

STRICTLY SINGULAR NON-COMPACT OPERATORS ON HEREDITARILY INDECOMPOSABLE BANACH SPACES

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ABSTRACT. An example is given of a strictly singular non-compact operator on a Hereditarily Indecomposable, reflexive, asymptotic ℓ_1 Banach space. The construction of this operator relies on the existence of transfinite c_0 -spreading models in the dual of the space.

1. INTRODUCTION

A Banach space is said to be *Hereditarily Indecomposable* (H.I.) if for every pair Y, Z of subspaces of X with $Y \cap Z = \{0\}$, the subspace $Y + Z$ is not closed (by a *subspace* of a Banach space we shall mean an infinite dimensional, closed linear subspace). The first example of an H.I. space was given by Gowers and Maurey [16] providing a negative solution to the famous unconditional basic sequence problem. The following important result was established in [16]: Every operator on a complex H.I. space is a strictly singular perturbation of a multiple of the identity (by the term *operator* we shall mean a bounded linear operator). Actually, the following characterization of complex H.I. spaces is given in [11]: X is H.I. if, and only if, every operator from a subspace of X into X is a strictly singular perturbation of the inclusion map. We recall that an operator on a Banach space is *strictly singular* if no restriction of it to a subspace is an isomorphism.

There has been an interest in investigating strictly singular operators on H.I. spaces because of their connection to the invariant subspace problem. Indeed, known results [10], [19] yield that if X is an H.I. space with the property that every strictly singular operator on X is compact, then every operator on X admits a non-trivial invariant subspace. It is therefore natural to investigate whether or not the known examples of H.I. spaces admit strictly singular, non-compact operators. Gowers [15] constructed an example of a strictly singular non-compact operator from a certain subspace of the Gowers-Maurey space into the whole space. An example of an operator (unpublished) with analogous properties was constructed by Argyros and Wagner on the Argyros-Deliyanni H.I. space [4].

Recently, Androulakis and Schlumprecht [3] gave an example of a strictly singular non-compact operator on the Gowers-Maurey space. We also note that examples of H.I. spaces admitting strictly singular non-compact operators were obtained by Argyros and Felouzis as a consequence of their deep dichotomy result [7].

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In the present paper we show that certain asymptotic ℓ_1 H.I. spaces constructed in [14] also admit strictly singular non-compact operators. This will be a consequence of the fact, established here, that their duals admit c_0^ω -spreading models. We recall the definition which requires the concept of the Schreier families $\{S_\xi\}_{\xi < \omega_1}$ [1] (defined in the next section).

Definition 1.1. Suppose that X is a Banach space with a basis (e_i) . A seminormalized block basis (x_i) of (e_i) is a c_0^ω (resp. ℓ_1^ω)-spreading model if there exists a constant $C > 0$ such that the following property is satisfied: For every $j \in \mathbb{N}$, every finite subset F of \mathbb{N} with $\min F \geq j$ and such that $(x_i)_{i \in F}$ is S_j -admissible, we have that $\|\sum_{i \in F} a_i x_i\| \leq C \max_{i \in F} |a_i|$ (resp. $\|\sum_{i \in F} a_i x_i\| \geq C \sum_{i \in F} |a_i|$) for every scalar sequence $(a_i)_{i \in F}$.

The Banach spaces discussed in this paper are Tsirelson-type spaces defined as the completion of c_{00} (the space of finitely supported real sequences) under norms given by suitable subsets of \mathcal{P} (the set of finitely supported signed measures μ on \mathbb{N} such that $|\mu(\{n\})| \leq 1$ for all $n \in \mathbb{N}$).

A subset \mathcal{N} of \mathcal{P} is said to be *norming* provided it satisfies the following:

- (1) $e_n^* \in \mathcal{N}$, for all $n \in \mathbb{N}$, where e_n^* denotes the point mass measure at n .
- (2) \mathcal{N} is symmetric, that is, if $\mu \in \mathcal{N}$, then $-\mu \in \mathcal{N}$.
- (3) \mathcal{N} is closed under restriction to intervals, that is, if $\mu \in \mathcal{N}$, then $\mu|_J \in \mathcal{N}$, for every interval J in \mathbb{N} .

The term norming is justified by the fact that one can define a norm $\|\cdot\|_{\mathcal{N}}$ on c_{00} in the following manner:

$$\left\| \sum_{i=1}^{\infty} a_i e_i \right\|_{\mathcal{N}} = \sup \left\{ \sum_{i=1}^{\infty} a_i \mu(\{i\}) : \mu \in \mathcal{N} \right\}$$

for every finitely supported scalar sequence (a_i) . Of course, (e_i) is the natural basis of c_{00} . Letting $X_{\mathcal{N}}$ denote the completion of $(c_{00}, \|\cdot\|_{\mathcal{N}})$, we see that (e_n) is a normalized, bimonotone basis for $X_{\mathcal{N}}$.

