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SUBSYSTEMS OF THE WALSH ORTHOGONAL SYSTEM WHOSE MULTIPLICATIVE COMPLETIONS ARE QUASIBASES FOR $L^p[0,1]$, $1 \leq p < +\infty$

M. G. GRIGORIAN AND ROBERT E. ZINK

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Abstract. If one discards some of the elements from the Walsh family, an ancient example of a system that serves as a Schauder basis for each of the L^p-spaces, with $1 < p < +\infty$, the residual system will not be a Schauder basis for any of those spaces. Nevertheless, Price has shown that each member of a large class of such subsystems is complete on subsets of $[0,1]$ that have measure arbitrarily close to 1. In the present work, it is shown that subsystems of this kind can be multiplicatively completed in such a way that the resulting systems are quasibases for each space $L^p[0,1]$, $1 \leq p < +\infty$, from which the earlier completeness result follows as a corollary.

1. In one of the earliest monographs treating the subject, Paley [9] showed that the Walsh system is a Schauder basis for each space $L^p[0,1], 1 < p < +\infty$. If some of the Walsh functions are deleted from the system, the residual family will no longer be a Schauder basis for any of these spaces, of course, but, as Price [10] has shown, the subfamily may be complete on a set of large measure.

In the present work, it is shown that subsystems of the type considered in [10] can be multiplicatively completed in such a way that the new systems are, in fact, quasibases for every space $L^p[0,1]$, with $1 \leq p < +\infty$. It thus follows that for each such Walsh subfamily, for each such p, and for every $\epsilon > 0$, there is a measurable subset, E_ϵ, of $[0,1]$, such that the subfamily is closed in $L^p(E_\epsilon)$, and $|E_\epsilon| > 1 - \epsilon$.

2. Let B be a Banach space, let B^* be the associated conjugate space, and let $\Phi = \{\varphi_n : n \in \mathbb{N}\}$ be a subset of B. Then Φ is a quasibasis for B, if there is a corresponding subset of B^*, $\{\psi_n^* : n \in \mathbb{N}\}$, such that for every $f \in B$, $\sum_{n=1}^{\infty} \psi_n^*(f)\varphi_n$ converges to f in the norm of B. This notion, introduced by Gelbaum [3] in 1958, is a weaker concept than that of a Schauder basis, since the sequence of coefficient functionals need not be unique [12, p. 278].

The Walsh system, an extension of the Rademacher system, may be obtained in the following manner.

Let r be the periodic function, of least period 1, defined on $[0,1)$ by

$$r = \chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}.$$

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The Rademacher system, $\mathcal{R} = \{r_n : n = 0, 1, \dots\}$, is defined by the conditions

$$r_n(x) = r(2^n x), \forall x \in \mathbb{R}, n = 0, 1, \dots,$$

and, in the ordering employed by Paley, the n^{th} element of the Walsh system, $\mathcal{W} = \{W_n : n = 0, 1, \dots\}$ is given by

$$W_n = \prod_{k=0}^{\infty} r_k^{n_k},$$

where $\sum_{k=0}^{\infty} n_k 2^k$ is the unique binary expansion of n, with each n_k either 0 or 1.

One shows that the Walsh subsystems introduced by Price can be multiplicatively completed so as to become quasibases, by constructing appropriate sequences of coefficient functionals. For this purpose one employs the Haar system $\{h_n : n \in \mathbb{N}\}$.

In the standard notation, one has $h_1 = \mathrm{id}_{[0,1]}$,

$$h_k^{(j)} = 2^{k/2}\left(\chi_{\left(\frac{2j-2}{2^{k+1}}, \frac{2j-1}{2^{k+1}}\right)} - \chi_{\left(\frac{2j-1}{2^{k+1}}, \frac{2j}{2^{k+1}}\right)}\right), \quad k = 0, 1, \dots; j = 1, \dots, 2^k;$$

and, for $n = 2^k + j$, $h_n = h_k^{(j)}$.

3. In his AMS Colloquium Lectures, Levinson [8] proved the following completeness theorem for families of exponential functions.

Let $S = \{n_j\}_{j=1}^{\infty}$ be an increasing sequence of natural numbers, let $\Lambda(n)$ be the number of elements of S that are less than n, and let

$$D(S) = \limsup_{\xi \to 1^-} \limsup_{n} \frac{\Lambda(n) - \Lambda(\xi n)}{(n - \xi n)}.$$

If $D(S) = 1$, then, for every $\epsilon > 0$, there is a measurable set $E_\epsilon \subset [0, 1]$ such that $|E_\epsilon| > 1 - \epsilon$ and $\left\{e^{2\pi i n_j(.)} : j \in \mathbb{N}\right\}$ is complete in $L^2(E_\epsilon)$.

The corresponding result for the Walsh functions involves a different set function which yields a weaker notion of density; viz., for $S \subset \mathbb{N}$, let

$$\rho(S) = \limsup_{k} \limsup_{n} \frac{\Lambda(n + k) - \Lambda(n)}{k}.$$

One always has $\rho(S) \geq D(S)$, and equality need not hold.

