INVERSE LIMITS OF ALGEBRAS
AS RETRACTS OF THEIR DIRECT PRODUCTS

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(Communicated by Carl G. Jockusch, Jr.)

Abstract. Inverse limits of modules and, more generally, of universal algebras, are not always pure in corresponding direct products. In this note we show that when certain set-theoretic properties are imposed, they even become direct summands.

Given a direct system \( \{ M_i \}_{i \in I} \) of modules, it is well known that \( \lim \rightarrow M_i \) is a pure quotient of the direct sum \( \bigoplus_{i \in I} M_i \). In contrast, the dual statement that inverse limits are pure submodules of corresponding direct products is not always true: For each prime number \( p \), we can construct a descending chain \( \{ A_n \}_{n \in \mathbb{N}} \) of divisible abelian groups whose intersection \( A \) is isomorphic to \( \mathbb{Z}/p\mathbb{Z} \) (see [2, Exercise 6, p. 101]). Since divisibility is inherited by pure subgroups and direct products and since \( A \) is not divisible, it follows that the inverse limit \( A \) of the divisible groups \( A_n \) is not pure in \( \prod_{n \in \mathbb{N}} A_n \). However, as we shall show in this note, when certain set-theoretic conditions are imposed on an inverse system of modules, the inverse limit is a direct summand of the corresponding direct product. This is motivated by the following observation: Let \( p \) be a prime number and let \( J_p \) be the \( p \)-adic group \( \lim \rightarrow \mathbb{Z}/p^n\mathbb{Z} \). As each \( \mathbb{Z}/p^n\mathbb{Z} \) is finite, \( J_p \) is linearly, and hence algebraically compact. (See [1] and [2].) Since, as can easily be proved, \( J_p \) is pure in \( \prod_{n \in \mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} \), it follows that the canonical monomorphism \( 0 \rightarrow \lim \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow \prod_{n \in \mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} \) splits.

The purpose of this note is to generalize this result in both set-theoretic and universal algebraic directions. We refer to [4] and [3] for the various undefined notions used here from the theory of large cardinals and universal algebra, respectively. Recall that a tree is a poset \( (T, <) \) such that for each \( t \in T \) the set \( \{ s \in T : s < t \} \) of the predecessors of \( t \) is well ordered by \( < \). A subalgebra \( B \) of an algebra \( A \) is a retract of \( A \) if there exists a homomorphism \( g : A \rightarrow B \) whose restriction to \( B \) is the identity on \( B \); such a \( g \) is called a retraction. A directed set \( \{ I; \leq \} \) is \( \lambda \)-directed for some infinite cardinal \( \lambda \), if every subset of \( I \) of size less than \( \lambda \) has an upper bound in \( I \).

First, we need

**Lemma 1.** A subalgebra \( B \) of an algebra \( A \) is a retract of \( A \) if and only if every system of equations over \( B \) and with a solution in \( A \) has a solution in \( B \).

**Proof.** (Cf. [2, Proposition 22.3].) Suppose \( B \) is a retract of \( A \), with retraction \( g \), and let \( \Sigma \) be a system of equations over \( B \) with a set of unknowns \( \{ x_s \}_{s \in S} \). If
\(\{a_s\}_{s \in S}\) is a solution of \(A\) in \(\Sigma\), then, clearly, \(\{g(a_s)\}_{s \in S}\) is a solution of \(\Sigma\) in \(B\). Conversely, let \(\Sigma\) be the system over \(B\)

\[xf((a_i)_{i \in r(f)}) = f((x_{a_i})_{i \in r(f)}), \]

\[xb = b\]

for any \(a_i \in A, b \in B\) and any operation \(f\) on \(A\) (with arity \(r(f)\)), and where the unknowns are indexed by \(A\). This system is solvable in \(B\) by \(x_a = g(a)\) \((a \in A)\). Thus, if \(x_a = g(a)\) \((a \in A)\) is a solution of \(\Sigma\) in \(B\), then the mapping \(g : A \rightarrow B\) is a retraction.