We shall next describe sufficient conditions on \mathcal{N} in order for $X_{\mathcal{N}}^*$ to admit c_0^ω -spreading models. We shall be using two infinite subsets $M = (m_i)_{i=0}^\infty$ and $N = (n_i)_{i=0}^\infty$ of \mathbb{N} satisfying the following requirements:

(1.1) $m_0 > 1$ and there exists an increasing sequence of integers $(s_i)_{i=0}^\infty$

so that $m_j = \prod_{i < j} m_i^{s_i}$ for all $j \geq 1$.

(1.2) $4f_j < n_j$ for all $j \geq 0$, where (f_j) is defined as follows: $f_0 = 1$,

while for $j \geq 1$, $f_j = \max \left\{ \sum_{i < j} \rho_i n_i : \rho_i \in \mathbb{N} \cup \{0\} (i < j), \prod_{i < j} m_i^{\rho_i} < m_j \right\}$.

In order to state our result we need to introduce some notation.

- Notation.*
- (1) Given μ, ν in \mathcal{P} , we write $\mu < \nu$ if $\max \text{supp } \mu < \min \text{supp } \nu$.
 - (2) A finite subset A of \mathcal{P} is S_p -admissible, $p \in \mathbb{N}$, if $A = \{\mu_1 < \dots < \mu_k\}$ and $\{\min \text{supp } \mu_i : i \leq k\} \in S_p$.
 - (3) Given $\mathcal{N} \subset \mathcal{P}$ and $j \in \mathbb{N} \cup \{0\}$ we set $\mathcal{N}_j = \{(1/m_j) \sum_{\mu \in A} \mu : A \subset \mathcal{N} \text{ is } S_{n_j} \text{-admissible}\}$.

$$(4) \mathcal{N}_\infty = \{ \theta \sum_{i=1}^k \mu_i : k \in \mathbb{N}, \theta \in (0, 1/m_0], \mu_1 < \dots < \mu_k, \exists \tau: \{1, \dots, k\} \rightarrow \mathbb{N} \cup \{0\}, 1 - 1, \mu_i \in \mathcal{N}_{\tau(i)} (i \leq k) \}.$$

The following definition will be important for our purposes.

Definition 1.2. A norming set \mathcal{N} is said to be (M, N) -Schreier if the following properties are satisfied:

- (1) $\mathcal{N} \subset \bigcup_{i=0}^\infty \mathcal{N}_i \cup \mathcal{N}_\infty \cup \{ \pm e_n^* : n \in \mathbb{N} \}.$
- (2) $\mathcal{N}_i \subset \mathcal{N},$ for $0 \leq i < \infty.$

The natural norming set of the mixed Tsirelson space $T(\frac{1}{m_i}, S_{n_i})_{i=0}^\infty$ [4] turns out to be (M, N) -Schreier. Further, one can check that if $N = (n_i)_{i=0}^\infty$ is M -good (this term is defined in [14]) ($M = (m_i)_{i=0}^\infty$) and $M^{(2)} = (m_{2i})_{i=0}^\infty, N^{(2)} = (n_{2i})_{i=0}^\infty$ satisfy conditions (1.1), (1.2), then the norming set \mathcal{N} of the H.I. space $X_{\mathcal{N}}$ constructed in [14] is $(M^{(2)}, N^{(2)})$ -Schreier.

The main result of this paper is the following:

Theorem 1.3. *Suppose \mathcal{N} is (M, N) -Schreier. Then $X_{\mathcal{N}}^*$ admits a c_0^ω -spreading model (x_i^*) . Moreover, for a suitably chosen infinite sequence of integers (l_i) , there exists a non-compact operator T on $X_{\mathcal{N}}$ satisfying $Tx = \sum_{i=1}^\infty x_{l_i}^*(x)e_i$ for all $x \in X_{\mathcal{N}}$. In case $X_{\mathcal{N}}$ is H.I., then T can, in addition, be taken to be strictly singular.*

In fact, under the assumptions of Theorem 1.3, given any sequence (a_i) in ℓ_∞ the formula $Sx = \sum_{i=1}^\infty a_i x_{l_i}^*(x)e_i, x \in X_{\mathcal{N}}$, defines an operator on $X_{\mathcal{N}}$. It follows from this that the space of operators on $X_{\mathcal{N}}$ contains a subspace isomorphic to ℓ_∞ (cf. [3]).

An immediate consequence of Theorem 1.3 is our next corollary.

