FERENC LUKÁCS TYPE THEOREMS IN TERMS OF THE ABEL-POISSON MEAN OF CONJUGATE SERIES

FERENC MÓRICZ

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Abstract. A theorem of Ferenc Lukács determines the generalized jumps of a periodic, Lebesgue integrable function \( f \) in terms of the partial sum of the conjugate series to the Fourier series of \( f \). The main aim of this paper is to prove an analogous theorem in terms of the Abel-Poisson mean. We also prove an estimate of the partial derivative (with respect to the angle) of the Abel-Poisson mean of an integrable function \( F \) at those points at which \( F \) is smooth. Finally, we reveal the intimate relation between these two results.

1. A theorem of Ferenc Lukács

Let \( f \) be a periodic, Lebesgue integrable function, in symbol: \( f \in L^1(\mathbb{T}) \), \( \mathbb{T} := [-\pi, \pi) \), with Fourier series

\[
f(x) \sim \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),
\]

where

\[
a_k := \frac{1}{\pi} \int_{\mathbb{T}} f(x) \cos kx \, dx \quad \text{and} \quad b_k := \frac{1}{\pi} \int_{\mathbb{T}} f(x) \sin kx \, dx
\]

are the Fourier coefficients of \( f \). We recall that the conjugate series to (1.1) is defined by

\[
\tilde{s}_n(f, x) := \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx).
\]

Denote by \( \tilde{s}_n(f, x) \) the \( n \)th partial sum of series (1.3). The following theorem was proved by Ferenc Lukács [2].

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Theorem 1. Let $f \in L^1(\mathbb{T})$ and $x \in \mathbb{T}$. If there exists a number $d_x(f)$ such that
\begin{equation}
\lim_{h \to 0^+} \frac{1}{h} \int_0^h \|[f(x+t) - f(x-t)] - d_x(f)]dt = 0,
\end{equation}
then
\begin{equation}
\lim_{n \to \infty} \frac{\hat{s}_n(f,x)}{\log n} = -\frac{1}{\pi} d_x(f).
\end{equation}

By ‘log’ we mean the natural logarithm.

It is clear that if the finite limit
\begin{equation}
d_x(f) := \lim_{t \to 0^+} [f(x+t) - f(x-t)]
\end{equation}
exists, then condition (1.4) is also satisfied with the same $d_x(f)$. In particular, if a periodic function $f$ is of bounded variation over $[-\pi, \pi]$, then (1.5) is satisfied at every point $x \in \mathbb{T}$ with
\begin{equation}
d_x(f) := f(x+0) - f(x-0).
\end{equation}
This means that the terms of the Fourier series of $f$ determine the (ordinary) jumps of $f$ at any point $x \in \mathbb{T}$ of discontinuity of first kind. Or equivalently, the terms of the Fourier series of $f$ determine the atoms of the finite Borel measure induced by $f$ on $\mathbb{T}$.

We note that Fejér [4] achieved the first result in the subject of determining the jumps of a function of bounded variation in terms of the partial sum of its Fourier series. Then Zygmund [4, p. 108] proved a Fejér type theorem via the Abel-Poisson mean of the Fourier series in question.

2. Main results

One of the aims of the present paper is to prove a Ferenc Lukács type theorem in terms of the Abel-Poisson mean of the conjugate series (1.3). We recall that the Abel-Poisson mean of (1.3) is defined by
\begin{equation}
\tilde{f}(r, x) := \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)r^k, \quad 0 \leq r < 1.
\end{equation}

The analogue of Theorem 1 reads as follows.

