

INVERSE LIMITS OF ALGEBRAS AS RETRACTS OF THEIR DIRECT PRODUCTS

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ABSTRACT. Inverse limits of modules and, more generally, of universal algebras, are not always pure in corresponding direct products. In this note we show that when certain set-theoretic properties are imposed, they even become direct summands.

Given a direct system $\{M_i\}_{i \in I}$ of modules, it is well known that $\varprojlim M_i$ is a pure quotient of the direct sum $\bigoplus_{i \in I} M_i$. In contrast, the dual statement that inverse limits are pure submodules of corresponding direct products is not always true: For each prime number p , we can construct a descending chain $\{A_n\}_{n \in \mathbb{N}}$ of divisible abelian groups whose intersection A is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ (see [2, Exercise 6, p. 101]). Since divisibility is inherited by pure subgroups and direct products and since A is not divisible, it follows that the inverse limit A of the divisible groups A_n is not pure in $\prod_{n \in \mathbb{N}} A_n$. However, as we shall show in this note, when certain set-theoretic conditions are imposed on an inverse system of modules, the inverse limit is a *direct summand* of the corresponding direct product. This is motivated by the following observation: Let p be a prime number and let J_p be the p -adic group $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$. As each $\mathbb{Z}/p^n\mathbb{Z}$ is finite, J_p is linearly, and hence algebraically compact. (See [1] and [2].) Since, as can easily be proved, J_p is pure in $\prod_n \mathbb{Z}/p^n\mathbb{Z}$, it follows that the canonical monomorphism $0 \rightarrow \varprojlim \mathbb{Z}/p^n\mathbb{Z} \rightarrow \prod_n \mathbb{Z}/p^n\mathbb{Z}$ splits.

The purpose of this note is to generalize this result in both set-theoretic and universal algebraic directions. We refer to [4] and [3] for the various undefined notions used here from the theory of large cardinals and universal algebra, respectively. Recall that a *tree* is a poset $(T, <)$ such that for each $t \in T$ the set $\{s \in T : s < t\}$ of the predecessors of t is well ordered by $<$. A subalgebra B of an algebra A is a *retract* of A if there exists a homomorphism $g : A \rightarrow B$ whose restriction to B is the identity on B ; such a g is called a *retraction*. A directed set $\{I; \leq\}$ is λ -*directed* for some infinite cardinal λ , if every subset of I of size less than λ has an upper bound in I .

First, we need

Lemma 1. *A subalgebra B of an algebra A is a retract of A if and only if every system of equations over B and with a solution in A has a solution in B .*

Proof. (Cf. [2, Proposition 22.3].) Suppose B is a retract of A , with retraction g , and let Σ be a system of equations over B with a set of unknowns $\{x_s\}_{s \in S}$. If

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$\{a_s\}_{s \in S}$ is a solution of A in Σ , then, clearly, $\{g(a_s)\}_{s \in S}$ is a solution of Σ in B . Conversely, let Σ be the system over B

$$\begin{aligned} x_{f((a_i)_{i \in r(f)})} &= f((x_{a_i})_{i \in r(f)}), \\ x_b &= b \end{aligned}$$

for any $a_i \in A, b \in B$ and any operation f on A (with arity $r(f)$), and where the unknowns are indexed by A . This system is solvable in A by $x_a = a$ ($a \in A$). Thus, if $x_a = g(a)$ ($a \in A$) is a solution of Σ in B , then the mapping $g : A \rightarrow B$ is a retraction. \square

Proposition 2. *Let α be a limit ordinal, κ be an infinite cardinal and $\{A_i; \sigma_i^j\}_{i \leq j < \alpha}$ be a well-ordered inverse system of algebras with $|\sigma_i^{i+1}(A_{i+1})| < \kappa < \text{cf}(\alpha)$. Then the inverse limit $\varprojlim A_i$ is a retract of $\prod_{i < \alpha} A_i$.*