Theorem A (Price). *Let $S = \{n_j\}_{j=1}^{\infty}$ be an increasing sequence of natural numbers, and let $\mathcal{W}_S = \{W_{n_j} : j \in \mathbb{N}\}$. If $\rho(S) = 1$, then, for every $\epsilon > 0$, there exists a measurable set $E_\epsilon \subset [0, 1]$ such that $|E_\epsilon| > 1 - \epsilon$ and $\mathcal{W}_S$ is complete in $L^2(E_\epsilon)$.*

At the time of this writing, it was not known whether it would be possible to replace $D(S)$ by $\rho(S)$ in Levinson's theorem.

The equivalence of the notions of completeness on a set of large measure and multiplicative completability was demonstrated in [11].

Theorem B. *Let E be a measurable set of finite, positive measure, and let $\Phi = \{\varphi_n : n = 1, 2, \dots\}$ be a subset of $L^2(E)$. The following conditions are equivalent:*

(BP) *There exists a bounded measurable function, m, such that $\{m\varphi_n : n = 1, 2, \dots\}$ is complete in $L^2(E)$;*

(M) *Φ is complete in measure on E;*

(T) *For every positive ϵ, there exists $E_\epsilon \subset E$ such that $|E_\epsilon| > |E| - \epsilon$, and Φ is complete in $L^2(E_\epsilon)$.*

The notion of multiplicative completion is due to Boas and Pollard [1]; the equivalence $(M) \leftrightarrow (T)$ is due to Talalyan [13], [14] who also showed that if Φ has these properties, then so also does every family obtained from Φ by deleting a finite number of its members. Subsequently, Goffman and Waterman [4] gave a new proof of the latter result of Talalyan and observed that it is always possible to make certain infinite deletions from a system that satisfies (T) so as to leave a residual system that also enjoys this property.

Although it was not shown there, adaptations of the arguments given in [11] can be used to establish a companion theorem to Theorem B in which the rôle of L^2 is played by any space L^p, with $1 \leq p < +\infty$. It follows that any system that is multiplicatively completable in $L^p(E)$ is closed in $L^p(E_\epsilon)$, for some sets E_ϵ with $|E_\epsilon| > |E| - \epsilon$, for every $\epsilon > 0$.

Moreover, Braun [2] has extended the work of Boas and Pollard in the following manner.

Theorem C. *Let E be a measurable set of finite, positive measure, and let $\Phi = \{\varphi_n : n = 1, 2, \ldots\}$ be a Schauder basis for some space $L^p(E)$, with $1 \leq p < +\infty$. Then, to every natural number N there corresponds a bounded measurable function, M, such that every element f of $L^p(E)$ can be represented by a series $\sum_{k=N+1}^{\infty} a_k M\varphi_k$ that converges to f in the L^p-norm.*

In other terminology, Theorem C asserts that the systems $\{M\varphi_k : k = N+1, \ldots\}$ are systems of representation for $L^p(E)$. Neither does the theorem claim nor are the arguments used to establish it sufficient to show that these systems are quasibases for $L^p(E)$; nevertheless, subsequent analysis of the problem has shown this to be the case [7].

4. Theorem. *Let $S = \{n_j\}_{j=1}^{\infty}$ be an increasing sequence of natural numbers such that $\rho(S) = 1$, and let $\mathcal{W}_S = \{W_{n_j} : j \in \mathbb{N}\}$ be the corresponding subsystem of the Walsh system. Then, there exists a bounded, measurable function, M, such that $\{MW_{n_j} : j \in \mathbb{N}\}$ is a quasibasis for each space $L^p[0,1], 1 \leq p < +\infty$.*

The demonstration of the theorem depends upon a proposition of Menshov-Talalyan type (see, for example, [14]), to which the following lemmata lead.

Lemma 1. *Let $S = \{n_j\}_{j=1}^{\infty}$ be an increasing sequence of natural numbers. The following are equivalent assertions:*

(α) $\rho(S) = 1$;

(β) *There exists an increasing sequence of natural numbers, $\{M_k\}_{k=1}^{\infty}$, such that*

$$\lim_j (M_{2j} - M_{2j-1}) = +\infty\,,$$

and, for every j, each integer in $[M_{2j-1}, M_{2j}]$ is an element of S.

Proof. In his demonstration of the completeness in measure of a Walsh subsystem $\mathcal{W}_S$, with $\rho(S) = 1$, Price showed that, for every natural number p, there is an increasing sequence of natural numbers, $\{j_i\}_{i=1}^{\infty}$, such that S contains all of the integers in $[j_i 2^p, (j_i + 1)2^p]$, $\forall i$; thus, $(\alpha) \rightarrow (\beta)$. The other implication is even simpler.