Theorem 2. If $f \in L^1(\mathbb{T})$ and the finite limit
\begin{equation}
\delta_x(f) := \lim_{h \to 0^+} \frac{1}{h} \int_0^h [f(x+t) - f(x-t)]dt
\end{equation}
exists at some point $x \in \mathbb{T}$, then
\begin{equation}
\lim_{r \to 1^-} \frac{\tilde{f}(r, x)}{\log(1-r)} = \frac{1}{\pi} \delta_x(f).
\end{equation}

The quantity $\delta_x(f)$ defined in (2.2) may be called the generalized jump of the function $f$ at the point $x$. The existence of $d_x(f)$ in (1.6) or even in (1.4) implies that of $\delta_x(f)$, and both numbers are equal. Observe the lack of the absolute value bars in (2.2). Loosely speaking, condition (2.2) expresses a differentiability property of an integral, while (1.4) resembles the definition of a Lebesgue point.
An immediate corollary of Theorem 2 is that if a periodic function \( f \) is of bounded variation over \([-\pi, \pi]\), then at every point \( x \in T \) we have
\[
\lim_{r \to 1^-} \frac{\tilde{f}(r, x)}{\log(1 - r)} = \frac{1}{\pi} \left[ f(x + 0) - f(x - 0) \right].
\]

We remind the reader that a function \( F \) is said to be smooth at some inner point \( x \) of the domain of \( F \) if the following limit relation holds:
\[
\Delta(F, x, h) := \frac{1}{h} [F(x + h) + F(x - h) - 2F(x)] \to 0 \quad \text{as} \quad h \to 0.
\]

The function class \( \lambda^* (T) \) is defined to consist of all periodic, continuous functions \( F \) such that (2.4) holds uniformly in \( x \in T \). The class \( \Lambda^*(T) \) is defined by the requirement that the ratio \( \Delta(F, x, h) \) is uniformly bounded in \( x \in T \) and \( h > 0 \).

These function classes (not only in the periodic case) were introduced by Zygmund [3] (see also [4, p. 43]).

Our second main result is the following

**Theorem 3.** (i) If a function \( F \in L^1(T) \) is smooth at some point \( x \in T \), then for the Abel-Poisson mean \( \tilde{F}(r, x) \) of the conjugate series to the Fourier series of \( F \) we have
\[
\frac{\partial \tilde{F}(r, x)}{\partial x} = o\{\log \frac{1}{1 - r}\} \quad \text{as} \quad r \to 1 - .
\]

(ii) If \( F \in \lambda^*(T) \), then (2.5) holds uniformly in \( x \in T \). If \( F \in \Lambda^*(T) \), then (2.5) holds with ‘\( O \)’ in place of ‘\( o \)’, uniformly in \( 0 \leq r < 1 \) and \( x \in T \).

We note that Theorem 3 can be considered as a counterpart of a theorem of Zygmund [4, p. 109] which provides an estimate of \( \frac{\partial^2 F(r, x)}{\partial x^2} \), whereby \( F(r, x) \) we denote the Abel-Poisson mean of the Fourier series of \( F \).

### 3. Auxiliary results on the conjugate Abel-Poisson kernel

We start with the representation
\[
\tilde{f}(r, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x - t)Q(r, t)dt,
\]
where
\[
Q(r, t) := \sum_{k=1}^{\infty} r^k \sin kt = \frac{r \sin t}{1 - 2r \cos t + r^2}, \quad 0 \leq r < 1,
\]
is the conjugate Abel-Poisson kernel. (See, for example, [4, p. 96].)

Since \( Q(r, t) \) is odd in \( t \), from (3.1) it follows that
\[
\tilde{f}(r, x) = \frac{1}{\pi} \int_0^{\pi} [f(x - t) - f(x + t)]Q(r, t)dt, \quad 0 \leq r < 1.
\]

The other crucial property is that \( Q(r, t) \) is positive for \( 0 < t < \pi \). From (3.2) it also follows that
\[
Q(r, t) = \frac{r \sin t}{(1 - r)^2 + 4r \sin^2 \frac{t}{2}} \leq \frac{1}{2 \tan \frac{t}{2}}, \quad 0 \leq r < 1 \quad \text{and} \quad 0 < t < \pi.
\]

**Lemma 1.** For \( 0 \leq r < 1 \), we have
\[
Q_r := \int_0^{\pi} Q(r, t)dt = \log \frac{1 + r}{1 - r}.
\]
Proof. By elementary calculus, we have

\[ Q_r = \left[ \frac{1}{2} \log \left\{ (1 - r)^2 + 4r \sin^2 \frac{t}{2} \right\} \right]_t=0, \]

which gives the right-hand side in (3.5).