Corollary 1.4. *Let M, N be infinite subsets of \mathbb{N} subject to conditions (1.1) and (1.2). Then the dual of the mixed Tsirelson space $T(\frac{1}{m_i}, S_{n_i})_{i=0}^\infty$ admits a c_0^ω -spreading model.*

Applying Theorem 1.3 to the H.I. space $X_{\mathcal{N}}$ constructed in [14] we obtain:

Corollary 1.5. *There exists an asymptotic ℓ_1 , reflexive H.I. Banach space that admits a strictly singular non-compact operator and whose dual admits a c_0^ω -spreading model.*

Remark. In an earlier version of this paper we actually showed that the dual of every subspace of the H.I. space constructed in [14] admits a c_0^ω -spreading model.

Standard duality arguments yield that if $X_{\mathcal{N}}^*$ admits a c_0^ω -spreading model, then $X_{\mathcal{N}}$ admits an ℓ_1^ω -spreading model. However, the converse is not true in general. Thus, our approach provides a new method of showing that certain mixed Tsirelson spaces admit ℓ_1^ω -spreading models. This problem has been studied in [6] where it is shown that every subspace of certain regular mixed Tsirelson spaces [2] contains an ℓ_1^ω -spreading model. Their approach is based on the finite representability of c_0 in such spaces [5]. The method of constructing c_0^ω -spreading models in $X_{\mathcal{N}}^*$ relies on the existence of normalized functionals in \mathcal{N} which belong simultaneously to different classes $\mathcal{N}_j, j \geq 0$. Our method is also applied in [6] in order to show that certain modified mixed Tsirelson spaces [5] also admit ℓ_1^ω -spreading models. We finally mention the result of D. Kutzarova and P.K. Lin [17] on the existence of ℓ_1 -spreading models in Schlumprecht’s space [22].

2. PRELIMINARIES

We shall make use of standard Banach space facts and terminology as may be found in [18]. Let X be a Banach space. A sequence (x_n) in X is *semi-normalized* if there exists $\delta > 0$ such that $\delta \leq \|x_n\| \leq 1$, for all n .

Given any set D , we let $[D]$ (resp. $D^{<\infty}$) denote the set of its infinite (resp. finite) subsets. Given $M \in [\mathbb{N}]$, the notation $M = (m_i)$ indicates that $M = \{m_1 < m_2 < \dots\}$. Let E and F be finite subsets of \mathbb{N} . We write $E < F$ if $\max E < \min F$.

Suppose now that X has a Schauder basis (e_n) . A sequence (u_n) of non-zero vectors in X is a *block basis* of (e_n) if there exist successive subsets $F_1 < F_2 < \dots$ of \mathbb{N} and a scalar sequence (a_n) so that $u_n = \sum_{i \in F_n} a_i e_i$, for every $n \in \mathbb{N}$. We adopt the notation $u_1 < u_2 < \dots$ to indicate that (u_n) is a block basis of (e_n) . We let $\text{supp } u_n$ denote the set $\{i \in F_n : a_i \neq 0\}$.

We shall next review the Schreier hierarchy $\{S_\xi\}_{\xi < \omega_1}$ [1]. Since we shall only be using the families $\{S_\xi\}_{\xi < \omega}$, we confine the definitions to the finite ordinal case.

The Schreier families. We let $S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$. Suppose S_ξ has been defined, $\xi < \omega$. We set

$$S_{\xi+1} = \left\{ \bigcup_{i=1}^n F_i : n \in \mathbb{N}, n \leq \min F_1, F_1 < \dots < F_n, F_i \in S_\xi (i \leq n) \right\} \cup \{\emptyset\}.$$

An important property shared by the Schreier families is that they are *hereditary*: If $F \in S_\xi$ and $G \subset F$, then $G \in S_\xi$. Another important property is that they are *spreading*: If $\{p_1, \dots, p_k\} \in S_\xi$, $p_1 < \dots < p_k$, and $q_1 < \dots < q_k$ are so that $p_i \leq q_i$ for all $i \leq k$, then $\{q_1, \dots, q_k\} \in S_\xi$. It is not hard to check that if $F_1 < \dots < F_n$ are members of S_α such that $\{\min F_i : i \leq n\}$ belongs to S_β , then $\bigcup_{i=1}^n F_i$ belongs to $S_{\alpha+\beta}$.

