FINITENESS OF REPRESENTATION DIMENSION

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Abstract. We will show that any module over an artin algebra is a direct summand of some module whose endomorphism ring is quasi-hereditary. As a conclusion, any artin algebra has a finite representation dimension.

M. Auslander introduced a concept of representation dimension of artin algebras in \[A\], which was a trial to give a reasonable way of measuring homologically how far an artin algebra is from being of finite representation type (\[X1\], \[FGR\]). His methods given there have been effectively applied not only for the representation theory of artin algebras \[ARS\], but also for the theory of quasi-hereditary algebras of Cline-Parshall-Scott \[CPS\] by Dlab-Ringel in \[DR2\]. Unfortunately, much seems to be unknown about representation dimension itself. In particular, Reiten asked in 1998 whether any artin algebra has a finite representation dimension or not (cf. §2.3(2)). In this paper, we will give a positive answer to this question (§1.2) by showing that any module is a direct summand of some module whose endomorphism ring is quasi-hereditary (§1.1). Our method is to construct a certain chain of subcategories of \(\text{mod} \Lambda\) (§2.2), which was applied to solve Solomon’s second conjecture on zeta functions of orders in \[I3\]. We will formulate it in terms of rejective subcategories (§2.1), which was effectively applied in \[I1\] to study the representation theory of orders and give a characterization of their finite Auslander-Reiten quivers in \[I2\].

Note. After the author submitted this paper, Professor Xi kindly informed him that Theorem 1.1 and Corollary 1.2 were stated in \[X2\] as conjectures, where the former was given by Ringel and Yamagata. He thanks Professor Xi and Professor Yamagata for valuable comments.

1.

In this paper, any module is assumed to be a left module. For an artin algebra \(\Lambda\) over \(R\), let \(\text{mod} \Lambda\) be the category of finitely generated left \(\Lambda\)-modules, \(J_\Lambda\) the Jacobson radical of \(\Lambda\), \(\text{dom.dim} \Lambda\) the dominant dimension of \(\Lambda\) \([T]\), \(I_\Lambda(X)\) the injective hull of the \(\Lambda\)-module \(X\) and \((\ )^* := \text{Hom}_R(\ , I_R(R/J_R)) : \text{mod} \Lambda \leftrightarrow \text{mod} \Lambda^{op}\) the duality. For \(X \in \text{mod} \Lambda\), we denote by \(\text{add} X\) the full subcategory of \(\text{mod} \Lambda\) consisting of direct summands of a finite direct sum of \(X\). The representation dimension of \(\Lambda\) is defined by \(\text{rep.dim} \Lambda := \inf \{\text{gl.dim} \Gamma \mid \Gamma \in A(\Lambda)\}\), where \(A(\Lambda)\) is the collection of all artin algebras \(\Gamma\) such that \(\text{dom.dim} \Gamma \geq 2\) and \(\text{End}_\Gamma(I_\Gamma(\Gamma))\) is...
Morita-equivalent to $\Lambda$. Then $\text{rep}.\dim \Lambda = \inf \{ \text{gl}.\dim \text{End}_{\Lambda}(M) \mid M \in \text{mod} \Lambda \text{ such that } \Lambda \oplus \Lambda^* \in \text{add } M \}$ holds by [A].

1.1. Theorem. Let $\Lambda$ be an artin algebra. Then any $M \in \text{mod} \Lambda$ is a direct summand of some $N \in \text{mod} \Lambda$ such that $\text{End}_{\Lambda}(N)$ is a quasi-hereditary algebra.

1.2. Corollary. Let $\Lambda$ be an artin algebra. Then $\text{rep}.\dim \Lambda$ has a finite value which is not greater than $2l-2$, where $l$ is the length of a $(\Lambda, \text{End}_{\Lambda}(\Lambda \oplus \Lambda^*))$-module $\Lambda \oplus \Lambda^*$.

2. In the rest of this paper, any subcategory $C'$ of an additive category $C$ is assumed to be full and closed under direct sums. Let $J_C$ be the Jacobson radical of $C$ and $[C']$ the ideal of $C$ consisting of morphisms which factor through some object in $C'$. Thus $J_C(X, X)$ forms the Jacobson radical of the ring $C(X, X)$ for any $X \in C$.

2.1. Let $C$ be an additive category and $C'$ a subcategory of $C$.

1) $C'$ is called a right rejective subcategory of $C$ if the inclusion functor $C' \to C$ has a right adjoint $F : C \to C'$ with a counit $\epsilon \in \text{HS}$ such that $\epsilon_X$ is a monomorphism for any $X \in C$ (cf. [II], 5.1). This is equivalent to that, for any $X \in C$, there exists a monomorphism $g \in C(Y, X)$ with $Y \in C'$ which induces an isomorphism $C(-, Y) \xrightarrow{\sim} [C'](-, X)$ on $C$ (cf. [II], 5.2).