Next, we examine the partial derivative

(3.6) \[ Q'(r, t) := \frac{\partial Q(r, t)}{\partial t} = \frac{r(1 + r^2) \cos t - 2r^2}{(1 - 2r \cos t + r^2)^2}. \]

It is clear that \( Q'(r, t) \) changes sign in the interval \((0, \pi)\) only once, namely, for \( t = \tau = \tau(r) \) satisfying

\[ \cos \tau = \frac{2r}{1 + r^2}, \]

so that \( \tau \to 0+ \) as \( r \to 1- \). Since \( 0 < \tau \leq \pi/2 \) for \( 0 \leq r < 1 \), we have

\[ 1 - \frac{\tau^2}{2} \leq \frac{2r}{1 + r^2}, \]

whence

(3.7) \[ \tau \geq \sqrt{2} \frac{1 - r}{\sqrt{1 + r^2}} \geq 1 - r. \]

Lemma 2. For \( 0 \leq r < 1 \), we have

(3.8) \[ \int_0^{\pi} t|Q'(r, t)|dt \leq 2 + \log \frac{\pi}{1 - r}. \]

Proof. By integration by parts, we obtain

\[ \int_0^{\pi} t|Q'(r, t)|dt = \int_0^\tau tQ'(r, t)dt - \int_\tau^{\pi} tQ'(r, t)dt \]
\[ = 2\tau Q(r, \tau) - \int_0^\tau Q(r, t)dt + \int_\tau^{\pi} Q(r, t)dt. \]

By virtue of (3.4) and (3.7), we conclude that

\[ \int_0^{\pi} t|Q'(r, t)|dt \leq \frac{2\tau \sin \tau}{4 \sin^2 \frac{\tau}{2}} + \int_\tau^{\pi} \frac{\sin t}{4 \sin^2 \frac{\pi}{2}} dt \]
\[ = \frac{\tau}{\tan \frac{\tau}{2}} + \int_\tau^{\pi} \frac{dt}{2 \tan \frac{\tau}{2}}, \]

whence (3.8) follows immediately.

4. Proof of Theorem 2

By Lemma 1, we have

\[ \lim_{r \to 1-} \frac{Q_r}{-\log(1 - r)} = 1. \]

Thus, it is enough to prove that

(4.1) \[ \lim_{r \to 1-} \left( \frac{\tilde{f}(r, x)}{Q_r} + \frac{1}{\pi} \delta_x(f) \right) = 0. \]
By (2.2), given any \( \varepsilon > 0 \) we can choose \( 0 < \eta < \pi/2 \) so that for the function

\[
I(h) := \int_{0}^{h} [f(x - t) - f(x + t) + \delta_x(f)] dt
\]

we have

\[
|I(h)| \leq \varepsilon h \quad \text{if} \quad 0 \leq h \leq \eta.
\]

(4.2)

Keeping (3.3), the notation in (3.5), and (4.2) in mind, we may write that

\[
\tilde{f}(r, x) = \frac{Q_r + 1}{\pi} \int_{0}^{\pi} \left[ f(x - t) - f(x + t) + \delta_x(f) \right] Q(r, t) dt
\]

(4.3)

\[
= \frac{1}{\pi Q_r} \left( \int_{0}^{\eta} + \int_{\eta}^{\pi} \right) =: J_1 + J_2, \quad \text{say}.
\]

First, integrating by parts and making use of (4.2), (3.4), (3.5) and (3.8) yield

\[
|J_1| \leq \frac{1}{\pi Q_r} \left( I(\eta) Q(r, \eta) + \int_{0}^{\eta} |I(t) Q'(r, t)| dt \right)
\]