\begin{proposition}
Let \(\alpha\) be a limit ordinal, \(\kappa\) be an infinite cardinal and \(\{A_i; \sigma^1_i\}_{i \leq j < \alpha}\) be a well-ordered inverse system of algebras with \(|\sigma^1_i(A_{i+1})| < \kappa < \text{cf}(\alpha)\). Then the inverse limit \(\lim A_i\) is a retract of \(\prod_{i < \alpha} A_i\).
\end{proposition}

\begin{proof}
We first show that every system of equations over \(\lim A_i\) and solvable in \(\prod_{i < \alpha} A_i\) has a solution in \(\lim A_i\). Let \(\Sigma\) be a system of equations over \(\lim A_i\) with unknowns \(\{a_s\}_{s \in S}\), let \(C\) be the set of all constants appearing in \(\Sigma\), and suppose \(\Sigma\) is solvable in \(\prod_{i < \alpha} A_i\) by \(\{a_s\}_{s \in S}\), say. For each \(i < \alpha\), let \(\Sigma^i\) be the system obtained from \(\Sigma\) by replacing each \(c \in C\) by its \(i\)-th coordinate \(c(i)\) in \(A_i\). Fix \(s\) in \(S\) and let \(T^i_s = \{\sigma^i_j(a_s(j)) : i \leq j < \alpha\}\), where \(a_s(j)\) is the \(j\)-th coordinate of \(a_s\).

Partial-order \(T^i_s = \bigcup_{i < \alpha} T^i_s\) by setting \(x < y\) when \(x \in T^i_s, y \in T^j_s\) and \(\sigma^i_s(y) = x\) for some \(j < i < \alpha\). It is easy to see that \((T^i_s, <)\) is a tree of height \(\alpha\). For any \(x = \sigma^i_s(a_s(j))\) in \(T^i_s\), we have \(x = a_s(i)\) or \(x = \sigma^i_{j+1}(a_s(j))\), so that \(T^i_s \subseteq \{a_s(i)\} \cup \sigma^i_{j+1}(A_{i+1})\), and therefore \(|T^i_s| < \kappa\). As \(\alpha\) is a limit ordinal, \(T^i_s\) has a branch \((\mu_s(i))_{i < \alpha}\), by [5] Proposition 2.32, p. 304]. Since \(\sigma^i(\mu_s(j)) = \mu_s(i)\) for all \(i \leq j < \alpha\), we obtain that \((\mu_s(i))_{i < \alpha} \in \lim A_i\) for each \(s \in S\). Now we have \(c(i) = \sigma^i_s(c(j))\) for all \(i \leq j\), since \(C \subseteq \lim A_i\), so that for all \(j \geq i\), \(\{\sigma^i_j(a_s(j))\}_{s \in S}\) is a solution of \(\Sigma^i\). By definition of \(T^i_s\), for each \(i < \alpha\), \(\mu_s(i) = \sigma^i_j(a_s(j))\) for some \(j \geq i\), i.e. \(\{\mu_s(i)\}_{s \in S}\) is a solution of \(\Sigma^i\). Since \((\mu_s(i))_{i < \alpha} \in \lim A_i\) for all \(s \in S\), we infer that \(\Sigma\) is solvable in \(\lim A_i\), as required. It now follows that \(\lim A_i\) is not empty (choose \(\Sigma\) with \(C = \emptyset\)) and therefore is a subalgebra of \(\prod_{i < \alpha} A_i\), and that it is a retract of \(\prod_{i < \alpha} A_i\), by Lemma [1].