A finite collection \mathcal{F} of finite subsets of \mathbb{N} is said to be S_ξ -admissible, $\xi < \omega$, if there exists an enumeration $\{I_k : k \leq n\}$ of \mathcal{F} such that $I_1 < \dots < I_n$ and the set $\{\min I_k : k \leq n\}$ is a member of S_ξ . A finite block basis $u_1 < \dots < u_n$ in a Banach space with a basis is S_ξ -admissible if $\{\text{supp } u_i : i \leq n\}$ is also. A Banach space X with a basis (e_n) is *asymptotic* ℓ_1 [20] if there exists $\delta > 0$ such that every S_1 -admissible block basis $(u_i)_{i=1}^p$ of (e_n) satisfies $\|\sum_{i=1}^p u_i\| \geq \delta \sum_{i=1}^p \|u_i\|$.

3. TREE REPRESENTATIONS OF FUNCTIONALS IN \mathcal{N}

In this section we describe tree representations of members of \mathcal{N} which turn out to be very useful in estimating the norm of certain functionals in \mathcal{N} .

We recall that a *tree* is a partially ordered finite set (\mathcal{T}, \leq) , such that for every $\alpha \in \mathcal{T}$, the set $\{\beta \in \mathcal{T} : \beta \leq \alpha\}$ is well ordered. The elements of \mathcal{T} are called *nodes*. A node of \mathcal{T} is *terminal* if it has no successors in \mathcal{T} . Given $\alpha \in \mathcal{T}$ which is not terminal, we denote by $D_\alpha(\mathcal{T})$ the set of the immediate successors of α in \mathcal{T} . A tree \mathcal{T} is *rooted* if it has a unique node α_0 (the root) such that $\alpha_0 \leq \alpha$ for all $\alpha \in \mathcal{T}$. A *branch* of \mathcal{T} is a maximal, under inclusion, well ordered subset. The *height* $o(\mathcal{T})$ of \mathcal{T} is the cardinality of its longest branch.

In the sequel, \mathcal{N} is a (M, N) -Schreier set of measures (see Definition 1.2).

Definition 3.1. A functional tree in \mathcal{N} is a subset $(x_\alpha^*)_{\alpha \in \mathcal{T}}$ of \mathcal{N} indexed by a rooted tree \mathcal{T} and such that the following are satisfied:

- (1) $\text{supp } x_\beta^* \subset \text{supp } x_\alpha^*$ whenever $\alpha < \beta$ in \mathcal{T} .

- (2) If $\alpha \in \mathcal{T}$ is non-terminal, then $(x_\beta^*)_{\beta \in D_\alpha(\mathcal{T})}$ is, under an appropriate enumeration, a finite block basis of (e_n^*) .

Definition 3.2. Let $x^* \in \mathcal{N}$. A tree representation of x^* is a functional tree $(x_\alpha^*)_{\alpha \in \mathcal{T}}$ in \mathcal{N} together with a function $\psi: \mathcal{T} \rightarrow [m_0, \infty) \cup \{0\}$ so that the following properties hold:

- (1) $x_{\alpha_0}^* = x^*$, where α_0 is the root of \mathcal{T} .
- (2) $\psi(\alpha) = 0$ if, and only if, α is terminal. In that case $x_\alpha^* = \pm e_{p_\alpha}^*$ for some $p_\alpha \in \mathbb{N}$.
- (3) If $\alpha \in \mathcal{T}$ is non-terminal, then $x_\alpha^* = (1/\psi(\alpha)) \sum_{\beta \in D_\alpha(\mathcal{T})} x_\beta^*$.
- (4) Every non-terminal $\alpha \in \mathcal{T}$ is either of type I or of type II. Specifically, α is of type I if $\psi(\alpha) = m_j$ and $(x_\beta^*)_{\beta \in D_\alpha(\mathcal{T})}$ is S_{n_j} -admissible for some $j \geq 0$. α is of type II if $D_\alpha(\mathcal{T})$ consists of type I nodes and $\psi|_{D_\alpha(\mathcal{T})}$ is $1 - 1$.

Notation. We set $\psi_\alpha = \prod_{\beta < \alpha} \psi(\beta)$ if α is not the root of \mathcal{T} , and $\psi_\alpha = 1$ if α is the root of \mathcal{T} .

It is not hard to see that every member of \mathcal{N} admits a (not necessarily unique) tree representation.

Remark. Suppose that $(x_\alpha^*)_{\alpha \in \mathcal{T}}$ is a tree representation for x^* with associated function ψ . The following facts can be easily established by induction on $o(\mathcal{T})$:

- (1) Let A be a subset of \mathcal{T} consisting of pairwise incomparable nodes. Then $(x_\alpha^*)_{\alpha \in A}$ is, under an appropriate enumeration, a block basis of (e_n^*) .
- (2) If A is additionally assumed to intersect every branch of \mathcal{T} , then $x^* = \sum_{\alpha \in A} \frac{1}{\psi_\alpha} x_\alpha^*$.

