1. Introduction

A Banach space is said to be Hereditarily Indecomposable (H.I.) if for every pair $Y$, $Z$ of subspaces of $X$ with $Y \cap Z = \{0\}$, the subspace $Y + Z$ is not closed (by a subspace of a Banach space we shall mean an infinite dimensional, closed linear subspace). The first example of an H.I. space was given by Gowers and Maurey providing a negative solution to the famous unconditional basic sequence problem. The following important result was established in: Every operator on a complex H.I. space is a strictly singular perturbation of a multiple of the identity (by the term operator we shall mean a bounded linear operator). Actually, the following characterization of complex H.I. spaces is given in: $X$ is H.I. if, and only if, every operator from a subspace of $X$ into $X$ is a strictly singular perturbation of the inclusion map. We recall that an operator on a Banach space is strictly singular if no restriction of it to a subspace is an isomorphism.

There has been an interest in investigating strictly singular operators on H.I. spaces because of their connection to the invariant subspace problem. Indeed, known results yield that if $X$ is an H.I. space with the property that every strictly singular operator on $X$ is compact, then every operator on $X$ admits a non-trivial invariant subspace. It is therefore natural to investigate whether or not the known examples of H.I. spaces admit strictly singular, non-compact operators. Gowers constructed an example of a strictly singular non-compact operator from a certain subspace of the Gowers-Maurey space into the whole space. An example of an operator (unpublished) with analogous properties was constructed by Argyros and Wagner on the Argyros-Deliyanni H.I. space.

Recently, Androulakis and Schlumprecht gave an example of a strictly singular non-compact operator on the Gowers-Maurey space. We also note that examples of H.I. spaces admitting strictly singular non-compact operators were obtained by Argyros and Felouzis as a consequence of their deep dichotomy result.
In the present paper we show that certain asymptotic \( \ell_1 \) H.I. spaces constructed in \cite{14} also admit strictly singular non-compact operators. This will be a consequence of the fact, established here, that their duals admit \( c_0^\infty \)-spreading models. We recall the definition which requires the concept of the Schreier families \( \{S_\xi\}_{\xi<\omega_1} \) (defined in the next section).

**Definition 1.1.** Suppose that \( X \) is a Banach space with a basis \( (e_i) \). A semi-normalized block basis \( (x_i) \) of \( (e_i) \) is a \( c_0^\infty \) (resp. \( \ell_1^\infty \))-spreading model if there exists a constant \( C > 0 \) such that the following property is satisfied: For every \( j \in \mathbb{N} \), every finite subset \( F \) of \( \mathbb{N} \) with \( \min F \geq j \) and such that \( (x_i)_{i \in F} \) is \( S_j \)-admissible, we have that 
\[
\| \sum_{i \in F} a_i x_i \| \leq C \max_{i \in F} |a_i| \quad (\text{resp.} \quad \| \sum_{i \in F} a_i x_i \| \geq C \sum_{i \in F} |a_i|)
\]
for every scalar sequence \( (a_i)_{i \in F} \).

The Banach spaces discussed in this paper are Tsirelson-type spaces defined as the completion of \( c_0^0 \) (the space of finitely supported real sequences) under norms given by suitable subsets of \( P \) (the set of finitely supported signed measures \( \mu \) on \( \mathbb{N} \) such that \( |\mu(i)| \leq 1 \) for all \( i \in \mathbb{N} \)).

A subset \( \mathcal{N} \) of \( P \) is said to be norming provided it satisfies the following:

1. \( e_n^* \in \mathcal{N} \), for all \( n \in \mathbb{N} \), where \( e_n^* \) denotes the point mass measure at \( n \).
2. \( \mathcal{N} \) is symmetric, that is, if \( \mu \in \mathcal{N} \), then \( -\mu \in \mathcal{N} \).
3. \( \mathcal{N} \) is closed under restriction to intervals, that is, if \( \mu \in \mathcal{N} \), then \( \mu|J \in \mathcal{N} \), for every interval \( J \) in \( \mathbb{N} \).

The term norming is justified by the fact that one can define a norm \( \|\cdot\|_{\mathcal{N}} \) on \( c_0^0 \) in the following manner:

\[
\left\| \sum_{i=1}^\infty a_i e_i \right\|_{\mathcal{N}} = \sup \left\{ \sum_{i=1}^\infty a_i \mu(\{i\}) : \mu \in \mathcal{N} \right\}
\]

for every finitely supported scalar sequence \( (a_i) \). Of course, \( (e_i) \) is the natural basis of \( c_0^0 \). Letting \( X_\mathcal{N} \) denote the completion of \( (c_0^0, \|\cdot\|_{\mathcal{N}}) \), we see that \( (e_i) \) is a normalized, bimonotone basis for \( X_\mathcal{N} \).

