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THE RANK OF HANKEL OPERATORS ON HARMONIC BERGMAN SPACES

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ABSTRACT. We show that on the harmonic Bergman spaces, the Hankel operators with nonconstant harmonic symbol cannot be of finite rank.

1. INTRODUCTION

Let D denote the open unit disk in the complex plane $\mathbb{C}$ and let dA denote normalized Lebesgue measure on D. The harmonic Bergman space L_h^2 is the space of harmonic functions on D which are in $L^2 = L^2(D, dA)$.

The set L_h^2 with the inner product

$$\langle u, v\rangle = \int_D u\overline{v}dA$$

and orthonormal basis $\{1, \sqrt{2}z, \sqrt{2}\overline{z}, ..., \sqrt{n+1}z^n, \sqrt{n+1}\overline{z}^n, ...\}$ is a Hilbert space.

If P is the orthogonal projection from L^2 onto L_h^2, the Hankel operator with symbol $f \in L^2$ is defined by

$$H_f : L_h^2 \to (L_h^2)^{\perp} \quad \text{such that} \quad H_f(u) = (I - P)(fu).$$

In this paper, we show that there are no Hankel operators with nonconstant harmonic symbols which are of finite rank. The finite rank Hankel operators on the Bergman space $L_a^2 = L^2 \cap Hol(D)$ are characterized in [1]. On L_a^2 we define the big and the small Hankel operators respectively by

$$H_f^{big}(\varphi) = (I - P_a)(f\varphi) \qquad \text{and} \qquad H_f^{sm}(\varphi) = Q_a(f\varphi)$$

where P_a and Q_a are the projections from L^2 onto L_a^2 and $\overline{L_a^2}$ respectively. If f is analytic, $H_{\overline{f}}^{big}$ is not of finite rank on L_a^2 unless f is a constant function. On the other hand, the small Hankel operator $H_{\overline{f}}^{sm}$ is of finite rank on L_a^2 if and only if the symbol is a linear combination of the reproducing kernels and some of their derivatives (see [1]). Similar results have been shown for the intermediate Hankel operators on the Bergman space (see [3]).

It is not surprising that linear combinations of reproducing kernels are always symbols of finite rank operators on the Bergman space. For $w \in D$, let h_w be the

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reproducing kernel in w for the complex conjugate Bergman space $\overline{L^2_a}$:

$$h_w(z) = h(w,z) = (1 - w\overline{z})^{-2}.$$

Then $\langle \overline{g}, h_w \rangle = \overline{g}(w)$ for all $\overline{g} \in \overline{L^2_a}$ and, by differentiating under the integral, we see that for $h_w^{(k)} = \frac{\partial^k}{\partial w^k} h(w,z)$

$$\overline{g}^{(k)}(w) = \frac{\partial^k \overline{g}}{\partial \overline{w}^k}(w) = \langle \overline{g}, h_w^{(k)} \rangle.$$

So, if $\varphi \in L^2_a$, and $\overline{g} \in \overline{L^2_a}$

$$\begin{aligned} \langle \overline{g}, H^{sm}_{h_w^{(k)}}(\varphi) \rangle &= \langle \overline{g}, Q_a(\varphi h_w^{(k)}) \rangle \\ &= (\overline{g\varphi})^{(k)}(w) \\ &= \sum_{j=0}^{k} C_k^j \overline{\varphi}^{(k-j)}(w) \langle \overline{g}, h_w^{(j)} \rangle. \end{aligned}$$

Thus $H^{sm}_{h_w^{(k)}}(\varphi) = \sum_{j=0}^{k} C_k^j \varphi^{(k-j)}(w) h_w^{(j)}$, so the range of the small Hankel operator with the symbol $h_w^{(k)}$ is spanned by the reproducing kernel h_w and its k first partial derivatives.