The proof of Theorem 1.3 requires the following lemma.

Lemma 3.3. *Let $(x_\alpha^*)_{\alpha \in \mathcal{T}}$ be a functional tree in \mathcal{N} and let $\phi: \mathcal{T} \rightarrow \mathbb{N}$ be a function such that if $\alpha \in \mathcal{T}$ is non-terminal, then $(x_\beta^*)_{\beta \in D_\alpha(\mathcal{T})}$ is $S_{\phi(\alpha)}$ -admissible. Then, for every subset A of \mathcal{T} consisting of pairwise incomparable nodes, the collection $(x_\alpha^*)_{\alpha \in A}$ is S_p -admissible, where $p = \max\{\sum_{\beta < \alpha} \phi(\beta) : \alpha \in A\}$.*

Proof. The proof of the lemma is done by induction on $o(\mathcal{T})$. If $o(\mathcal{T}) = 1$ the assertion of the lemma is trivial. Assuming the assertion true when $o(\mathcal{T}) < k, k > 1$, let \mathcal{T} be such that $o(\mathcal{T}) = k$. Let α_0 be the root of \mathcal{T} and set $\mathcal{T}_\alpha = \{\beta \in \mathcal{T} : \alpha \leq \beta\}$, for all $\alpha \in D_{\alpha_0}(\mathcal{T})$. Our assumptions yield that $(x_\alpha^*)_{\alpha \in D_{\alpha_0}(\mathcal{T})}$ is $S_{\phi(\alpha_0)}$ -admissible. We can assume that $|A| \geq 2$ and set $A_\alpha = \mathcal{T}_\alpha \cap A$, for all $\alpha \in D_{\alpha_0}(\mathcal{T})$. Since $o(\mathcal{T}_\alpha) < k$ our induction hypothesis implies that $(x_\beta^*)_{\beta \in A_\alpha}$ is S_{p_α} -admissible, where $p_\alpha = \max\{\sum_{\alpha \leq \beta < \gamma} \phi(\beta) : \gamma \in A_\alpha\}$, for all $\alpha \in A$. Note that $p_\alpha \leq p - \phi(\alpha_0)$, for all $\alpha \in A$. This completes the inductive step as $(x_\alpha^*)_{\alpha \in D_{\alpha_0}(\mathcal{T})}$ is $S_{\phi(\alpha_0)}$ -admissible. \square

4. PROOF OF THEOREM 1.3

The existence of a c_0^ω -spreading model (x_k^*) in $X_{\mathcal{N}}^*$ follows after establishing Lemma 4.1 and Corollary 4.4. The former shows that a natural candidate for (x_k^*) satisfies an upper c_0^ω -estimate. The latter implies that this particular sequence (x_k^*) is semi-normalized, based on a decomposition result (Lemma 4.3) for the elements of \mathcal{N} , and thus it is indeed a c_0^ω -spreading model. It will be crucial for the entire proof that $x_k^* \in \mathcal{N}_j$, for all $j \leq k$.

Remark. Observe that if $x_1^* < \dots < x_p^*$ is an $S_{n_i+n_j}$ -admissible sequence in \mathcal{N} , for some $i \geq 0, j \geq 0$, then $\frac{1}{m_i m_j} \sum_{k=1}^p x_k^* \in \mathcal{N}$. Indeed, we may decompose $\{1, \dots, p\}$ into successive subsets $F_1 < \dots < F_t$ so that $(x_k^*)_{k \in F_r}$ is S_{n_j} -admissible for every $r \leq t$, while $(x_{\min F_r}^*)_{r=1}^t$ is S_{n_i} -admissible. If we set $y_r^* = \frac{1}{m_j} \sum_{k \in F_r} x_k^*$, then $y_r^* \in \mathcal{N}$ for all $r \leq t$, by condition (2) of Definition 1.2. Clearly, $(y_r^*)_{r=1}^t$ is S_{n_i} -admissible and thus $\frac{1}{m_i} \sum_{r=1}^t y_r^* \in \mathcal{N}$, again by condition (2) of Definition 1.2.

A similar inductive argument now implies the following: Let $q = \sum_{i=0}^t a_i n_i$ for some $t \in \mathbb{N} \cup \{0\}$ and $a_i \in \mathbb{N} \cup \{0\}, i \leq t$. Assume that $(x_l^*)_{l=1}^k$ is an S_q -admissible collection of functionals in \mathcal{N} . Then $(1/\prod_{i \leq t} m_i^{a_i}) \sum_{l=1}^k x_l^* \in \mathcal{N}$.

