimension or not (cf. §2.3(2)). In this paper, we will give a positive answer to this question (§1.2) by showing that any module is a direct summand of some module whose endomorphism ring is quasi-hereditary (§1.1). Our method is to construct a certain chain of subcategories of $\text{mod } \Lambda$ (§2.2), which was applied to solve Solomon's second conjecture on zeta functions of orders in [I3]. We will formulate it in terms of rejective subcategories (§2.1), which was effectively applied in [I1] to study the representation theory of orders and give a characterization of their finite Auslander-Reiten quivers in [I2].

Note. After the author submitted this paper, Professor Xi kindly informed him that Theorem 1.1 and Corollary 1.2 were stated in [X2] as conjectures, where the former was given by Ringel and Yamagata. He thanks Professor Xi and Professor Yamagata for valuable comments.

1.

In this paper, any module is assumed to be a left module. For an artin algebra Λ over R , let $\text{mod } \Lambda$ be the category of finitely generated left Λ -modules, J_Λ the Jacobson radical of Λ , $\text{dom.dim } \Lambda$ the dominant dimension of Λ [T], $I_\Lambda(X)$ the injective hull of the Λ -module X and $(\)^* := \text{Hom}_R(\ , I_R(R/J_R)) : \text{mod } \Lambda \leftrightarrow \text{mod } \Lambda^{op}$ the duality. For $X \in \text{mod } \Lambda$, we denote by $\text{add } X$ the full subcategory of $\text{mod } \Lambda$ consisting of direct summands of a finite direct sum of X . The representation dimension of Λ is defined by $\text{rep.dim } \Lambda := \inf\{\text{gl.dim } \Gamma \mid \Gamma \in A(\Lambda)\}$, where $A(\Lambda)$ is the collection of all artin algebras Γ such that $\text{dom.dim } \Gamma \geq 2$ and $\text{End}_\Gamma(I_\Gamma(\Gamma))$ is

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Morita-equivalent to Λ . Then $\text{rep.dim } \Lambda = \inf\{\text{gl.dim } \text{End}_\Lambda(M) \mid M \in \text{mod } \Lambda \text{ such that } \Lambda \oplus \Lambda^* \in \text{add } M\}$ holds by [A].

1.1. Theorem. *Let Λ be an artin algebra. Then any $M \in \text{mod } \Lambda$ is a direct summand of some $N \in \text{mod } \Lambda$ such that $\text{End}_\Lambda(N)$ is a quasi-hereditary algebra.*

1.2. Corollary. *Let Λ be an artin algebra. Then $\text{rep.dim } \Lambda$ has a finite value which is not greater than $2l - 2$, where l is the length of a $(\Lambda, \text{End}_\Lambda(\Lambda \oplus \Lambda^*))$ -module $\Lambda \oplus \Lambda^*$.*

2.

In the rest of this paper, any subcategory \mathcal{C}' of an additive category \mathcal{C} is assumed to be full and closed under direct sums. Let $\mathcal{J}_\mathcal{C}$ be the Jacobson radical of \mathcal{C} and $[\mathcal{C}']$ the ideal of \mathcal{C} consisting of morphisms which factor through some object in \mathcal{C}' . Thus $\mathcal{J}_\mathcal{C}(X, X)$ forms the Jacobson radical of the ring $\mathcal{C}(X, X)$ for any $X \in \mathcal{C}$.

2.1. Let \mathcal{C} be an additive category and \mathcal{C}' a subcategory of \mathcal{C} .

(1) \mathcal{C}' is called a *right rejective subcategory* of \mathcal{C} if the inclusion functor $\mathcal{C}' \rightarrow \mathcal{C}$ has a right adjoint $\mathbb{F} : \mathcal{C} \rightarrow \mathcal{C}'$ with a counit ϵ [HS] such that ϵ_X is a monomorphism for any $X \in \mathcal{C}$ (cf. [I1], 5.1). This is equivalent to that, for any $X \in \mathcal{C}$, there exists a monomorphism $g \in \mathcal{C}(Y, X)$ with $Y \in \mathcal{C}'$ which induces an isomorphism $\mathcal{C}(\ , Y) \xrightarrow{g} [\mathcal{C}'](\ , X)$ on \mathcal{C} (cf. [I1], 5.2).

(2) $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \dots \subseteq \mathcal{C}_0 = \mathcal{C}$ is called a *right rejective chain* if $\mathcal{J}_{\mathcal{C}_n / [\mathcal{C}_{n+1}]} = 0$ holds and \mathcal{C}_{n+1} is a right rejective subcategory of \mathcal{C}_n for any n ($0 \leq n < m$). In this case, if $\Gamma := \mathcal{C}(M, M)$ is an artin algebra for an additive generator M of \mathcal{C} , then Γ is a quasi-hereditary algebra with a heredity chain $0 = [\mathcal{C}_m](M, M) \subseteq [\mathcal{C}_{m-1}](M, M) \subseteq \dots \subseteq [\mathcal{C}_0](M, M) = \Gamma$.

Dually, we define a left rejective subcategory and a left rejective chain.

2.1.1. Let \mathcal{C}' be a right rejective subcategory of \mathcal{C} and \mathcal{C}'' a subcategory of \mathcal{C}' . Then $\mathcal{C}' / [\mathcal{C}'']$ is a right rejective subcategory of $\mathcal{C} / [\mathcal{C}'']$ since the isomorphism $\mathcal{C}(\ , \mathbb{F}(X)) \xrightarrow{\epsilon_X} [\mathcal{C}'](\ , X)$ induces an isomorphism $[\mathcal{C}''](\ , \mathbb{F}(X)) \xrightarrow{\epsilon_X} [\mathcal{C}''](\ , X)$. Moreover, if \mathcal{C}'' is a right rejective subcategory of \mathcal{C}' , then it is a right rejective subcategory of \mathcal{C} .