In the harmonic Bergman space, there is an analogy between the small Hankel operators and the Toeplitz operators $T_f : u \in L^2_h \to P(fu)$. In fact, $\overline{L^2_h} = L^2_h$ and so the analogue of H^{sm}_f is $\Gamma^{sm}_f(u) = P(fu)$ which is the usual definition of a Toeplitz operator. In spite of this analogy, the argument above does not work here for the simple reason that the product of two harmonic functions is not harmonic. Furthermore, it is shown in [4] that the only compact Toeplitz operator with a harmonic symbol on the harmonic Bergman spaces is the zero operator. So, no Toeplitz operator with a non-zero harmonic symbol is of finite rank on L^2_h.

In the next section, we study Hankel operators with a monomial symbol, then show that the Hankel operators with analytic or antianalytic symbol are not of finite rank on the harmonic Bergman space. From this we conclude that there are no finite rank Hankel operators with nonconstant harmonic symbols. In this paper, we exclude the trivial case when the symbol is a constant function.

2. Rank of Hankel operators

Let $k \in \mathbb{Z}$. We say that the polynomial p_k is quasihomogenous of degree k if it can be written as

$$p_k(re^i\theta) = q(r)e^{ik\theta}$$

where $q(r)$ is a real polynomial and r and θ are the polar coordinates of the complex variable z.

We can immediately see that two quasihomogenous polynomials of different degrees are orthogonal in L^2_h. The following lemma uses the fact that the image of a monomial by a Hankel operator with a monomial symbol is a quasihomogenous polynomial.

Lemma 2.1. *Let n be a positive integer. Then:*

(a) $H_{\overline{z}^n}(z^k) \quad \perp \quad H_{\overline{z}^n}(z^j) \qquad$ *if $j \neq k$,*

(b) $\|H_{\overline{z}^n}(z^k)\| = \begin{cases} \frac{k}{(n+1)\sqrt{k+n+1}} & \text{if } k \leq n, \\ \\ \frac{n}{(k+1)\sqrt{k+n+1}} & \text{if } k \geq n. \end{cases}$

Proof. It is easy to calculate the projection of the monomials

$$P(\overline{z}^n z^k) = \begin{cases} \frac{n-k+1}{n+1}\overline{z}^{n-k} & \text{if } k \leq n, \\ \\ \frac{k-n+1}{k+1} z^{k-n} & \text{if } k \geq n. \end{cases}$$

With this formula, notice that $H_{\overline{z}^n}(z^k)$ is a quasihomogenous polynomial of degree $k-n$. This proves (a). An immediate calculation gives the value of the L^2-norm of $H_{\overline{z}^n}(z^k)$ in (b). □

Observing the degree of each quasihomogenous polynomial, we remark that $H_{\overline{z}^n}(z^k)$ is not orthogonal to $H_{\overline{z}^m}(z^j)$ if and only if $k-n$ equals $j-m$.

Proposition 2.2. *Let φ and ψ be two functions in L_a^2. The Hankel operators H_φ and $H_{\overline{\psi}}$ are not of finite rank on the harmonic Bergman space L_h^2.*

Proof. Let us show that H_φ is not of finite rank. The same proof works for an antianalytic symbol. Let $\varphi = \sum_{k=0}^{\infty} a_k z^k$. We may assume, without loss of generality, that $\varphi(0) = 0$, because $H_{\varphi-\varphi(0)} \equiv H_\varphi$. We denote by s_0 the least index such that

$$a_1 = a_2 = ... = a_{s_0-1} = 0 \text{ and } a_{s_o} \neq 0.$$

Fix n:

$$H_\varphi(\overline{z}^n) = \sum_{k=s_0}^{\infty} a_k (I-P)(z^k \overline{z}^n) = \sum_{k=s_0}^{\infty} a_k H_{\overline{z}^n}(z^k).$$