We next turn our attention to cardinals with the tree property stating that the cardinal satisfies König’s Lemma. Recall that \(\aleph_0\) and weakly compact (e.g. measurable) cardinals have the tree property, whereas \(\aleph_1\) and singular cardinals do not.

\begin{proposition}
Let \(\alpha\) be a limit ordinal, \(\kappa\) be an infinite cardinal with the tree property, and \(\{A_i; \sigma^1_i\}_{i \leq j < \alpha}\) be a well-ordered inverse system of algebras with \(|\sigma^1_{i+1}(A_{i+1})| < \kappa \leq \text{cf}(\alpha)\). Then \(\lim A_i\) is a retract of \(\prod_{i < \alpha} A_i\).
\end{proposition}

\begin{proof}
If \(\kappa < \text{cf}(\alpha)\), use Proposition 2. Suppose that \(\kappa = \text{cf}(\alpha)\) with \(\alpha = \sum_{i < \alpha} \alpha_i\), where \(\alpha_i < \alpha\). Then, using the tree property of \(\kappa\) and an argument similar to that of Proposition 2 we obtain that \(\lim A_i\) is a subalgebra of \(\prod_{i < \alpha} A_i\), and that \(\lim A_{\alpha_i}\), the inverse limit of the inverse family \(\{A_{\alpha_i}; \sigma^1_{\alpha_i}\}_{i \leq \kappa < \alpha}\), is a retract of \(\prod_{i \leq \kappa} A_i\). Let \(\varphi : \prod_{i < \kappa} A_i \to \prod_{i < \kappa} A_{\alpha_i}\) be the canonical projection. Then (see for example the proof of [3] Lemma 7, p. 133]), the restriction \(\varphi\) of \(\varphi\) to \(\lim A_i\) is an isomorphism \(\lim A_i \to \lim A_{\alpha_i}\) and we have \(\varphi f = g \psi\), where \(f : \lim A_i \to \prod_{i < \kappa} A_i\) and \(g : \lim A_{\alpha_i} \to \prod_{i < \kappa} A_{\alpha_i}\) are the inclusions mappings. If \(\pi : \prod_{i < \kappa} A_{\alpha_i} \to \lim A_{\alpha_i}\) is
such that πg is the identity, then \( \psi^{-1} \pi \varphi f = \psi^{-1} \pi g \psi \) is the identity mapping on \( \varprojlim A_i \), and so \( \varprojlim A_i \) is a retract of \( \prod_{i<\alpha} A_i \).

The conclusion of Proposition 3 can be arrived at for a wider class of inverse systems, provided \( \kappa \) is a compact cardinal. An infinite cardinal \( \lambda \) is compact if, for any set \( S \), every \( \lambda \)-complete filter on \( S \) can be extended to a \( \lambda \)-complete ultrafilter. Thus \( \aleph_0 \) is compact, and it is known that uncountable compact cardinals are necessarily measurable. We have

**Proposition 4.** Let \( \{A_i; \sigma_i^I\} \) be an inverse system of algebras and let \( \kappa \) be a compact cardinal such that \( \{|I| \leq \} \) is \( \kappa \)-directed and \( |\bigcup_{j \geq 1} \sigma_i^I(A_j)| < \kappa \), for every \( i \in I \). Then \( \varprojlim A_i \) is a retract of \( \prod_{i \in I} A_i \).