We shall next describe sufficient conditions on \( \mathcal{N} \) in order for \( X_\mathcal{N} \) to admit \( c_0^\infty \)-spreading models. We shall be using two infinite subsets \( M = (m_i)_{i=0}^\infty \) and \( N = (n_i)_{i=0}^\infty \) of \( \mathbb{N} \) satisfying the following requirements:

1. \( 1 \) \( m_0 > 1 \) and there exists an increasing sequence of integers \( (s_i)_{i=0}^\infty \) so that \( m_j = \prod_{i<j} m_i^{s_i} \) for all \( j \geq 1 \).

2. \( 4f_j < n_j \) for all \( j \geq 0 \), where \( (f_j) \) is defined as follows: \( f_0 = 1 \), while for \( j \geq 1, f_j = \max \left\{ \sum_{i<j} \rho_i n_i : \rho_i \in \mathbb{N} \cup \{0\} (i < j), \prod_{i<j} m_i^{s_i} < m_j \right\} \).

In order to state our result we need to introduce some notation.

**Notation.**

1. Given \( \mu, \nu \) in \( P \), we write \( \mu < \nu \) if \( \max \text{supp} \mu < \min \text{supp} \nu \).
2. A finite subset \( A \) of \( P \) is \( S_p \)-admissible, \( p \in \mathbb{N} \), if \( A = \{\mu_1 < \cdots < \mu_k\} \) and \( \{\min \text{supp} \mu_i : i \leq k\} \in S_p \).
3. Given \( \mathcal{N} \subset \mathcal{P} \) and \( j \in \mathbb{N} \cup \{0\} \) we set \( \mathcal{N}_j = \{1/m_j \sum_{\mu \in A} \mu : A \subset \mathcal{N} \text{ is } S_{n_j} \text{ -admissible}\} \).
strictly singular non-compact operators

\[ N_{\infty} = \{ \theta \sum_{i=1}^{k} \mu_i : k \in \mathbb{N}, \theta \in (0, 1/m_0], \mu_1 < \cdots < \mu_k, \exists \tau : \{1, \ldots, k\} \rightarrow \mathbb{N} \cup \{0\}, 1 - \tau, \mu_i \in N_{\tau(i)} (i \leq k) \}. \]

The following definition will be important for our purposes.

**Definition 1.2.** A norming set \( N \) is said to be \((M, N)\)-Schreier if the following properties are satisfied:

1. \( N \subset \bigcup_{i=0}^{\infty} N_i \cup N_{\infty} \cup \{ \pm e_n^* : n \in \mathbb{N} \}. \)

2. \( N_i \subset N, \) for \( 0 \leq i < \infty. \)

The natural norming set of the mixed Tsirelson space \( T(\frac{1}{m_i}, S_n)_{i=0}^{\infty} \) turns out to be \((M, N)\)-Schreier. Further, one can check that if \( N = (n_i)_{i=0}^{\infty} \) is \( M \)-good (this term is defined in [14]) \( M = (m_i)_{i=0}^{\infty} \) and \( M^{(2)} = (m_{2i})_{i=0}^{\infty}, N^{(2)} = (n_{2i})_{i=0}^{\infty} \) satisfy conditions (1.1), (1.2), then the norming set \( N \) of the H.I. space \( X_N \) constructed in [14] is \((M^{(2)}, N^{(2)})\)-Schreier.

The main result of this paper is the following:

**Theorem 1.3.** Suppose \( N \) is \((M, N)\)-Schreier. Then \( X_N \) admits a \( c_0^* \)-spreading model \((x_i^*)\). Moreover, for a suitably chosen infinite sequence of integers \((\ell_i)\), there exists a non-compact operator \( T \) on \( X_N \) satisfying \( Tx = \sum_{i=1}^{\infty} x_i^* (x) e_i \) for all \( x \in X_N \). In case \( X_N \) is H.I., then \( T \) can, in addition, be taken to be strictly singular.

In fact, under the assumptions of Theorem 1.3 given any sequence \((a_i)\) in \( \ell_\infty \) the formula \( Sx = \sum_{i=1}^{\infty} a_i x_i^* (x) e_i, x \in X_N \), defines an operator on \( X_N \). It follows from this that the space of operators on \( X_N \) contains a subspace isomorphic to \( \ell_\infty \) (cf. [3]).

An immediate consequence of Theorem 1.3 is our next corollary.