Notation. We set $p_k = \sum_{i < k} s_i n_i$, for all $k \in \mathbb{N}$ (see (1.1)). Note that $p_k \leq 2f_k$ by the choice of N .

Lemma 4.1. *Let $F_1 < F_2 < \dots < F_t$ be successive subsets of \mathbb{N} such that $F_k \in S_{p_k}$ and set $x_k^* = \frac{1}{m_k} \sum_{i \in F_k} e_i^*$, for all $k \in \mathbb{N}$. Assume $(x_k^*)_{k=1}^\infty$ is semi-normalized. Then $(x_k^*)_{k=1}^\infty$ is a c_0^ω -spreading model.*

Proof. We first observe that for all $l \leq k$ we can find $z_1^* < \dots < z_l^*$ in \mathcal{N} , S_{p_l} -admissible so that $x_k^* = \frac{1}{m_l} \sum_{i=1}^l z_i^*$. Indeed, we may write $F_k = \bigcup_{i=1}^t H_i$, where $H_1 < \dots < H_t$ belong to $S_{p_k-p_l}$ and $(H_i)_{i=1}^t$ is S_{p_l} -admissible. Set $z_i^* = (m_l/m_k) \sum_{r \in H_i} e_r^*, i \leq t$. But since $p_k - p_l = \sum_{l \leq j < k} s_j n_j$, and $z_i^* = \frac{1}{\prod_{l \leq j < k} m_j^{s_j}} \sum_{r \in H_i} e_r^*, i \leq t$, our preceding remark yields that $(z_i^*)_{i=1}^t$ is an S_{p_l} -admissible family in \mathcal{N} which clearly satisfies $x_k^* = (1/m_l) \sum_{i=1}^t z_i^*$.

Now let $k_0 \in \mathbb{N}$. Let $F \in \mathbb{N}^{<\infty}$ with $\min F \geq k_0$ and so that $(x_k^*)_{k \in F}$ is $S_{f_{k_0}}$ -admissible. According to our initial observation, for each $k \in F$ there exists an $S_{p_{k_0}}$ -admissible family $(y_i^*)_{i \in G_k}$ in \mathcal{N} , so that $x_k^* = \frac{1}{m_{k_0}} \sum_{i \in G_k} y_i^*$. Note that $p_{k_0} \leq 2f_{k_0}$. We now obtain, since $4f_{k_0} < n_{k_0}$, that $(y_i^*)_{i \in G}$ is $S_{n_{k_0}}$ -admissible, where $G = \bigcup_{k \in F} G_k$. Of course, $\sum_{k \in F} x_k^* = \frac{1}{m_{k_0}} \sum_{i \in G} y_i^*$. Therefore, $\sum_{k \in F} x_k^* \in \mathcal{N}$, by condition (2) of Definition 1.2. The proof is now complete since $n \leq f_n$ for all $n \in \mathbb{N}$. □

We shall also make use of the following numerical result.

Lemma 4.2. *Assume that $(a_i)_{i=0}^{k-1}$ are positive integers satisfying $\prod_{i < k} m_i^{a_i} < m_k$. Then $\sum_{i < k} a_i n_i < \sum_{i < k} s_i n_i$.*

Proof. By induction on k . The case $k = 1$ is trivial since $a_0 < s_0$. Assume the assertion holds for some $k \geq 1$ and let the integers $(a_i)_{i=0}^k$ satisfy $\prod_{i < k+1} m_i^{a_i} < m_{k+1}$. Observe that $m_{k+1} = m_k^{s_k+1}$. Clearly, $a_k \leq s_k$. We shall distinguish between two cases: $a_k = s_k$ and $a_k < s_k$. If the former, then $\prod_{i < k} m_i^{a_i} < m_k$. By the induction hypothesis we obtain $\sum_{i < k} a_i n_i < \sum_{i < k} s_i n_i$. The assertion now follows as $a_k = s_k$.

When $a_k < s_k$, we obtain that $\prod_{i < k} m_i^{a_i} < m_k^{s_k - a_k + 1}$. Our assumptions on N allow us to deduce that $\sum_{i < k} a_i n_i \leq 2(s_k - a_k + 1)f_k$. It follows that $\sum_{i < k+1} a_i n_i \leq 2(s_k - a_k + 1)f_k + a_k n_k$. But also, $2(s_k - a_k + 1)f_k + a_k n_k < s_k n_k$. Indeed, the latter inequality follows easily as $4f_k < n_k$. Hence, $\sum_{i < k+1} a_i n_i < s_k n_k$ from which the assertion follows. The inductive step as well as the proof of the lemma are now complete. □

Lemma 4.3. *Let $k \in \mathbb{N}$ and $x^* \in \mathcal{N}$ such that $k \leq \min \text{supp } x^*$. Then there exist $p \in \mathbb{N}$, a partition $\{I_1, I_2\}$ of $\{1, \dots, p\}$, functionals $x_1^* < \dots < x_p^*$ in \mathcal{N} and scalars $(\lambda_i)_{i=1}^p$ in $(0, 1]$ so that the following are satisfied:*

- (1) $x^* = \sum_{i=1}^p \lambda_i x_i^*$.
- (2) $x_i^* = \pm e_{j_i}^*$ for all $i \in I_1$ and $\{j_i : i \in I_1\} \in S_{p_k-1}$.
- (3) $\lambda_i \leq 1/m_k$ for all $i \in I_2$.