Lemma 2. *Let $S = \{n_k\}_{k=1}^{\infty}$ be an increasing sequence of natural numbers, with $\rho(S) = 1$, and let $\{M_k\}_{k=1}^{\infty}$ be a corresponding sequence, of the type guaranteed by Lemma 1; let Δ_0 be an interval of the form $[\frac{i-1}{2^m}, \frac{i}{2^m}]$, for some $m \in \mathbb{N}$ and*

$1 \le i \le 2^m$; let k_0 be a natural number greater than 1; and let the real numbers ϵ, p, and γ satisfy the conditions $0 < \epsilon < 1$, $p \ge 1$, and $\gamma \ne 0$. Then, there exist a measurable set $E \subset \Delta_0$, and a Walsh polynomial

$$Q = \sum_{k=k_0}^{K} c_{n_k} W_{n_k} \in \operatorname{Span} \mathcal{W}_S$$

such that

$$Q(x) = \begin{cases} \gamma, & \text{if } x \in E; \\ 0, & \text{if } x \notin \Delta_0; \end{cases}$$

$$|E| > (1-\epsilon)|\Delta_0|;$$

and

$$\max\left\{ \left[\int_0^1 \Big| \sum_{k=k_0}^{m} c_{n_k} W_{n_k} \Big|^p dt \right]^{1/p} : k_0 \le m \le K \right\} \le \begin{cases} 4|\gamma|\epsilon^{-1/2} A_2 |\Delta_0|^{1/2}, & \text{if } p = 1, \\ 4|\gamma|\epsilon^{-1/q} A_p |\Delta_0|^{1/p}, & \text{if } p > 1, \end{cases}$$

where A_p is a constant depending only upon p, and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $\nu = 1 + [\log_2(1/\epsilon)]$, where $[x]$ denotes the greatest integer less than or equal to x. Because Δ_0 is a dyadic interval, there is a natural number μ_1 such that

$$S_{\mu_1}(\gamma\chi_{\Delta_0}) = \sum_{j=0}^{\mu_1} a_j W_j = \gamma\chi_{\Delta_0}\,,$$

where the a_j are the Walsh-Fourier coefficients and $S_n f$ denotes the n^{th} partial sum of the Walsh-Fourier series.

Suppose that the greatest power of 2 that appears in the binary expansion of μ_1 is 2^l. If m is any positive multiple of 2^{l+1}, then, for every $j \in [0, \mu_1]$, one has $W_{m+j} = W_m W_j$. From this observation and from the definition of the sequence $\{M_k\}_{k=1}^{\infty}$, there follows the existence of natural numbers k_1 and $m_1 > \mu_1$ such that, for all $j \in [0, \mu_1]$, both

$$W_{m_1+j} = W_{m_1} W_j$$

and

$$m_1 + j \in [M_{2k_1-1}, M_{2k_1}]\ .$$

Setting

$$\Delta_0^{(-)} = \{x \in \Delta_0 : W_{m_1}(x) = -1\}\,,$$
$$\Delta_0^{(+)} = \{x \in \Delta_0 : W_{m_1}(x) = +1\}\,,$$

and

$$b_t^{(1)} = \begin{cases} a_j, & \text{if } t = m_1 + j, \text{ for some } j \in [0, \mu_1]; \\ 0, & \text{otherwise}; \end{cases}$$

one has

$$Q_1 =: \sum_{t=M_{2k_1-1}}^{M_{2k_1}} b_t^{(1)} W_t = \sum_{j=0}^{\mu_1} a_j W_{m_1+j}$$
$$= \left(\sum_{j=0}^{\mu_1} a_j W_j \right) W_{m_1}$$
$$= \gamma \chi_{\Delta_0^{(+)}} - \gamma \chi_{\Delta_0^{(-)}} \ ,$$

and, since W_{m_1} is orthogonal to $\sum_{j=0}^{\mu_1} a_j W_j$,

$$|\Delta_0^{(-)}| = |\Delta_0^{(+)}| = \frac{1}{2} |\Delta_0| \, .$$

Let $\Delta_1 = \Delta_0^{(-)}$. Then Δ_1 is a finite union of congruent, dyadic intervals; thus, there exists a natural number μ_2 such that

$$S_{\mu_2}(2\gamma \chi_{\Delta_1}) = \sum_{j=0}^{\mu_2} a_j^{(2)} W_j = 2\gamma \chi_{\Delta_1},$$

where the $a_j^{(2)}$ are the Walsh-Fourier coefficients.

Choose natural numbers k_2 and m_2 such that $k_2 > k_1$, $m_2 > 2 \max\{m_1, \mu_2\}$, and, $\forall j \in [0, \mu_2]$, both

$$W_{m_2+j} = W_{m_2} W_j$$

and

$$m_2 + j \in [M_{2k_2-1}, M_{2k_2}].$$

Setting

$$b_t^{(2)} = \begin{cases} a_j^{(2)}, & \text{if } t = m_2 + j, \text{ and } \ j \in [0, \mu_2]; \\ 0, & \text{otherwise}; \end{cases}$$

$$\Delta_1^{(-)} = \{ x \in \Delta_1 : W_{m_2}(x) = -1 \} \, ;$$

and

$$\Delta_1^{(+)} = \{ x \in \Delta_1 : W_{m_2}(x) = +1 \} \, ;$$

one has

$$Q_2 =: \sum_{t=M_{2k_2-1}}^{M_{2k_2}} b_t^{(2)} W_t = \sum_{j=0}^{\mu_2} a_j^{(2)} W_{m_2+j}$$
$$= \left(\sum_{j=0}^{\mu_2} a_j^{(2)} W_j \right) W_{m_2}$$
$$= 2\gamma \chi_{\Delta_1} W_{m_2} = 2\gamma \left(\chi_{\Delta_1^{(+)}} - \chi_{\Delta_1^{(-)}} \right),$$