**Corollary 1.4.** Let \( M, N \) be infinite subsets of \( \mathbb{N} \) subject to conditions (1.1) and (1.2). Then the dual of the mixed Tsirelson space \( T(\frac{1}{m_i}, S_n)_{i=0}^{\infty} \) admits a \( c_0^* \)-spreading model.

Applying Theorem 1.3 to the H.I. space \( X_N \) constructed in [14] we obtain:

**Corollary 1.5.** There exists an asymptotic \( \ell_1 \), reflexive H.I. Banach space that admits a strictly singular non-compact operator and whose dual admits a \( c_0^* \)-spreading model.

Remark. In an earlier version of this paper we actually showed that the dual of every subspace of the H.I. space constructed in [14] admits a \( c_0^* \)-spreading model.

Standard duality arguments yield that if \( X_N \) admits a \( c_0^* \)-spreading model, then \( X_N \) admits an \( \ell_1^* \)-spreading model. However, the converse is not true in general. Thus, our approach provides a new method of showing that certain mixed Tsirelson spaces admit \( \ell_1^* \)-spreading models. This problem has been studied in [6] where it is shown that every subspace of certain regular mixed Tsirelson spaces [2] contains an \( \ell_1^* \)-spreading model. Their approach is based on the finite representability of \( c_0 \) in such spaces [5]. The method of constructing \( c_0^* \)-spreading models in \( X_N \) relies on the existence of normalized functionals in \( N \) which belong simultaneously to different classes \( N_j, j \geq 0. \) Our method is also applied in [6] in order to show that certain modified mixed Tsirelson spaces [5] also admit \( \ell_1^* \)-spreading models. We finally mention the result of D. Kutzarova and P.K. Lin [17] on the existence of \( \ell_1 \)-spreading models in Schlumprecht’s space [22].
2. Preliminaries

We shall make use of standard Banach space facts and terminology as may be found in [15]. Let $X$ be a Banach space. A sequence $(x_n)$ in $X$ is semi-normalized if there exists $\delta > 0$ such that $\delta \leq \|x_n\| \leq 1$, for all $n$.

Given any set $D$, we let $[D]$ (resp. $D^{<\infty}$) denote the set of its infinite (resp. finite) subsets. Given $M \in [\mathbb{N}]$, the notation $M = (m_i)$ indicates that $M = \{m_1 < m_2 < \cdots \}$. Let $E$ and $F$ be finite subsets of $\mathbb{N}$. We write $E < F$ if $\max E < \min F$.

Suppose now that $X$ has a Schauder basis $(e_n)$. A sequence $(u_n)$ of non-zero vectors in $X$ is a block basis of $(e_n)$ if there exist successive subsets $F_1 < F_2 < \cdots$ of $\mathbb{N}$ and a scalar sequence $(a_n)$ so that $u_n = \sum_{i \in F_n} a_i e_i$, for every $n \in \mathbb{N}$. We adopt the notation $u_1 < u_2 < \cdots$ to indicate that $(u_n)$ is a block basis of $(e_n)$. We let $\text{supp} u_n$ denote the set $\{i \in F_n : a_i \neq 0\}$.

We shall next review the Schreier hierarchy $\{S_\xi\}_{\xi < \omega}$. Since we shall only be using the families $\{S_\xi\}_{\xi < \omega}$, we confine the definitions to the finite ordinal case.

The Schreier families. We let $S_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$. Suppose $S_\xi$ has been defined, $\xi < \omega$. We set

$$S_{\xi+1} = \bigcup_{i=1}^n F_i : n \in \mathbb{N}, n \leq \text{min} F_1, F_1 < \cdots < F_n, F_i \in S_\xi (i \leq n) \bigcup \{\emptyset\}.$$ 

An important property shared by the Schreier families is that they are hereditary: If $F \in S_\xi$ and $G \subseteq F$, then $G \in S_\xi$. Another important property is that they are spreading: If $\{p_1, \ldots, p_k\} \in S_\xi$, $p_1 < \cdots < p_k$, and $q_1 < \cdots < q_k$ are so that $p_i \leq q_i$ for all $i \leq k$, then $\{q_1, \ldots, q_k\} \in S_\xi$. It is not hard to check that if $F_1 < \cdots < F_n$ are members of $S_\alpha$ such that $\{\min F_i : i \leq n\}$ belongs to $S_\beta$, then $\bigcup_{i=1}^n F_i$ belongs to $S_{\alpha + \beta}$.