(4.4)

\[
\leq \frac{\varepsilon}{\pi Q_r} \left( \eta Q(r, \eta) + \int_{0}^{\eta} t |Q'(r, t)| dt \right)
\]

\[
\leq \frac{\varepsilon}{-\pi \log(1 - r)} \left\{ \frac{\eta}{2 \tan \frac{\pi}{4}} + 2 + \log \pi - \log(1 - r) \right\}
\]

\[
\leq \frac{\varepsilon}{-\pi \log(1 - r)} \left\{ \frac{9}{2} - \log(1 - r) \right\} \leq \frac{11 \varepsilon}{2 \pi} \quad \text{if} \quad r \geq \frac{e - 1}{e},
\]

since \( \log \pi \leq 3/2 \) and \( -\log(1 - r) \geq 1 \) if \( 0 < (1-r) \leq 1/e \).

Second, by (3.4) and (3.5) we obtain

\[
|J_2| \leq \frac{1}{\pi Q_r} \int_{\eta}^{\pi} \left[ f(x - t) - f(x + t) + \delta_x(f) \right] Q(r, t) dt
\]

(4.5)

\[
\leq \frac{1}{\pi Q_r} \frac{1}{2 \tan \frac{\pi}{4}} \left( \int_{T} |f(t)| dt + 2\pi |\delta_x(f)| \right)
\]

\[
= O \frac{1}{-\log(1 - r)} \quad \text{as} \quad r \to 1 - .
\]

Combining (4.3)-(4.5) gives (4.1), which is equivalent to (2.3) which was to be proved.

5. Proof of Theorem 3

Part (i). We make use of (3.1) with \( F \) in place of \( f \) and the series representation of \( Q(r, t) \) in (3.2) (which allows a term-by-term differentiation) to obtain

\[
\frac{\partial \tilde{F}(r, x)}{\partial x} = \frac{1}{\pi} \int_{\mathbb{T}} F(t) Q'(r, x - t) dt = \frac{1}{\pi} \int_{\mathbb{T}} F(x - t) Q'(r, t) dt.
\]

(5.1)

Since \( Q'(r, t) \) is even in \( t \) (see (3.6)) and

\[
\int_{0}^{\pi} Q'(r, t) dt = Q(r, \pi) - Q(r, 0) = 0,
\]

from (5.1) we conclude that

\[
\frac{\partial \tilde{F}(r, x)}{\partial x} = \frac{1}{\pi} \int_{0}^{\pi} \left[ F(x + t) + F(x - t) - 2F(x) \right] Q'(r, t) dt, \quad 0 \leq r < 1.
\]
By assumption, $F$ is smooth at $x$. Thus, given any $\varepsilon > 0$ there exists $\eta > 0$ so that (cf. (2.4) as to the notation)

\begin{equation}
|t\Delta(F, x, t)| = |F(x + t) + F(x - t) - 2F(x)| \leq \varepsilon t \quad \text{if } 0 \leq t \leq \eta.
\end{equation}

Accordingly, we decompose the integral in (5.2) as follows:

\begin{equation}
\frac{\partial \tilde{F}(r, x)}{\partial x} = \frac{1}{-\pi \log(1 - r)}(\int_0^\eta + \int_\eta^{\pi}) t\Delta(F, x, t)Q'(r, t)dt =: I_1 + I_2, \quad \text{say}.
\end{equation}

By (5.3) and (3.8), we have

\begin{equation}
|I_1| \leq \frac{1}{-\pi \log(1 - r)} \int_0^\eta \varepsilon |tQ'(r, t)|dt = O(\varepsilon) \quad \text{as } r \to 1 - .
\end{equation}

On the other hand, we estimate $|Q'(r, t)|$ (see (3.6)) in the same way as in the case of inequality (3.4) to obtain

\begin{equation}
|Q'(r, t)| \leq \frac{1}{4r \sin \frac{\pi}{2} t}, \quad 0 \leq r < 1 \quad \text{and } 0 < t \leq \pi.
\end{equation}