so that

$$Q_1 + Q_2 = \begin{cases} \gamma, & \text{on } \Delta_0^{(+)} \cup \Delta_1^{(+)}, \\ -3\gamma, & \text{on } \Delta_1^{(-)}, \\ 0, & \text{otherwise }, \end{cases}$$

and, because $m_2 > 2m_1$,

$$|\Delta_1^{(+)}| = |\Delta_1^{(-)}| = \frac{1}{2}|\Delta_1| = \frac{1}{4}|\Delta_0|.$$

Proceeding inductively, suppose that, for all $i \in [1, \nu - 1]$, one has found natural numbers m_i and k_i, Walsh polynomials Q_i, and sets Δ_i, such that, $\forall\, i \in [1, \nu - 1]$,

$$m_i \in [M_{2k_i - 1}, M_{2k_i}],$$

$$m_{i+1} > 2m_i \quad \text{and} \quad k_{i+1} > k_i,$$

$$\Delta_{i-1}^{(+)} = \{x \in \Delta_{i-1} : W_{m_i}(x) = +1\} \text{ and } \Delta_{i-1}^{(-)} = \{x \in \Delta_{i-1} : W_{m_i}(x) = -1\},$$

$$Q_i = \sum_{t=M_{2k_i-1}}^{M_{2k_i}} b_t^{(i)} W_t = 2^{i-1}\gamma\chi_{\Delta_{i-1}} W_{m_i},$$

$$\Delta_i = \Delta_{i-1}^{(-)},$$

and

$$|\Delta_i^{(+)}| = |\Delta_i^{(-)}| = \frac{1}{2}|\Delta_i| = \frac{1}{2^{i+1}}|\Delta_0|, \text{ for } 0 \le i < \nu - 1.$$

Let $\Delta_{\nu-1} = \Delta_{\nu-2}^{(-)}$. As in the earlier work, there is a Walsh polynomial

$$P_\nu = \sum_{j=0}^{\mu_\nu} a_j^{(\nu)} W_j,$$

such that

$$P_\nu = 2^{\nu-1}\gamma\chi_{\Delta_{\nu-1}}.$$

Choose natural numbers $k_\nu > k_{\nu-1}$ and $m_\nu > 2\max\{m_{\nu-1}, \mu_\nu\}$ so that both

$$W_{m_\nu + j} = W_{m_\nu} W_j$$

and

$$m_\nu + j \in [M_{2k_\nu - 1}, M_{2k_\nu}], \quad \forall j \in [0, \mu_\nu].$$

Let

$$b_t^{(\nu)} = \begin{cases} a_j^{(\nu)}, & \text{if } t = m_\nu + j, \text{ and } j \in [0, \mu_\nu]; \\ 0, & \text{otherwise}; \end{cases}$$

and let

$$Q_\nu = \sum_{t=M_{2k_\nu - 1}}^{M_{2k_\nu}} b_t^{(\nu)} W_t.$$

Then

$$Q_\nu = P_\nu W_{m_\nu} = 2^{\nu-1}\gamma\{\chi_{\Delta_{\nu-1}^{(+)}} - \chi_{\Delta_{\nu-1}^{(-)}}\},$$

where

$$\Delta_{\nu-1}^{(+)}(\Delta_{\nu-1}^{(-)}) = \{x \in \Delta_{\nu-1} : W_{m_\nu}(x) = +1(-1)\}.$$

Finally, let

$$c_t = \begin{cases} b_t^{(i)}, & \text{if } M_{2k_i-1} \leq t \leq M_{2k_i}, 1 \leq i \leq \nu; \\ 0, & \text{otherwise ;} \end{cases}$$

in the ordering of $\mathcal{W}_S$, let $n_{k_0} = M_{2k_1-1}$ and $n_K = M_{2k_\nu}$; let

$$Q = \sum_{i=1}^{\nu} Q_i = \sum_{k=k_0}^{K} c_{n_k} W_{n_k},$$

and let

$$E = \Delta_0 \diagdown \Delta_{\nu-1}^{(-)}.$$

Then

$$Q = \gamma\chi_{\Delta_0} - 2^\nu \gamma \chi_{\Delta_{\nu-1}^{(-)}},$$

so that

$$Q(x) = \begin{cases} \gamma, & \text{if } x \in E; \\ 0, & \text{if } x \notin \Delta_0; \end{cases}$$

and

$$|E| = (1 - 2^{-\nu})|\Delta_0| < \epsilon.$$

Thus, for $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \|Q\|_p &= \left(\int_0^1 |Q(t)|^p dt \right)^{1/p} \\ &\leq |\gamma||\Delta_0|^{1/p} + 2^\nu |\gamma||\Delta_{\nu-1}^{(-)}|^{1/p} = |\gamma||\Delta_0|^{1/p}(1 + 2^{\nu/q}) \\ &\leq |\gamma||\Delta_0|^{1/p}(1 + 2^{1/q}\epsilon^{-1/q}) < |\gamma||\Delta_0|^{1/p} 2^{1+1/q}\epsilon^{-1/q} \\ &< 4|\gamma||\Delta_0|^{1/p}\epsilon^{-1/q}. \end{aligned}$$