Proof. We first show that every system of equations over $\varprojlim A_i$ and solvable in $\prod_{i < \alpha} A_i$ has a solution in $\varprojlim A_i$. Let Σ be a system of equations over $\varprojlim A_i$ with unknowns $\{x_s\}_{s \in S}$, let C be the set of all constants appearing in Σ , and suppose Σ is solvable in $\prod_{i < \alpha} A_i$ by $\{a_s\}_{s \in S}$, say. For each $i < \alpha$, let Σ^i be the system obtained from Σ by replacing each c in C by its i -th coordinate $c(i)$ in A_i . Fix s in S and let $T_s^i = \{\sigma_i^j(a_s(j)) : i \leq j < \alpha\}$, where $a_s(j)$ is the j -th coordinate of a_s . Partial-order $T_s = \bigcup_{i < \alpha} T_s^i$ by setting $x < y$ when $x \in T_s^i, y \in T_s^j$ and $\sigma_i^j(y) = x$ for some $i < j < \alpha$. It is easy to see that $(T_s, <)$ is a tree of height α . For any $x = \sigma_i^j(a_s(j))$ in T_s^i , we have $x = a_s(i)$ or $x = \sigma_i^{i+1} \sigma_{i+1}^j(a_s(j)) \in \sigma_i^{i+1}(A_{i+1})$, so that $T_s^i \subseteq \{a_s(i)\} \cup \sigma_i^{i+1}(A_{i+1})$, and therefore $|T_s^i| < \kappa$. As α is a limit ordinal, T_s has a branch $(\mu_s(i))_{i < \alpha}$, by [5, Proposition 2.32, p. 304]. Since $\sigma_i^j(\mu_s(j)) = \mu_s(i)$ for all $i \leq j < \alpha$, we obtain that $(\mu_s(i))_{i < \alpha} \in \varprojlim A_i$ for each $s \in S$. Now we have $c(i) = \sigma_i^j(c(j))$ for all $i \leq j$, since $C \subseteq \varprojlim A_i$, so that for all $j \geq i$, $\{\sigma_i^j(a_s(j))\}_{s \in S}$ is a solution of Σ^i . By definition of T_s , for each $i < \alpha$, $\mu_s(i) = \sigma_i^j(a_s(j))$ for some $j \geq i$, i.e. $\{\mu_s(i)\}_{s \in S}$ is a solution of Σ^i . Since $(\mu_s(i))_{i < \alpha} \in \varprojlim A_i$ for all s in S , we infer that Σ is solvable in $\varprojlim A_i$, as required. It now follows that $\varprojlim A_i$ is not empty (choose Σ with $C = \emptyset$) and therefore is a subalgebra of $\prod_{i < \alpha} A_i$, and that it is a retract of $\prod_{i < \alpha} A_i$, by Lemma 1. \square

We next turn our attention to cardinals with the tree property stating that the cardinal satisfies König's Lemma. Recall that \aleph_0 and weakly compact (e.g. measurable) cardinals have the tree property, whereas \aleph_1 and singular cardinals do not.

Proposition 3. *Let α be a limit ordinal, κ be an infinite cardinal with the tree property, and $\{A_i; \sigma_i^j\}_{i \leq j < \alpha}$ be a well-ordered inverse system of algebras with $|\sigma_i^{i+1}(A_{i+1})| < \kappa \leq \text{cf}(\alpha)$. Then $\varprojlim A_i$ is a retract of $\prod_{i < \alpha} A_i$.*

Proof. If $\kappa < \text{cf}(\alpha)$, use Proposition 2. Suppose that $\kappa = \text{cf}(\alpha)$ with $\alpha = \sum_{t < \kappa} \alpha_t$, where $\alpha_t < \alpha$. Then, using the tree property of κ and an argument similar to that of Proposition 2, we obtain that $\varprojlim A_i$ is a subalgebra of $\prod_{i < \alpha} A_i$, and that $\varprojlim A_{\alpha_t}$, the inverse limit of the inverse family $\{A_{\alpha_t}; \sigma_{\alpha_t}^s\}_{t \leq s < \kappa}$, is a retract of $\prod_{t < \kappa} A_{\alpha_t}$. Let $\varphi : \prod_{i < \alpha} A_i \rightarrow \prod_{t < \kappa} A_{\alpha_t}$ be the canonical projection. Then (see for example the proof of [3, Lemma 7, p. 133]), the restriction ψ of φ to $\varprojlim A_i$ is an isomorphism $\varprojlim A_i \rightarrow \varprojlim A_{\alpha_t}$ and we have $\varphi f = g \psi$, where $f : \varprojlim A_i \rightarrow \prod_{i < \alpha} A_i$ and $g : \varprojlim A_{\alpha_t} \rightarrow \prod_{t < \kappa} A_{\alpha_t}$ are the inclusions mappings. If $\pi : \prod_{t < \kappa} A_{\alpha_t} \rightarrow \varprojlim A_{\alpha_t}$ is

such that πg is the identity, then $\psi^{-1}\pi\varphi f = \psi^{-1}\pi g\psi$ is the identity mapping on $\varprojlim A_i$, and so $\varprojlim A_i$ is a retract of $\prod_{i<\alpha} A_i$. \square

The conclusion of Proposition 3 can be arrived at for a wider class of inverse systems, provided κ is a compact cardinal. An infinite cardinal λ is *compact* if, for any set S , every λ -complete filter on S can be extended to a λ -complete ultrafilter. Thus \aleph_0 is compact, and it is known that uncountable compact cardinals are necessarily measurable. We have

Proposition 4. *Let $\{A_i; \sigma_i^j\}_{i \in I}$ be an inverse system of algebras and let κ be a compact cardinal such that $\{I; \leq\}$ is κ -directed and $|\bigcup_{j>i} \sigma_i^j(A_j)| < \kappa$, for every $i \in I$. Then $\varprojlim A_i$ is a retract of $\prod_{i \in I} A_i$.*