2) $0 = C_m \subseteq C_{m-1} \subseteq \cdots \subseteq C_0 = C$ is called a right rejective chain if $J_{C_n/[C_{n+1}]} = 0$ holds and $C_{n+1}$ is a right rejective subcategory of $C_n$ for any $n (0 \leq n < m)$. In this case, if $\Gamma := C(M, M)$ is an artin algebra for an additive generator $M$ of $C$, then $\Gamma$ is a quasi-hereditary algebra with a heredity chain $0 = [C_m](M, M) \subseteq \cdots \subseteq [C_0](M, M) = \Gamma$.

Dually, we define a left rejective subcategory and a left rejective chain.

2.1.1. Let $C'$ be a right rejective subcategory of $C$ and $C''$ a subcategory of $C'$. Then $C'/[C'']$ is a right rejective subcategory of $C/[C'']$ since the isomorphism $C(-, F(X)) \xrightarrow{\sim} [C'](-, X)$ induces an isomorphism $[C''](-, F(X)) \xrightarrow{\sim} [C''](-, X)$. Moreover, if $C''$ is a right rejective subcategory of $C'$, then it is a right rejective subcategory of $C$.

2.1.2. Proof of 2.1.2. $C_{m-1}$ is also a right rejective subcategory of $C$ by 2.1.1. Let $F$ be the right adjoint of the inclusion $C_{m-1} \to C$. Then $I := [C_{m-1}](M, M)$ is isomorphic to a projective $\Gamma$-module $C(M, F(M))$, and $IJ_{C_{m-1}} = 0$ holds by $J_{C_{m-1}} = 0$. Since $[C_{m-1}]/C_{m-1}$ holds, $I$ is a heredity ideal of $\Gamma$. Since $0 = C_{m-1}/[C_{m-1}] \subseteq C_{m-2}/[C_{m-1}] \subseteq \cdots \subseteq C_0/[C_{m-1}] = C/[C_{m-1}]$ is again a right rejective chain by 2.1.1, we obtain the assertion inductively.

2.2. Our results 1.1 and 1.2 immediately follow from the following lemma (put $M := \Lambda \oplus \Lambda^*$ for 1.2).

Lemma. Let $\Lambda$ be an artin algebra and $M \in \text{mod} \Lambda$. Put $M_0 := M$, $M_{n+1} := M_n \text{End}_{\Lambda}(M_n) \subseteq M_n$ and take a large $m$ such that $M_m = 0$. Then $0 = C_m \subseteq C_{m-1} \subseteq \cdots \subseteq C_0 = C$ gives a right rejective chain for $C_n := \text{add } \bigoplus_{i=0}^{m-1} M_i$. Thus $\Gamma := \text{End}_{\Lambda}(N)$ is a quasi-hereditary algebra for $N := \bigoplus_{i=0}^{m-1} M_i$ such that $\text{gl}.\dim \Gamma \leq 2m - 2$. 

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Thus \( \Gamma = \text{End} \) is an isomorphism on interesting question whether any artin algebra \( \Lambda \) satisfies \( \text{rep} \).

From the viewpoint of the finitistic global dimension conjecture, it is an

\[ X \]

Remark.

2.3.

Remark. (1) The dual version of 2.2 is the following lemma, which gives a variation of the theorem of Auslander and Dlab-Ringel in \([A]\) and \([DR2]\) by putting \( M := \Lambda \).

Lemma. Let \( \Lambda \) be an artin algebra and \( M \in \text{mod} \Lambda \). Put \( M_0 := M, M_{n+1} := M_n/M_{n}/x \in M_n \mid x_{\text{End} \Lambda(M_n)} = 0 \) and take a large \( m \) such that \( M_m = 0 \). Then

\[ 0 = C_m \subseteq C_{m-1} \subseteq \cdots \subseteq C = \mathcal{G} \]

gives a left rejective chain for \( \mathcal{C} := \text{add} \oplus_{l=n}^{m-1} M_l \).
Thus \( \Gamma := \text{End} \Lambda(N) \) is a quasi-hereditary algebra for \( N := \oplus_{l=n}^{m-1} M_l \) such that

\[ \text{gl.dim} \Gamma \leq 2m - 2. \]

(2) By a result of Igusa-Todorov (\([IT]\), 0.8), \( \text{rep.dim} \Lambda \leq 3 \) implies \( \text{fin.dim} \Lambda < \infty \).
Thus, from the viewpoint of the finitistic global dimension conjecture, it is an interesting question whether any artin algebra \( \Lambda \) satisfies \( \text{rep.dim} \Lambda \leq 3 \) or not \([A]\).

References


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