Since Q is a Walsh polynomial, it is its own Walsh-Fourier series; that is,

$$c_t = \int_0^1 Q(x) W_t(x) dx, \quad \forall t;$$

thus, from Paley's theorem, $\|S_n Q\|_p \leq A_p \|Q\|_p$, $\forall n \in \mathbb{N}$ and $p > 1$, where A_p is a constant depending only upon p. Hence

$$\max \left\{ \| \sum_{k=k_0}^{m} c_{n_k} W_{n_k} \|_p : k_0 \leq m \leq K \right\} \leq 4 A_p \epsilon^{-1/q} |\gamma||\Delta_0|^{1/p}, \forall p > 1,$$

and

$$\begin{aligned} \max \left\{ \| \sum_{k=k_0}^{m} c_{n_k} W_{n_k} \|_1 : k_0 \leq m \leq K \right\} &\leq \max \left\{ \| \sum_{k=k_0}^{m} c_{n_k} W_{n_k} \|_2 : k_0 \leq m \leq K \right\} \\ &\leq 4 A_2 \epsilon^{-1/2} |\gamma||\Delta_0|^{1/2}. \end{aligned}$$

Lemma 3. *Let f be a (dyadic) step function on $[0,1]$,*

$$f = \sum_{i=1}^{2^n} \gamma_i \chi_{\Delta_i},$$

where

$$\Delta_i = \left(\frac{i-1}{2^n}, \frac{i}{2^n}\right), i = 1, \dots, 2^n,$$

and let $\{W_{n_k} : k \in \mathbb{N}\}$ be a subsystem of the Walsh system that satisfies the conditions of the Theorem. To each $p \geq 1$, each $\epsilon \in (0,1)$, and each $K_0 \in \mathbb{N}$, there corresponds a Walsh polynomial

$$Q = \sum_{k=K_0+1}^{K} c_{n_k} W_{n_k}$$

and a measurable set $E \subset [0,1]$ such that

$$|E| > 1 - \epsilon,$$

$$Q(x) = f(x), \ \forall x \in E,$$

and

$$\max\left\{ \| \sum_{k=K_0+1}^{m} c_{n_k} W_{n_k} \|_{L^p(e)} : K_0 < m \leq K \right\} \leq \epsilon + \|f\|_{L^p(e)},$$

for every measurable set $e \subset E$.

Proof. Without loss of generality, one may assume that, for all $i \in [1, 2^n]$,

$$4|\gamma_i|\{A_2 \epsilon^{-1/2} |\Delta_i|^{1/2} + A_p \epsilon^{-1/q} |\Delta_i|^{1/p}\} < \epsilon, \text{ for } p > 1, \frac{1}{p} + \frac{1}{q} = 1,$$

since, if necessary, one could refine the dyadic partition upon which f is defined.

Successive applications of Lemma 2 yield measurable sets $E_i \subset \Delta_i$ and Walsh polynomials

$$Q_i = \sum_{k=K_{i-1}+1}^{K_i} c_{n_k}^{(i)} W_{n_k},$$

such that for all $i = 1, \dots, 2^n$,

$$Q_i(x) = \begin{cases} \gamma_i, & \text{if } x \in E_i; \\ 0, & \text{if } x \notin \Delta_i; \end{cases}$$

$$|E_i| > (1-\epsilon)|\Delta_i|;$$

and

$$\max\left\{ \| \sum_{k=K_{i-1}+1}^{j} c_{n_k} W_{n_k} \|_p : K_{i-1} + 1 \leq j \leq K_i \right\}$$
$$\leq \begin{cases} 4|\gamma_i| A_p \epsilon^{-1/q} |\Delta_i|^{1/p}, & \text{for } p > 1; \\ 4|\gamma_i| A_2 \epsilon^{-1/2} |\Delta_i|^{1/2}, & \text{for } p = 1\ . \end{cases}$$

Let $E = \bigcup_{i=1}^{2^n} E_i$, let $K = K_{2^n}$, and let

$$Q = \sum_{i=1}^{2^n} \sum_{k=K_{i-1}+1}^{K_i} c_{n_k}^{(i)} W_{n_k} = \sum_{k=K_0+1}^{K} c_{n_k} W_{n_k},$$

where

$$c_{n_k} = c_{n_k}^{(i)}, \text{ for } K_{i-1}+1 \le k \le K_i, \ i = 1, \dots, 2^n.$$

Then

$$|E| > 1 - \epsilon$$

and

$$Q(x) = f(x), \quad \forall\, x \in E.$$

To each $j \in [K_0+1, K]$ there corresponds a unique i_j such that $1 \le i_j \le 2^n$ and

$$K_{i_j-1} + 1 \le j \le K_{i_j};$$

thus,

$$\sum_{k=K_0+1}^{j} c_{n_k} W_{n_k} = \sum_{i=1}^{i_j-1} Q_i + \sum_{k=K_{i_j-1}+1}^{j} c_{n_k} W_{n_k}.$$