A finite collection $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is said to be $S_\xi$-admissible, $\xi < \omega$, if there exists an enumeration $\{I_k : k \leq n\}$ of $\mathcal{F}$ such that $I_1 < \cdots < I_n$ and the set $\{\min I_k : k \leq n\}$ is a member of $S_\xi$. A finite block basis $u_1 < \cdots < u_n$ in a Banach space with a basis is $S_{\xi}$-admissible if $\{\text{supp} u_i : i \leq n\}$ is also a Banach space $X$ with a basis $(e_n)$ is asymptotic $\ell_1$ [20] if there exists $\delta > 0$ such that every $S_1$-admissible block basis $(u_i)_{i=1}^p$ of $(e_n)$ satisfies $\|\sum_{i=1}^p u_i\| \geq \delta \sum_{i=1}^p \|u_i\|$.

3. Tree representations of functionals in $\mathcal{N}$

In this section we describe tree representations of members of $\mathcal{N}$ which turn out to be very useful in estimating the norm of certain functionals in $\mathcal{N}$.

We recall that a tree is a partially ordered finite set $(\mathcal{T}, \leq)$, such that for every $\alpha \in \mathcal{T}$, the set $\{\beta \in \mathcal{T} : \beta \leq \alpha\}$ is well ordered. The elements of $\mathcal{T}$ are called nodes. A node of $\mathcal{T}$ is terminal if it has no successors in $\mathcal{T}$. Given $\alpha \in \mathcal{T}$ which is not terminal, we denote by $D_\alpha(\mathcal{T})$ the set of the immediate successors of $\alpha$ in $\mathcal{T}$. A tree $\mathcal{T}$ is rooted if it has a unique node $a_0$ (the root) such that $a_0 \leq \alpha$ for all $\alpha \in \mathcal{T}$. A branch of $\mathcal{T}$ is a maximal, under inclusion, well ordered subset. The height $h(\mathcal{T})$ of $\mathcal{T}$ is the cardinality of its longest branch.

In the sequel, $\mathcal{N}$ is a $(M, N)$-Schreier set of measures (see Definition 1.2).

Definition 3.1. A functional tree in $\mathcal{N}$ is a subset $(x^*_\alpha)_{\alpha \in \mathcal{T}}$ of $\mathcal{N}$ indexed by a rooted tree $\mathcal{T}$ and such that the following are satisfied:

1. $\text{supp} x^*_\beta \subseteq \text{supp} x^*_\alpha$ whenever $\alpha < \beta$ in $\mathcal{T}$. 

(2) If $\alpha \in T$ is non-terminal, then $(x^*_\beta)_{\beta \in D_\alpha(T)}$ is, under an appropriate enumeration, a finite block basis of $(e_n^*)$.

**Definition 3.2.** Let $x^* \in \mathcal{N}$. A tree representation of $x^*$ is a functional tree $(x^*_\alpha)_{\alpha \in T}$ in $\mathcal{N}$ together with a function $\psi : T \to \{m_0, \infty\} \cup \{0\}$ so that the following properties hold:

1. $x^*_{\alpha_0} = x^*$, where $\alpha_0$ is the root of $T$.
2. $\psi(\alpha) = 0$ if, and only if, $\alpha$ is terminal. In that case $x^*_\alpha = \pm e^*_p$ for some $p \in \mathbb{N}$.
3. If $\alpha \in T$ is non-terminal, then $x^*_\alpha = (1/\psi(\alpha)) \sum_{\beta \in D_\alpha(T)} x^*_\beta$.
4. Every non-terminal $\alpha \in T$ is either of type I or of type II. Specifically, $\alpha$ is of type I if $\psi(\alpha) = m_j$ and $(x^*_\beta)_{\beta \in D_\alpha(T)}$ is $S_{\alpha_j}$-admissible for some $j \geq 0$. $\alpha$ is of type II if $D_\alpha(T)$ consists of type I nodes and $\psi|D_\alpha(T)$ is $1 - 1$.

**Notation.** We set $\psi_\alpha = \prod_{\beta < \alpha} \psi(\beta)$ if $\alpha$ is not the root of $T$, and $\psi_\alpha = 1$ if $\alpha$ is the root of $T$.

It is not hard to see that every member of $\mathcal{N}$ admits a (not necessarily unique) tree representation.

**Remark.** Suppose that $(x^*_\alpha)_{\alpha \in T}$ is a tree representation for $x^*$ with associated function $\psi$. The following facts can be easily established by induction on $o(T)$:

1. Let $A$ be a subset of $T$ consisting of pairwise incomparable nodes. Then $(x^*_\alpha)_{\alpha \in A}$ is, under an appropriate enumeration, a block basis of $(e_n^*)$.
2. If $A$ is additionally assumed to intersect every branch of $T$, then $x^* = \sum_{\alpha \in A} \psi_\alpha x^*_\alpha$.