**Proof.** We first show that \( \varprojlim A_i \) is non-empty. For each \( i \in I \), let \( p_i \in A_i, \pi_i : \prod_{i \in I} A_j \to A_i \) be the \( i \)-th canonical projection, and let \( T_i = \{\sigma_i^I(p_j) : i,j \in I, i \leq j\} \). For every \( J \in |I| < \kappa = \{S \subseteq I : |S| < \kappa\} \), let \( X_J = \{x \in \prod_{i \in I} T_i : \sigma_i^I(p_j) = p_i\} \) for all \( i,j \in I \) and \( i \leq j \). Since \( I \) is \( \kappa \)-directed and \( \kappa \) is regular (compact cardinals are regular), \( \emptyset \subset X_{\varnothing} \subset X_{J'}, J \subset \bigcap_{J < \lambda} X_{J'}, \) whenever \( J \in |I| < \kappa \) and \( \lambda \) is a cardinal less than \( \kappa \). It follows that the set \( \{X_I\}_{I \in |I| < \kappa} \) generates on \( \prod_{i \in I} T_i \) a \( \kappa \)-complete proper filter, which, as \( \kappa \) is compact, can be extended to a \( \kappa \)-complete ultrafilter \( U \). For each \( Y \subseteq U \), let \( Y = \{x \in T_i : x = (x_i)_{i \in I} \in Y\} \) and \( U_i = \{Y_i : Y_i \in U\} \). As in the proof of [3] Theorem 1, p. 132, we obtain that \( U_i \) is a \( \kappa \)-complete ultrafilter on \( T_i \). By hypothesis \( |T_i| < \kappa \), so that \( U_i \) is a principal generated by a singleton \( \{y_i\} \), say. Now, for all \( i,j \in I \), \( \pi_i^{-1}(\{y_i\}), \pi_j^{-1}(\{y_j\}) \) and \( X_{(i,j)} \) are in \( U \), so that \( \pi_i^{-1}(\{y_i\}) \cap \pi_j^{-1}(\{y_j\}) \cap X_{(i,j)} \) is in \( U \). Therefore, if \( i \leq j \), there exists \( x = (x_i)_{i \in I} \in X_{(i,j)} \) such that \( \sigma_i^I(y_i) = \sigma_j^I(x_j) = x_i = y_i \). This proves that \( \varprojlim A_i \) is non-empty, and thus a subalgebra of \( \prod_{i \in I} A_i \). Next, let \( \Sigma \) be a system of equations over \( \varprojlim A_i \) with unknowns \( \{x_s\}_{s \in S} \) and constants \( \{c\}_{c \in C} \), and suppose it is solvable in \( \prod_{i \in I} A_i \) by \( \{a_s\}_{s \in S} \). For each \( i \in I \), let \( \Sigma_i \) be the system obtained from \( \Sigma \) by replacing each \( c \) in \( C \) by its \( i \)-th coordinate \( c(i) \) in \( A_i \). Fix \( s \in S \), and set \( B_s^i = \{\sigma_i^I(a_s(j)) : j \in I, i \leq j\} \). It is easy to see that \( \{B_s^i\}_{i \in I} \) can be regarded as an inverse system of non-empty sets with bonding maps \( \sigma_i^I(i \leq j) \). By the first part of this proof, \( \varprojlim B_s^i \) is non-empty. Clearly, if \( \mu_s \in \varprojlim B_s^i \), then \( \mu_s \) is a solution of \( \Sigma \) in \( \varprojlim A_i \). Now use Lemma 1.

**Remarks.** The first part of the foregoing proof (that \( \varprojlim A_i \) is non-empty) is a straightforward adaptation of an argument of Grätzer [3] Theorem 1, p. 132], where he used ultrafilters to prove the classical theorem that inverse limits of finite non-empty sets are non-empty. Indeed, since \( \aleph_0 \) is compact, Proposition 4 generalizes both [3] Theorem 1, p. 132] (and hence König’s Graph Lemma) and the observation on \( J_\rho \) mentioned above. As was pointed out by the referee, Proposition 2 and its proof provide an alternative proof of [3] Theorem 1, p. 132] for inverse systems \( \{A_i; \sigma_i^I\}_{i<\alpha} \): if \( cf(\alpha) = \omega \), then, for some countable cofinal subset \( \{i_n : n \in \mathbb{N}\} \) of \( \alpha \), \( \varprojlim A_i \) is isomorphic to \( \varprojlim A_{i_n} \), which, using the proof of Proposition 2 with König’s Lemma, is non-empty. If \( cf(\alpha) > \omega \), then the same follows again by using \( \kappa = \omega \) in Proposition 2.

**Corollary 5.** Let \( \kappa \) be a compact cardinal and let \( \{A_i\}_{i \in I} \) be an inverse system of algebras such that \( \{|I| \leq \} \) is \( \kappa \)-directed and \( |A_i| < \kappa \) for all \( i \in I \). Then \( \varprojlim A_i \) is a retract of \( \prod_{i \in I} A_i \).
ACKNOWLEDGMENTS

The author gratefully acknowledges the support of King Fahd University of Petroleum and Minerals. He also thanks the referee for comments that led to several improvements.

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