Proof. Let $(x_\alpha^*)_{\alpha \in \mathcal{T}}$ be a tree representation for x^* with associated function ψ . Let \mathfrak{B} denote the set of all branches of \mathcal{T} . Given $b \in \mathfrak{B}$ let $\alpha(b)$ denote the smallest node $\beta \in b$ such that $\psi_\beta \geq m_k$, or, if such a β does not exist, let $\alpha(b)$ be the terminal node of b .

We let $A = \{\alpha(b) : b \in \mathfrak{B}\}$. It is not hard to check that A consists of pairwise incomparable nodes of \mathcal{T} . Since A intersects all branches of \mathcal{T} , we have that $x^* = \sum_{\alpha \in A} \frac{1}{\psi_\alpha} x_\alpha^*$. We now set $A_1 = \{\alpha \in A : \psi_\alpha < m_k\}$ and $A_2 = A \setminus A_1$. It is clear that A_1 consists of terminal nodes of \mathcal{T} . Let $(x_i^*)_{i=1}^p$ be an enumeration of $(x_\alpha^*)_{\alpha \in A}$ such that $x_1^* < \dots < x_p^*$. We now define $I_r = \{i \leq p : x_i^* \in (x_\alpha^*)_{\alpha \in A_r}\}$ for $r \leq 2$. We finally set $\lambda_i = 1/\psi_\alpha$ ($i \leq p$) if $x_i^* = x_\alpha^*$ for some $\alpha \in A$.

We need only show that $(x_i^*)_{i \in I_1}$ is S_{p_k-1} -admissible. The rest of the required properties are straightforward. To this end, we set $\mathcal{R} = \bigcup_{\alpha \in A_1} \{\beta \in \mathcal{T} : \beta \leq \alpha\}$. The key point is that if $\beta \in \mathcal{R}$ is of type II in \mathcal{T} , then $|D_\beta(\mathcal{R})| \leq k$. Indeed, $\psi|_{D_\beta(\mathcal{R})}$ is 1-1 as β is of type II. On the other hand $\psi(\gamma) < m_k$ and $\psi(\gamma) \in M$ for all $\gamma \in D_\beta(\mathcal{R})$. It follows that $|D_\beta(\mathcal{R})| \leq k$. In particular, $(x_\gamma^*)_{\gamma \in D_\beta(\mathcal{R})}$ is S_1 -admissible.

Define $\phi : \mathcal{R} \rightarrow \mathbb{N}$ by

$$\phi(\beta) = \begin{cases} n_i, & \text{if } \beta \text{ is of type I and } \psi(\beta) = m_i, \text{ for some } i \geq 0, \\ n_0, & \text{if } \beta \text{ is of type II,} \\ 1, & \text{if } \beta \in A_1. \end{cases}$$

We now have that $(x_\gamma^*)_{\gamma \in D_\beta(\mathcal{R})}$ is $S_{\phi(\beta)}$ -admissible, for every non-terminal $\beta \in \mathcal{R}$. Since $\prod_{\beta < \alpha} \psi(\beta) < m_k$, for all $\alpha \in A_1$, Lemma 4.2 yields $\sum_{\beta < \alpha} \phi(\beta) < p_k$, for all $\alpha \in A_1$. Therefore, $(x_\alpha^*)_{\alpha \in A_1}$ is S_{p_k-1} -admissible by Lemma 3.3. This completes the entire proof. □

Corollary 4.4. *For every $x^* \in \mathcal{N}$ and $k \in \mathbb{N}$ we have that $\{i \in \mathbb{N}, i \geq k : |x^*(e_i)| \geq 2/m_k\}$ belongs to S_{p_k-1} .*

Proof. We can assume that $\min \text{supp } x^* \geq k$. We next apply Lemma 4.3 to conclude that $\{i \in \mathbb{N}, i \geq k : |x^*(e_i)| \geq 2/m_k\}$ is contained in $\{j_i : i \leq p\}$ which belongs to S_{p_k-1} . □

Proof of Theorem 1.3. We first show that given k, l in \mathbb{N} there exists $F_k \in S_{p_k}$, $l < \min F_k$, so that letting $x_k^* = \frac{1}{m_k} \sum_{i \in F_k} e_i^*$ we have that $1/3 \leq \|x_k^*\| \leq 1$. Once this is accomplished and since we can take the F_k 's to be successive, it will follow from Lemma 4.1 that (x_k^*) is a c_0^ω -spreading model.