The proof of Theorem 1.3 requires the following lemma.

**Lemma 3.3.** Let $(x^*_\alpha)_{\alpha \in T}$ be a functional tree in $\mathcal{N}$ and let $\phi : T \to \mathbb{N}$ be a function such that if $\alpha \in T$ is non-terminal, then $(x^*_\beta)_{\beta \in D_\alpha(T)}$ is $S_{\phi(\alpha)}$-admissible. Then, for every subset $A$ of $T$ consisting of pairwise incomparable nodes, the collection $(x^*_\alpha)_{\alpha \in A}$ is $S_p$-admissible, where $p = \max\{\sum_{\beta < \alpha} \phi(\beta) : \alpha \in A\}$.

**Proof.** The proof of the lemma is done by induction on $o(T)$. If $o(T) = 1$ the assertion of the lemma is trivial. Assuming the assertion true when $o(T) < k$, $k > 1$, let $T$ be such that $o(T) = k$. Let $\alpha_0$ be the root of $T$ and set $T_\alpha = \{\beta \in T : \beta < \alpha\}$, for all $\alpha \in D_{\alpha_0}(T)$. Our assumptions yield that $(x^*_\alpha)_{\alpha \in D_{\alpha_0}(T)}$ is $S_{\phi(\alpha_0)}$-admissible. We can assume that $|A| \geq 2$ and set $A_\alpha = T_\alpha \cap A$, for all $\alpha \in D_{\alpha_0}(T)$. Since $o(T_\alpha) < k$ our induction hypothesis implies that $(x^*_\alpha)_{\beta \in A_\alpha}$ is $S_p$-admissible, where $p_\alpha = \max\{\sum_{\alpha \leq \beta < \gamma} \phi(\beta) : \gamma \in A_\alpha\}$, for all $\alpha \in A$. Note that $p_\alpha \leq p - \phi(\alpha_0)$, for all $\alpha \in A$. This completes the inductive step as $(x^*_\alpha)_{\alpha \in D_{\alpha_0}(T)}$ is $S_{\phi(\alpha_0)}$-admissible.

4. **Proof of Theorem 1.3**

The existence of a $c_0^*$-spreading model $(x^*_k)$ in $X^*_N$ follows after establishing Lemma 4.1 and Corollary 4.3. The former shows that a natural candidate for $(x^*_k)$ satisfies an upper $c_0^*$-estimate. The latter implies that this particular sequence $(x^*_k)$ is semi-normalized, based on a decomposition result (Lemma 4.3) for the elements of $\mathcal{N}$, and thus it is indeed a $c_0^*$-spreading model. It will be crucial for the entire proof that $x^*_k \in \mathcal{N}_j$, for all $j \leq k$. 

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Let the choice of \( (H_j \text{ where } m_j) \) and set \( x_k^* = \frac{1}{m_k} \sum_{i=1}^{m_k} z_i^{*} \). Then \( x_k^* \in \mathcal{N} \) and thus \( \frac{1}{m_k} \sum_{i=1}^{m_k} y_i^* \in \mathcal{N} \), again by condition (2) of Definition 1.2.

A similar inductive argument now implies the following: Let \( (x_k^*)_{k=1}^{\infty} \) be a \( \mathcal{S} \)-admissible sequence in \( \mathcal{N} \). Then \( (1/\prod_{i \leq t} m_i) \sum_{i=1}^{k} x_i^* \in \mathcal{N} \).

### Notation

We set \( p_k = \sum_{i<k} s_in_i \), for all \( k \in \mathbb{N} \) (see (1.1)). Note that \( p_k \leq 2f_k \) by the choice of \( N \).

### Lemma 4.1

Let \( F_1 < F_2 < \cdots < \) be successive subsets of \( \mathbb{N} \) such that \( F_k \in S_{p_k} \) and set \( x_k^* = \frac{1}{m_k} \sum_{i \in F_k} e_i^* \), for all \( k \in \mathbb{N} \). Assume \( (x_k^*)_{k=1}^{\infty} \) is semi-normalized. Then \( (x_k^*)_{k=1}^{\infty} \) is a \( c_0 \)-spreading model.