Since Q coincides with f on E, one has for each measurable set $e \subset E$,

$$\begin{aligned}\left(\int_e |\sum_{k=K_0+1}^{j} c_{n_k} W_{n_k}|^p dx\right)^{1/p} &\le \left(\int_e |\sum_{i=1}^{i_j-1} Q_i|^p dx\right)^{1/p} \\ &\quad + \left(\int_0^1 |\sum_{k=K_{i_j-1}+1}^{j} c_{n_k} W_{n_k}|^p dx\right)^{1/p} \\ &\le \left(\int_e |f|^p dx\right)^{1/p} + \epsilon, \quad \forall\, p \ge 1.\end{aligned}$$

To complete the proof of the theorem, one employs the methods devised by Braun.

Choose $p > 1$, and let $\{h_n : n = 1, 2, \dots\}$ be the Haar system. For each n, let $f_n = h_n / \|h_n\|_p$, and let $\{g_n : n = 1, 2, \dots\}$ be the family of coefficient functionals associated with $\{f_n : n = 1, 2, \dots\}$. By virtue of Lemma 3, one may, following Braun, construct a measurable function M, with $0 \le M(t) \le 1$, for all t in $[0, 1]$, and a double sequence of $\mathcal{W}_S$-polynomials $\{P_{kj}\}_{k=1, j=k}^{\infty\ \ \infty}$,

$$P_{kj} = \sum_{i=n_{k-1}(j)+1}^{n_k(j)} a_i W_{n_i},$$

with

$$\begin{aligned}N = n_0(1) < n_1(1) = &n_0(2) < n_1(2) < n_2(2) = n_0(3) < \dots \\ &n_0(j) < n_1(j) < \cdots < n_j(j) = n_0(j+1) < \dots,\end{aligned}$$

and a sequence $\{E_n\}_{n=1}^{\infty}$ of measurable subsets of $[0, 1]$, such that

(i) $|[0, 1] \setminus E_n| < \delta_n$, $\{\delta_n\}_{n=1}^{\infty} \downarrow 0$;

(ii) $\|f_k - M \sum_{j=k}^{\ell} P_{kj}\|_p \le 2^{-\ell}$, $\forall\, k$, $\forall \ell \ge k$;

(iii) $\sup\{\|M\sum_{i=n_{k-1}(\ell)+1}^{s} a_iW_{n_i}\|_p : s \le n_k(\ell)\} \le 2^{-\ell+2}$, if $\ell > k$;
(iv) $\sup\{\|M\sum_{i=n_{\ell-1}(\ell)+1}^{s} a_iW_{n_i}\|_{L^p([0,1]\setminus E_\ell)} : s \le n_\ell(\ell)\} \le 2^{-\ell+2}$;

and, for every measurable set $e \subset E_\ell$,

(v) $\sup\{\|M\sum_{i=n_{\ell-1}(\ell)+1}^{s} a_iW_{n_i}\|_{L^p(e)} : s \le n_\ell(\ell)\} \le 2^{-\ell+1} + \|f_\ell\|_{L^p(e)}$.

One associates with $\Phi = \{MW_{n_i} : i > N\}$ the system $\Psi = \{\psi_i : i > N\}$, where

$$\psi_i = a_ig_k, \text{ if } n_{k-1}(\ell) < i \le n_k(\ell), \text{ for some } \ell \ge k.$$

Then, with the coefficient functionals defined on $L^r[0,1]$ by setting

$$b_i(\cdot) = \int_0^1 (\cdot)\psi_i dt, \text{ for } i = N+1, N+2, \dots,$$

one finds that Φ is a quasibasis for each space $L^r[0,1]$, with $1 \le r \le p$.

The abbreviated demonstration below essentially duplicates the argument employed in [7, 437ff].

Let $r \in [1,p]$, let f be an arbitrary element of $L^r[0,1]$, let $S_n(f)$ be the nth partial sum of the series

$$\sum_{i=N+1}^{n} b_i(f)MW_{n_i},$$

and let $\sum_{k=1}^{\infty} c_kf_k$ be the expansion of f in the p-normalized Haar system.

The estimation of $\|f - S_n(f)\|_r$ depends upon two critical properties of the sequence $\{c_k\}_{k=1}^{\infty}$:

$$(\alpha) \qquad |c_k| \le k\|f\|_r, \qquad \forall k \in \mathbb{N},$$

since

$$|c_k| = |c_k(f)| = |\int_0^1 fg_kdt| \le \|f\|_r\|g_k\|_{r'}, \quad \frac{1}{r} + \frac{1}{r'} = 1,$$

and

$$\begin{aligned}\|g_k\|_{r'} = \|h_k\|_p\|h_k\|_{r'} &= |h_k\|_\infty^2|\Delta_k|^{\frac{1}{p}+\frac{1}{r'}}, \text{ where } |\Delta_k| \text{ is the support of } h_k,\\ &= |\Delta_k|^{\frac{1}{p}+\frac{1}{r'}-1} \le k^{1-(\frac{1}{p}+\frac{1}{r'})} \le k;\end{aligned}$$

and

$$(\beta) \qquad \lim_k |c_k|/\|g_k\|_{r'} = 0,$$

since

$$\begin{aligned}|c_k| = |c_k(f)| = |\int_0^1 fg_kdt| &= |\int_0^1 (f - \sum_{j=1}^{k-1} c_jf_j)g_kdt|\\ &\le \|f - \sum_{j=1}^{k-1} c_jf_j\|_r\|g_k\|_{r'},\end{aligned}$$

and since $\{f_n : n = 1, 2, \dots\}$ is a Schauder basis for $L^r[0,1]$ (as well as for $L^p[0,1]$).