Hence $H_\varphi(\overline{z}^n)$ is the sum of quasihomogenous polynomials of degree $k-n$, where $k \geq s_0$. We need the following result, which is easy to check. If m and n are given, then

$$\langle H_\varphi(\overline{z}^n), H_{\overline{z}^m}(z^{s_0})\rangle = \begin{cases} a_{s_0}\|H_{\overline{z}^n}(z^{s_0})\|^2 & \text{if } m = n, \\ \\ 0 & \text{if } m > n. \end{cases}$$

Now, suppose that there exist scalars $\lambda_1, \lambda_2, ..., \lambda_p$, such that

$$\sum_{n=1}^{p} \lambda_n H_\varphi(\overline{z}^n) = 0.$$

From the previous remark

$$0 = \langle \sum_{n=1}^{p} \lambda_n H_\varphi(\overline{z}^n), H_{\overline{z}^p}(z^{s_0})\rangle = \lambda_p a_{s_0} \|H_{\overline{z}^p}(z^{s_0})\|^2,$$

which implies $\lambda_p = 0$. Continuing with the same method we find

$$\lambda_p = \lambda_{p-1} = ... = \lambda_1 = 0,$$

and this shows that the set $\{H_\varphi(\overline{z}^n)\}_{n=1}^{p}$ is linearly independent for all $p > 0$. Finally, H_φ is not of finite rank if $\varphi \in L_a^2$. □

Corollary 2.3. *If $f \in L_h^2$, then the Hankel operator H_f is not of finite rank on the harmonic Bergman space.*

Proof. Let $f \in L_h^2$. Hence, $f = \varphi + \overline{\psi}$, where φ and ψ are from L_a^2. For all functions $g \in L_a^2$, $H_\varphi(g) = 0$ and $H_{\overline{\psi}}(\overline{g}) = 0$. So we have

$$H_f\Big|_{L_a^2} = H_{\overline{\psi}} \qquad \text{and} \qquad H_f\Big|_{\overline{L_a^2}} = H_\varphi.$$

But $H_f\Big|_{L_a^2}$ is of finite rank only if $\psi = 0$, in which case $\varphi \neq 0$ and so $H_f\Big|_{\overline{L_a^2}}$ is not of finite rank. So, the Hankel operator H_f cannot be of finite rank. □

Notice that from Lemma 2.1, one can deduce very easily a result about the compacity on L_h^2 of Hankel operators, when the symbol is continuous in the closed disk.

Proposition 2.4. *The Hankel operator $H_{\overline{z}^n}$ is a Hilbert-Schmidt operator, so it is compact on the harmonic Bergman space L_h^2.*

Proof. By definition, $H_{\overline{z}^n}$ is a Hilbert-Schmidt operator if $Tr(H^*_{\overline{z}^n} H_{\overline{z}^n}) < \infty$. This is easy to prove, using the formulas from Lemma 2.1. We apply $H^*_{\overline{z}^n} H_{\overline{z}^n}$ to the elements of the orthonormal basis of L_h^2 and we have

$$\begin{aligned} Tr(H^*_{\overline{z}^n} H_{\overline{z}^n}) &= \sum_{k=0}^{\infty} (k+1) \| H_{\overline{z}^n}(z^k) \|^2 \\ &= \sum_{k=0}^{n} \frac{k^2(k+1)}{(k+n+1)(n+1)^2} + \sum_{k=n+1}^{\infty} \frac{n^2}{(k+n+1)(k+1)}. \end{aligned}$$

It is clear that this sum is convergent. Thus, $H_{\overline{z}^n}$ is a Hilbert-Schmidt operator, hence compact. □

Proceeding in the same way, we see that H_{z^n} is also compact.

Corollary 2.5. *Any Hankel operator with a continuous symbol in the closed disk is compact on the harmonic Bergman space L_h^2.*

This corollary is due to the fact that polynomials are dense in $C(\overline{D})$. So, the compacity of H_{z^n} and $H_{\overline{z}^n}$ easily implies the compacity of Hankel operators with continuous symbol. This property has been proved in [2] using another method.

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