**Proof.** We first observe that for all \( l \leq k \) we can find \( z_l^* < \cdots < z_k^* \) in \( \mathcal{N} \), \( S_{p_l} \)-admissible so that \( x_k^* = \frac{1}{m_k} \sum_{i=1}^{m_k} z_i^* \). Indeed, we may write \( F_k = \bigcup_{i=1}^{H_i} H_i \), where \( H_1 < \cdots < H_t \) belong to \( S_{p_{k-1}} \) and \( (H_j)_{j=1}^{t} \) is \( S_{p_l} \)-admissible. Set \( z_i^* = (m_i/m_k) \sum_{r \in H_i} e_r^* \), \( i \leq t \). But since \( p_k - p_l = \sum_{t \leq j < k} s_jn_j \), and \( z_i^* = \prod_{i \leq j < k} m_i \sum_{r \in H_i} e_r^* \), \( i \leq t \), our preceding remark yields that \( (z_i^*)_{i=1}^{t} \) is an \( S_{p_l} \)-admissible family in \( \mathcal{N} \) which clearly satisfies \( x_k^* = (1/m_k) \sum_{i=1}^{t} z_i^* \).

Now let \( k_0 \in \mathbb{N} \). Let \( F \in \mathbb{N}^{<\infty} \) with \( \min F \geq k_0 \) and so that \( (x_k^*)_{k \in F} \) is \( S_{f_{k_0}} \)-admissible. According to our initial observation, for each \( k \in F \) there exists an \( S_{f_{k_0}} \)-admissible family \( (y_i^*)_{i \in G_k} \) in \( \mathcal{N} \), so that \( x_k^* = \frac{1}{m_{k_0}} \sum_{i \in G_k} y_i^* \). Note that \( p_{k_0} \leq 2f_{k_0} \). We now obtain, since \( 4f_{k_0} < n_{k_0} \), that \( (y_i^*)_{i \in G} \) is \( S_{n_{k_0}} \)-admissible, where \( G = \bigcup_{k \in F} G_k \). Of course, \( \sum_{k \in F} x_k^* = \frac{1}{m_{k_0}} \sum_{i \in G} y_i^* \). Therefore, \( \sum_{k \in F} x_k^* \in \mathcal{N} \), by condition (2) of Definition 1.2. The proof is now complete since \( n \leq f_n \) for all \( n \in \mathbb{N} \).

We shall also make use of the following numerical result.

### Lemma 4.2

Assume that \( (a_i)_{i=0}^{k-1} \) are positive integers satisfying \( \prod_{i<k} m_i^{a_i} < m_k \). Then \( \sum_{i<k} a_in_i < \sum_{i<k} s_in_i \).

**Proof.** By induction on \( k \). The case \( k = 1 \) is trivial since \( a_0 < s_0 \). Assume the assertion holds for some \( k \geq 1 \) and let the integers \( (a_i)_{i=1}^{k} \) satisfy \( \prod_{i<k+1} m_i^{a_i} < m_{k+1} \). Observe that \( m_{k+1} = m_k^{s_k} \). Clearly, \( a_k \leq s_k \). We shall distinguish between two cases: \( a_k = s_k \) and \( a_k < s_k \). If the former, then \( \prod_{i<k} m_i^{a_i} < m_k \).

By the induction hypothesis we obtain \( \sum_{i<k} a_in_i < \sum_{i<k} s_in_i \). The assertion now follows as \( a_k = s_k \). When \( a_k < s_k \), we obtain that \( \prod_{i<k} m_i^{a_i} < m_k^{s_k-a_k+1} \). Our assumptions on \( N \) allow us to deduce that \( \sum_{i<k} a_i n_i \leq 2(s_k-a_k+1)f_k \). It follows that \( \sum_{i<k+1} a_i n_i \leq 2(s_k-a_k+1)f_k + a_k n_k \). But also, \( 2(s_k-a_k+1)f_k + a_k n_k < s_k n_k \). Indeed, the latter inequality follows easily as \( 4f_k < n_k \). Hence, \( \sum_{i<k+1} a_i n_i < s_k n_k \) from which the assertion follows. The inductive step as well as the proof of the lemma are now complete.
Lemma 4.3. Let \( k \in \mathbb{N} \) and \( x^* \in \mathcal{N} \) such that \( k \leq \text{min supp } x^* \). Then there exist \( p \in \mathbb{N} \), a partition \( \{I_1, I_2\} \) of \( \{1, \ldots, p\} \), functionals \( x_1^* < \cdots < x_p^* \) in \( \mathcal{N} \) and scalars \( (\lambda_i)_{i=1}^p \) in \( [0,1] \) so that the following are satisfied:

1. \( x^* = \sum_{i=1}^p \lambda_i x_i^* \).
2. \( x_i^* = \pm e_i^* \) for all \( i \in I_1 \) and \( \{j_i : i \in I_1\} \in S_{p_k-1} \).
3. \( \lambda_i \leq 1/m_k \) for all \( i \in I_2 \).