If $n = n_k(\ell)$, for some natural numbers k and ℓ (with $\ell \geq k$), then, for $k < \ell$,

$$\begin{aligned}\|f - S_n(f)\|_r &= \|f - \sum_{j=1}^{k}\sum_{i=j}^{\ell} c_j MP_{ji} - \sum_{j=k+1}^{\ell-1}\sum_{i=j}^{\ell-1} c_j MP_{ji}\|_r \\ &\leq \|f - \sum_{j=1}^{\ell-1} c_j f_j\|_r + \sum_{j=1}^{k} |c_j|\,\|f_j - \sum_{i=j}^{\ell} MP_{ji}\|_r \\ &\quad + \sum_{j=k+1}^{\ell-1} |c_j|\,\|f_j - \sum_{i=j}^{\ell-1} MP_{ji}\|_r, \\ &\leq \|f - \sum_{j=1}^{\ell-1} c_j f_j\|_r + 2^{-\ell+1}\sum_{j=1}^{\ell-1}|c_j|,\end{aligned}$$

and, if $k = \ell$, then a similar estimate yields

$$\|f - S_n(f)\|_r \leq \|f - \sum_{j=1}^{\ell} c_j f_j\|_r + 2^{-\ell}\sum_{j=1}^{\ell}|c_j|.$$

From (α), one has

$$\sum_{j=1}^{\ell}|c_j| \leq (\ell(\ell+1)/2)\|f\|_r$$

so that

$$\|f - S_n(f)\|_r \leq \|f - \sum_{j=1}^{n-1} c_j f_j\|_r + 2^{-\ell}\ell(\ell+1), \text{ for } n = n_k(\ell).$$

Finally, if $n_{k-1}(\ell) < n < n_k(\ell)$ and $k < \ell$, then

$$\begin{aligned}|f - S_n(f)\|_r &\leq |f - \sum_{j=1}^{\ell-1} c_j f_j\|_r + 2^{-\ell}\ell(\ell+1)\|f\|_r + \|c_k \sum_{i=n_{k-1}(\ell)+1}^{n} a_i MW_{n_i}\|_p \\ &\leq \|f - \sum_{j=1}^{\ell-1} c_j f_j\|_r + 2^{-\ell}[\ell(\ell+1)\|f\|_r + 4k].\end{aligned}$$

On the other hand, if $k = \ell$, then

$$\|f - S_n(f)\|_r \leq \|f - \sum_{j=1}^{\ell-1} c_j f_j\|_r + 2^{-\ell}\ell(\ell+1)\|f\|_r + \|c_\ell \sum_{i=n_{\ell-1}(\ell)+1}^{n} a_i MW_{n_i}\|_r\ .$$

Setting

$$\sigma_{\ell n} = \sum_{i=n_{\ell-1}(\ell)+1}^{n} a_i MW_{n_i},$$

one has

$$\begin{aligned}\|\sigma_{\ell n}\|_r^r &= \|\sigma_{\ell n}\|_{L^r([0,1]\setminus E_\ell)}^r + \|\sigma_{\ell n}\|_{L^r(E_\ell)}^r \\ &\leq 2^{(-\ell+2)r} + \|\sigma_{\ell n}\|_{L^r(E_\ell\cap\Delta_\ell)}^r + \|\sigma_{\ell n}\|_{L^r(E_\ell\setminus\Delta_\ell)}^r,\end{aligned}$$

by virtue of condition (iv), and, from condition (v), follow

$$\begin{aligned}\|\sigma_{\ell n}\|^r_{L^r(E_\ell\setminus\Delta_t)} &\le \|\sigma_{\ell n}\|^r_{L^p(E_\ell\setminus\Delta_\ell)}\\ &\le (2^{-\ell+1}+\|f_\ell\|_{L^p(E_\ell\setminus\Delta_\ell)})^r = 2^{(-\ell+1)r}\end{aligned}$$

and

$$\begin{aligned}\|\sigma_{\ell n}\|^r_{L^r(E_\ell\cap\Delta_\ell)} &\le \|\sigma_{\ell n}\|^r_{L^p(E_\ell\cap\Delta_\ell)}|\Delta_\ell|^{1-\frac{r}{p}}\\ &\le 2^r|\Delta_\ell|^{1-\frac{r}{p}}(2^{(-\ell+1)r}+\|f_\ell\|^r_p) = 2^r|\Delta_\ell|^{1-\frac{r}{p}}(2^{(-\ell+1)r}+1).\end{aligned}$$