Proof. We first show that $\varprojlim A_i$ is non-empty. For each $i \in I$, let $p_i \in A_i, \pi_i : \prod_{j \in I} A_j \rightarrow A_i$ be the i -th canonical projection, and let $T_i = \{\sigma_i^j(p_j) : i, j \in I, i \leq j\}$. For every $J \in [I]^{<\kappa} = \{S \subseteq I : |S| < \kappa\}$, let $X_J = \{x \in \prod_{i \in I} T_i : \sigma_i^j(p_j) = p_i, \text{ for all } i, j \in J \text{ and } i \leq j\}$. Since I is κ -directed and κ is regular (compact cardinals are regular), $\emptyset \subset X_{\bigcup_{\tau < \lambda} J_\tau} \subseteq \bigcap_{\tau < \lambda} X_{J_\tau}$, whenever $J_\tau \in [I]^{<\kappa}$ and λ is a cardinal less than κ . It follows that the set $\{X_J\}_{J \in [I]^{<\kappa}}$ generates on $\prod_{i \in I} T_i$ a κ -complete proper filter, which, as κ is compact, can be extended to a κ -complete ultrafilter U . For each $Y \in U$, let $Y_i = \{x_i \in T_i : x = (x_i)_{i \in I} \in Y\}$ and let $U_i = \{Y_i : Y \in U\}$. As in the proof of [3, Theorem 1, p. 132], we obtain that U_i is a κ -complete ultrafilter on T_i . By hypothesis $|T_i| < \kappa$, so that U_i is a principal generated by a singleton $\{y_i\}$, say. Now, for all $i, j \in I, \pi_i^{-1}(\{y_i\}), \pi_j^{-1}(\{y_j\})$ and $X_{\{i,j\}}$ are in U , so that $\pi_i^{-1}(\{y_i\}) \cap \pi_j^{-1}(\{y_j\}) \cap X_{\{i,j\}} \in U$. Therefore, if $i \leq j$, there exists $x = (x_i)_{i \in I} \in X_{\{i,j\}}$ such that $\sigma_i^j(y_j) = \sigma_i^j(x_j) = x_i = y_i$. This proves that $\varprojlim A_i$ is non-empty, and thus a subalgebra of $\prod_{i \in I} A_i$. Next, let Σ be a system of equations over $\varprojlim A_i$ with unknowns $\{x_s\}_{s \in S}$ and constants $\{c\}_{c \in C}$, and suppose it is solvable in $\prod_{i \in I} A_i$ by $\{a_s\}_{s \in S}$. For each $i \in I$, let Σ^i be the system obtained from Σ by replacing each c in C by its i -th coordinate $c(i)$ in A_i . Fix s in S , and set $B_i^s = \{\sigma_i^j(a_s(j)) : j \in I, i \leq j\}$. It is easy to see that $\{B_i^s\}_{i \in I}$ can be regarded as an inverse system of non-empty sets with bonding maps σ_i^j ($i \leq j$). By the first part of this proof, $\varprojlim B_i^s$ is non-empty. Clearly, if $\mu_s \in \varprojlim B_i^s$, then $\{\mu_s\}_{s \in S}$ is a solution of Σ in $\varprojlim A_i$. Now use Lemma 1. \square

Remarks. The first part of the foregoing proof (that $\varprojlim A_i$ is non-empty) is a straightforward adaptation of an argument of Grätzer [3, Theorem 1, p. 132], where he used ultrafilters to prove the classical theorem that inverse limits of finite non-empty sets are non-empty. Indeed, since \aleph_0 is compact, Proposition 4 generalizes both [3, Theorem 1, p. 132] (and hence König's Graph Lemma) and the observation on J_p mentioned above. As was pointed out by the referee, Proposition 2 and its proof provide an alternative proof of [3, Theorem 1, p. 132] for inverse systems $\{A_i; \sigma_i^j\}_{i < \alpha}$: If $\text{cf}(\alpha) = \omega$, then, for some countable cofinal subset $\{i_n : n \in \mathbb{N}\}$ of α , $\varprojlim A_i$ is isomorphic to $\varprojlim A_{i_n}$ which, using the proof of Proposition 2 with König's Lemma, is non-empty. If $\text{cf}(\alpha) > \omega$, then the same follows again by using $\kappa = \omega$ in Proposition 2.

Corollary 5. *Let κ be a compact cardinal and let $\{A_i\}_{i \in I}$ be an inverse system of algebras such that $\{I; \leq\}$ is κ -directed and $|A_i| < \kappa$ for all $i \in I$. Then $\varprojlim A_i$ is a retract of $\prod_{i \in I} A_i$. \square*

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