Proof. Let \( (x^*_\alpha)_{\alpha \in \mathcal{T}} \) be a tree representation for \( x^* \) with associated function \( \psi \). Let \( \mathcal{B} \) denote the set of all branches of \( \mathcal{T} \). Given \( b \in \mathcal{B} \) let \( \alpha(b) \) denote the smallest node \( \beta \in b \) such that \( \psi_\beta \geq m_k \), or, if such a \( \beta \) does not exist, let \( \alpha(b) \) be the terminal node of \( b \).

We let \( A = \{\alpha(b) : b \in \mathcal{B}\} \). It is not hard to check that \( A \) consists of pairwise incomparable nodes of \( \mathcal{T} \). Since \( A \) intersects all branches of \( \mathcal{T} \), we have that \( x^* = \sum_{\alpha \in A} e_{\alpha}^* \). We now set \( A_1 = \{\alpha \in A : \psi_\alpha < m_k\} \) and \( A_2 = A \setminus A_1 \). It is clear that \( A_1 \) consists of terminal nodes of \( \mathcal{T} \). Let \( (x^*_\alpha)_{\alpha \in A} \) be an enumeration of \( (x^*_\alpha)_{\alpha \in A} \) such that \( x_1^* < \cdots < x_p^* \). We now define \( I_r = \{i \leq p : x_i^* \in (x^*_\alpha)_{\alpha \in A_r}\} \) for \( r \leq 2 \). We finally set \( \lambda_i = 1/\psi_\alpha (i \leq p) \) if \( x_i^* = x^*_\alpha \) for some \( \alpha \in A \).

We need only show that \( (x_i^*)_{i \in I_1} \) is \( S_{p_k-1} \)-admissible. The rest of the required properties are straightforward. To this end, we set \( \mathcal{R} = \bigcup_{\alpha \in A_1} \{\beta \in \mathcal{T} : \beta \leq \alpha\} \). The key point is that if \( \beta \in \mathcal{R} \) is of type II in \( \mathcal{T} \), then \( |D_\beta(\mathcal{R})| \leq k \). Indeed, \( \psi|D_\beta(\mathcal{R}) \) is 1-1 as \( \beta \) is of type II. On the other hand \( \psi(\gamma) < m_k \) and \( \psi(\gamma) \in M \) for all \( \gamma \in D_\beta(\mathcal{R}) \). It follows that \( |D_\beta(\mathcal{R})| \leq k \). In particular, \( (x^*_\alpha)_{\gamma \in D_\beta(\mathcal{R})} \) is \( S_{1} \)-admissible.

We define \( \phi : \mathcal{R} \to \mathbb{N} \) by

\[
\phi(\beta) = \begin{cases} n_i, & \text{if } \beta \text{ is of type I and } \psi(\beta) = m_i, \text{ for some } i \geq 0, \\
n_0, & \text{if } \beta \text{ is of type II,} \\
1, & \text{if } \beta \in A_1.
\end{cases}
\]

We now have that \( (x^*_\alpha)_{\gamma \in D_\beta(\mathcal{R})} \) is \( S_{\beta(\mathcal{R})} \)-admissible, for every non-terminal \( \beta \in \mathcal{R} \). Since \( \prod_{\beta \leq \alpha} \psi(\beta) < m_k \) for all \( \alpha \in A_1 \), Lemma 4.2 yields \( \sum_{\beta \leq \alpha} \phi(\beta) < p_k \), for all \( \alpha \in A_1 \). Therefore, \( (x^*_\alpha)_{\alpha \in A_1} \) is \( S_{p_k-1} \)-admissible by Lemma 4.3 This completes the entire proof.

Corollary 4.4. For every \( x^* \in \mathcal{N} \) and \( k \in \mathbb{N} \) we have that \( \{i \in \mathbb{N} : |x^*(e_i)| \geq 2/m_k\} \) belongs to \( S_{p_k-1} \).

Proof. We can assume that \( \text{min supp } x^* \geq k \). We next apply Lemma 4.3 to conclude that \( \{i \in \mathbb{N} : |x^*(e_i)| \geq 2/m_k\} \) is contained in \( \{j_i : i \leq p\} \) which belongs to \( S_{p_k-1} \).

Proof of Theorem 1.3. We first show that given \( k, l \in \mathbb{N} \) there exists \( F_k \in S_{p_k} \), \( l < \min F_k \), so that letting \( x^*_k = \frac{1}{m_k} \sum_{i \in F_k} e_i^* \) we have that \( 1/3 \leq \|x^*_k\| \leq 1 \). Once this is accomplished and since we can take the \( F_k \)'s to be successive, it will follow from Lemma 4.11 that \( (x^*_k) \) is a \( c_0 \)-spreading model.