A standard property of the repeated averages hierarchy [9] (Lemma 2.3 of [13]) allows us to find $F_k \in S_{p_k}$ with $\max\{k, l\} < \min F_k$, positive scalars $(a_i)_{i \in F_k}$ with $\sum_{i \in F_k} a_i = 1$ and such that $\sum_{i \in G} a_i < \frac{1}{m_k}$ for every $G \in S_{p_k-1}$. It follows readily from Corollary 4.4 that for every $x^* \in \mathcal{N}$, the set $\{i \in F_k : |x^*(e_i)| \geq 2/m_k\}$ belongs to S_{p_k-1} . Letting $x_k = \sum_{i \in F_k} a_i e_i$, we conclude that $\|x_k\| \leq \frac{3}{m_k}$. But since $x_k^* \in \mathcal{N}$ and $x_k^*(x_k) = 1/m_k$, we have that $\frac{1}{3} \leq \|x_k^*\| \leq 1$, as desired.

For the moreover assertions, we may assume, by passing to a subsequence of the x_k^* 's, that if $F \in \mathbb{N}^{<\infty}$, $\min F > j$, and $(x_i^*)_{i \in F}$ is S_{n_j} -admissible, then $\|\sum_{i \in F} x_i^*\| \leq 1$.

We define a linear map $T: c_{00} \rightarrow c_{00}$ by the formula $Tx = \sum_i x_i^*(x)e_i$. We are going to show that there exists a constant $C > 0$ such that $\|Tx\|_{\mathcal{N}} \leq C\|x\|_{\mathcal{N}}$, for all $x \in c_{00}$. It will then follow that T extends to an operator on $X_{\mathcal{N}}$, still denoted by T , which satisfies $Tx = \sum_i x_i^*(x)e_i$, for all $x \in X_{\mathcal{N}}$. T is the desired operator. Indeed, T is non-compact because (x_i^*) is semi-normalized. On the other hand if $\text{Ker}(T)$ is infinite-dimensional (which is the case if $\mathbb{N} \setminus \bigcup_i \text{supp } x_i^*$ is infinite) and $X_{\mathcal{N}}$ is H.I., then T is strictly singular.

It thus remains to establish that T is bounded on c_{00} with respect to the $\|\cdot\|_{\mathcal{N}}$ norm. To this end, fix a normalized $x \in X_{\mathcal{N}}$ and let $x^* \in \mathcal{N}$. If $x^* = \pm e_n^*$ for some $n \in \mathbb{N}$, then we easily see that $|x^*(Tx)| \leq 1$. If the support of x^* contains at least two elements, set $H_1 = \{i \in \text{supp } x^* : \frac{2}{m_1} \leq |x^*(e_i)| \leq \frac{1}{m_0}\}$ and $H_k = \{i \in \text{supp } x^* : \frac{2}{m_k} \leq |x^*(e_i)| < \frac{2}{m_{k-1}}\}$, for $k \geq 2$. Note that $\text{supp } x^* = \bigcup_{k=1}^{\infty} H_k$. We also put $G_k = \{i \in H_k : i \geq k\}$. Corollary 4.4 yields that $G_k \in S_{n_k}$, as $p_k < n_k$ for all $k \geq 1$, and therefore $\sum_{i \in G_k} |x_i^*(x)| \leq 2$. We now have

$$\begin{aligned} |x^*(Tx)| &= \left| \sum_{k=1}^{\infty} \sum_{i \in H_k} x_i^*(x)x^*(e_i) \right| \\ &\leq \sum_{i \in H_1} |x_i^*(x)||x^*(e_i)| + \sum_{k=2}^{\infty} \left(\sum_{i \in G_k} |x_i^*(x)| + \sum_{i \in H_k \setminus G_k} |x_i^*(x)| \right) \frac{2}{m_{k-1}} \\ &\leq \frac{2}{m_0} + \sum_{k=2}^{\infty} \left(\frac{4}{m_{k-1}} + 2\frac{k-1}{m_{k-1}} \right) \leq \sum_{k \geq 0} \frac{4+2k}{m_k} = D. \end{aligned}$$

To complete the proof of the theorem we need only take $C = \max\{D, 1\}$. \square

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