2.1.2. *Proof of 2.1(2).* \mathcal{C}_{m-1} is also a right rejective subcategory of \mathcal{C} by 2.1.1. Let \mathbb{F} be the right adjoint of the inclusion $\mathcal{C}_{m-1} \rightarrow \mathcal{C}$. Then $I := [\mathcal{C}_{m-1}](M, M)$ is isomorphic to a projective Γ -module $\mathcal{C}(M, \mathbb{F}(M))$, and $I\mathcal{J}_\Gamma I = 0$ holds by $\mathcal{J}_{\mathcal{C}_{m-1}} = 0$. Since $[\mathcal{C}_{m-1}]^2 = [\mathcal{C}_{m-1}]$ holds, I is a heredity ideal of Γ . Since $0 = \mathcal{C}_{m-1} / [\mathcal{C}_{m-1}] \subseteq \mathcal{C}_{m-2} / [\mathcal{C}_{m-1}] \subseteq \dots \subseteq \mathcal{C}_0 / [\mathcal{C}_{m-1}] = \mathcal{C} / [\mathcal{C}_{m-1}]$ is again a right rejective chain by 2.1.1, we obtain the assertion inductively. □

2.2. Our results 1.1 and 1.2 immediately follow from the following lemma (put $M := \Lambda \oplus \Lambda^*$ for 1.2).

Lemma. *Let Λ be an artin algebra and $M \in \text{mod } \Lambda$. Put $M_0 := M$, $M_{n+1} := M_n \mathcal{J}_{\text{End}_\Lambda(M_n)} \subsetneq M_n$ and take a large m such that $M_m = 0$. Then $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \dots \subseteq \mathcal{C}_0 = \mathcal{C}$ gives a right rejective chain for $\mathcal{C}_n := \text{add } \bigoplus_{l=n}^{m-1} M_l$. Thus $\Gamma := \text{End}_\Lambda(N)$ is a quasi-hereditary algebra for $N := \bigoplus_{l=0}^{m-1} M_l$ such that $\text{gl.dim } \Gamma \leq 2m - 2$.*

Proof. (i) Note that there exists a surjection $f_{n,l} \in \text{Hom}_\Lambda(\bigoplus M_n, M_l)$ for any $n < l$.
 (ii) Define a functor $\mathbb{F}_n : \text{mod } \Lambda \rightarrow \text{mod } \Lambda$ by

$$\mathbb{F}_n(X) := \sum_{Y \in \mathcal{C}_n, f \in \mathcal{J}_{\text{mod } \Lambda}(Y, X)} f(Y).$$

Then a natural transformation $\epsilon : \mathbb{F}_n \rightarrow 1$ is defined by the inclusion $\epsilon_X : \mathbb{F}_n(X) \rightarrow X$. By (i), $\mathbb{F}_n(M_n) = M_n \mathcal{J}_{\text{End}_\Lambda(M_n)} = M_{n+1} \in \mathcal{C}_{n+1}$ holds. Thus $\mathcal{J}_{\mathcal{C}_n}(\cdot, X) = [\mathcal{C}_{n+1}](\cdot, X) = \mathcal{C}_n(\cdot, \mathbb{F}_n(X))\epsilon_X$ holds on \mathcal{C}_n for any indecomposable $X \in \mathcal{C}_n - \mathcal{C}_{n+1}$.

(iii) Fix indecomposable $X \in \mathcal{C}_n$. Put $Y := \mathbb{F}_n(X)$ and $g := \epsilon_X$ if $X \notin \mathcal{C}_{n+1}$, and $Y := X$ and $g := 1_X$ if $X \in \mathcal{C}_{n+1}$. By (ii), $Y \in \mathcal{C}_{n+1}$ and $\mathcal{C}_n(\cdot, Y) \xrightarrow{g} [\mathcal{C}_{n+1}](\cdot, X)$ is an isomorphism on \mathcal{C}_n . Thus \mathcal{C}_{n+1} is a right rejective subcategory of \mathcal{C}_n . Since $\mathcal{J}_{\mathcal{C}_n/[\mathcal{C}_{n+1}]} = 0$ holds by (ii), our chain is right rejective. Now $\text{gl.dim } \Gamma \leq 2m - 2$ follows from [DR1]. □

2.3. *Remark.* (1) The dual version of 2.2 is the following lemma, which gives a variation of the theorem of Auslander and Dlab-Ringel in [A] and [DR2] by putting $M := \Lambda$.

Lemma. *Let Λ be an artin algebra and $M \in \text{mod } \Lambda$. Put $M_0 := M$, $M_{n+1} := M_n / \{x \in M_n \mid x \mathcal{J}_{\text{End}_\Lambda(M_n)} = 0\}$ and take a large m such that $M_m = 0$. Then $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \dots \subseteq \mathcal{C}_0 = \mathcal{C}$ gives a left rejective chain for $\mathcal{C}_n := \text{add } \bigoplus_{l=n}^{m-1} M_l$. Thus $\Gamma := \text{End}_\Lambda(N)$ is a quasi-hereditary algebra for $N := \bigoplus_{l=0}^{m-1} M_l$ such that $\text{gl.dim } \Gamma \leq 2m - 2$.*

(2) By a result of Igusa-Todorov ([IT], 0.8), $\text{rep.dim } \Lambda \leq 3$ implies $\text{fin.dim } \Lambda < \infty$. Thus, from the viewpoint of the finitistic global dimension conjecture, it is an interesting question whether any artin algebra Λ satisfies $\text{rep.dim } \Lambda \leq 3$ or not [A].

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