Thus, from these estimates and (β),

$$\begin{aligned}\|c_\ell\sum_{i=n_{\ell-1}(\ell)+1}^{n} a_i MW_{n_i}\|_r &\le (2^{(-\ell+2)r+1}+2^{r+1}|\Delta_\ell|^{1-\frac{r}{p}})^{\frac{1}{r}}|c_\ell|\\ &= (2^{(-\ell+2)r+1}+2^{r+1}|\Delta_\ell|^{1-\frac{r}{p}})^{\frac{1}{r}}|\Delta_\ell|^{\frac{1}{r}-\frac{1}{p}}\theta(\ell),\end{aligned}$$

with $\lim_\ell \theta(\ell)=0$, and it follows that

$$\lim_n \|S_n(f)-f\|_r = 0.$$

Since $p>1$ is otherwise arbitrary, Φ is a quasibasis for every space $L^p[0,1]$, with $p\in[1,+\infty)$.

As has been mentioned, in the discussion following Theorem B, it follows that $\mathcal{W}_S$ is complete (or total) in measure, on $[0,1]$ and, thus also, for each $\epsilon>0$, $\mathcal{W}_S$ is closed in $L^p(E_\epsilon)$ for some measurable set $E_\epsilon\subset[0,1]$, with $|E_\epsilon|>1-\epsilon$.

5. The theorem is, in a sense, complementary to earlier results of Grigorian [5], [6] who has established the following proposition.

Let S be an increasing sequence of natural numbers, with $\rho(S)=1$, and let $\mathcal{W}_S$ be the corresponding family of Walsh functions. For every $\epsilon>0$, there exists a measurable set $E_\epsilon\subset[0,1]$ such that $|E_\epsilon|>1-\epsilon$ and $\mathcal{W}_S$ is a system of representation, in $L^1(E_\epsilon)$, in the sense of convergence a.e., as well as in the sense of convergence in the L^1-norm.

As these results and the theorem of this article seem to indicate, the subsystems $\mathcal{W}_S$, with $\rho(S)=1$, are incredibly rich. On the other hand, Walsh families of this kind may be quite sparse indeed, for there are sets $S\subset\mathbb{N}$, of asymptotic density 0, such that $\rho(S)=1$.

References

1. R. P. Boas and Harry Pollard, *The multiplicative completion of sets of functions*, Bull. Amer. Math. Soc. **54** (1948), 518–522. MR **10:**189b
2. Ben-Ami Braun, *On the multiplicative completion of certain basic sequences in L^p, $1<p<\infty$*, Trans. Amer. Math. Soc. **176** (1973), 499–508. MR **47:**2331
3. Bernard R. Gelbaum, *Notes on Banach spaces and bases*, An. Acad. Brasil **30** (1958), 29–36. MR **20:**5419
4. Casper Goffman and Daniel Waterman, *Basic sequences in the space of measurable functions*, Proc. Amer. Math. Soc. **11** (1960), 211–213. MR **22:**2886
5. M. Zh. Grigoryan, *Convergence almost everywhere of Walsh-Fourier series of integrable functions*, Izv. Akad. Nauk. Arm. SSR Ser. Math. **18** (4) (1983), 291–304 (Russian). MR **85a:**42034
6. M. G. Grigorian, *Convergence of Walsh-Fourier series in the L^1 metric and almost everywhere*; English translation in. Soviet Math. Izv. VUZ **34** (11) (1990), 9–20.

7. K. S. Kazarian and Robert E. Zink, *Some ramifications of a theorem of Boas and Pollard concerning the completion of a set of functions in L^2*, Trans. Amer. Math. Soc. **349** (1997), 4367–4383. MR **99d:**42050
8. N. Levinson, *Gap and Density Theorems*, Amer. Math. Soc. Colloq. Publ. **26**, Amer. Math. Soc., Providence, 1940. MR **2:**180d
9. R. E. A. C. Paley, *A remarkable set of orthogonal functions*, Proc. London Math. Soc. **34** (1932), 241–279.
10. J. J. Price, *A density theorem for Walsh functions*, Proc. Amer. Math. Soc. **18** (1967), 209–211. MR **35:**656
11. J. J. Price and Robert E. Zink, *On sets of functions that can be multiplicatively completed*, Ann. Math. **82** (1965), 139–145. MR **31:**1349
12. Ivan Singer, *Bases in Banach Spaces II*, Springer Verlag, Berlin, Heidelberg, New York, 1981. MR **82k:**46024
13. A. A. Talalyan, *On the convergence almost everywhere of subsequences of partial sums of general orthogonal series*, Izv. Akad. Nauk. Arm. SSR Izv. Fiz.–Mat. Tehn Nauki **10** (1957), 17–34. MR **19:**742b
14. ———, *The representation of measurable functions by series*; English translation in Russian Math. Surveys **15** (1960), 77–136.

Department of Mathematics, Erevan State University, Alex Manoogian Str., 375049 Yerevan, Armenia

E-mail address: gmarting@ysu.am

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907-1968

E-mail address: zink@math.purdue.edu