A standard property of the repeated averages hierarchy [9] (Lemma 2.3 of [13]) allows us to find \( F_k \in S_{p_k} \) with \( \max\{k,l\} < \min F_k \), positive scalars \( (a_i)_{i \in F_k} \), \( \sum_{i \in F_k} a_i = 1 \) and such that \( \sum_{i \in F_k} a_i < \frac{1}{m_k} \) for every \( G \in S_{p_k-1} \). It follows readily from Corollary 4.3 that for every \( x^* \in \mathcal{N} \), the set \( \{i \in F_k : |x^*(e_i)| \geq 2/m_k\} \) belongs to \( S_{p_k-1} \). Letting \( x_k = \sum_{i \in F_k} a_i e_i \), we conclude that \( \|x_k\| \leq \frac{1}{m_k} \). But since \( x^*_k \in \mathcal{N} \) and \( x^*_k(x_k) = 1/m_k \), we have that \( \frac{1}{3} \leq \|x^*_k\| \leq 1 \), as desired.
For the moreover assertions, we may assume, by passing to a subsequence of the $x^*_1$’s, that if $F \in \mathbb{N}^{<\infty}$, $\min F > j$, and $(x^*_i)_{i \in F}$ is $S_{n_j}$-admissible, then $\|\sum_{i \in F} x^*_i\| \leq 1$.

We define a linear map $T: c_{00} \to c_{00}$ by the formula $Tx = \sum x^*_i(e_i)$. We are going to show that there exists a constant $C > 0$ such that $\|Tx\|_{\mathcal{N}} \leq C\|x\|_{\mathcal{N}}$, for all $x \in c_{00}$. It will then follow that $T$ extends to an operator on $X_{\mathcal{N}}$, still denoted by $T$, which satisfies $Tx = \sum x^*_i(e_i)$, for all $x \in X_{\mathcal{N}}$. $T$ is the desired operator. Indeed, $T$ is non-compact because $(x^*_i)$ is semi-normalized. On the other hand if $\text{Ker}(T)$ is infinite-dimensional (which is the case if $\mathbb{N} \setminus \cup_i \text{supp} x^*_i$ is infinite) and $X_{\mathcal{N}}$ is H.I., then $T$ is strictly singular.

It thus remains to establish that $T$ is bounded on $c_{00}$ with respect to the $\|\cdot\|_{\mathcal{N}}$ norm. To this end, fix a normalized $x \in X_{\mathcal{N}}$ and let $x^* \in \mathcal{N}$. If $x^* = \pm e^n$ for some $n \in \mathbb{N}$, then we easily see that $|x^*(Tx)| \leq 1$. If the support of $x^*$ contains at least two elements, set $H_1 = \{i \in \text{supp} x^*: \frac{2}{m_0} \leq |x^*(e_i)| \leq \frac{1}{m_0}\}$ and $H_k = \{i \in \text{supp} x^*: \frac{2}{m_k} \leq |x^*(e_i)| < \frac{2}{m_k-1}\}$, for $k \geq 2$. Note that $\text{supp} x^* = \bigcup_{k=1}^{\infty} H_k$. We also put $G_k = \{i \in H_k: i \geq k\}$. Corollary 4.1 yields that $G_k \in S_{n_k}$, as $p_k < n_k$ for all $k \geq 1$, and therefore $\sum_{i \in G_k} |x^*_i(x)| \leq 2$. We now have

$$|x^*(Tx)| = \left|\sum_{k=1}^{\infty} \sum_{i \in H_k} x^*_i(x)x^*(e_i)\right|$$

$$\leq \sum_{i \in H_1} |x^*_i(x)||x^*(e_i)| + \sum_{k=2}^{\infty} \left(\sum_{i \in G_k} |x^*_i(x)| + \sum_{i \in H_k \setminus G_k} |x^*_i(x)|\right)\frac{2}{m_{k-1}}$$

$$\leq \frac{2}{m_0} + \sum_{k=2}^{\infty} \left(\frac{4}{m_{k-1}} + \frac{k-1}{m_{k-1}}\right) = \sum_{k=0}^{\infty} \frac{4+2k}{m_k} = D.$$ 

To complete the proof of the theorem we need only take $C = \max\{D, 1\}$. \hfill \Box

Acknowledgments

We thank Spiros Argyros for his suggestions regarding the material discussed in this paper.

References


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