Taking into account this and the first equality in (5.3) gives

\begin{equation}
|I_2| \leq \frac{1}{-\pi \log(1 - r)} \frac{1}{4r \sin \frac{\pi}{2} t} \int_\eta^{\pi} |F(x + t) + F(x - t) - 2F(x)|dt
\leq \frac{1}{-\pi \log(1 - r)} \frac{1}{4r \sin \frac{\pi}{2} t} \left( \int_{\eta}^{\pi} |F(t)|dt + 2\pi |F(x)| \right)
= O\left( \frac{1}{-\log(1 - r)} \right) \quad \text{as } r \to 1 - .
\end{equation}

Combining (5.4)-(5.6) yields (2.5) to be proved.

Part (ii). The obvious extension to uniformity completes the proof of the first statement when $F \in \lambda_*(T)$. The proof of the second statement when $F \in \Lambda_+(T)$ runs along the same lines, the details of which are left to the reader. \hfill \Box

6. CONCLUDING REMARKS

Remark 1. We start with a function $f \in L^1(T)$. Being interested in the Abel-Poisson mean of the conjugate series to the Fourier series of $f$, there is no loss of generality if we assume that $a_0 = 0$ in (1.2). Then the integral $F$ of $f$ defined by

\begin{equation}
F(u) := \int_0^u f(t)dt, \quad u \in \mathbb{R},
\end{equation}

is also periodic. Furthermore, we consider such a point $x \in T$ at which $\delta_x(f) = 0$.

We claim that if we apply Theorem 3 to this $F$ at this point $x$, then conclusion (2.5) coincides with (2.3). In fact, by (6.1), condition (2.2) can be rewritten in the form

\[ \delta_x(f) = \lim_{h \to 0^+} \frac{1}{h}[F(x + h) + F(x - h) - 2F(x)], \]

whence we see that $\delta_x(f) = 0$ is the same thing as the smoothness of $F$ at $x$. 

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On the other hand, if $A_k$ and $B_k$ are the Fourier coefficients of $F$, then an integration by parts gives (see, for example, [4, p. 42]) that
\[ A_k = -\frac{b_k}{k} \quad \text{and} \quad B_k = \frac{a_k}{k}, \quad k = 1, 2, \ldots. \]

Hence it follows immediately that
\[ \frac{\partial \tilde{F}(r, x)}{\partial x} = \tilde{f}(r, x), \quad 0 \leq r < 1. \]

This justifies our claim made above.

In other words, we have found another proof of Theorem 2 by means of Theorem 3 in the particular case when $\delta_x(f) = 0$.

**Remark 2.** To prove Theorem 2 in the general case when $\delta_x(f) \neq 0$, we could have proceeded in the following way, as well. Imitating the argument of Fejér in [1], we introduce a new function $g$ by setting
\[ g(\xi) := f(\xi) - \frac{1}{\pi} \delta_x(f) \phi(\xi - x), \quad \xi \in \mathbb{T}, \tag{6.2} \]

where the auxiliary function $\phi$ is defined by
\[ \phi(t) := \frac{1}{2} (\pi - t) \quad \text{for} \quad 0 < t < 2\pi, \]

$\phi(0) = \phi(2\pi) := 0$, and $\phi$ is continued periodically for all $t \in \mathbb{R}$. Since the ordinary jump of the function $\phi(\xi - x)$ at the point $\xi := x$ equals $\pi$, we have $\delta_x(g) = 0$. By Remark 1, we conclude that
\[ \lim_{r \to 1^-} \frac{s_n(g, x)}{\log(1 - r)} = 0. \tag{6.3} \]

By (6.2), it remains only to take into account that the Fourier series of $\phi(\xi - x)$ (as a function of $\xi$) and its conjugate series are given by
\[ \phi(\xi - x) \sim \sum_{k=1}^{\infty} \frac{\sin k(\xi - x)}{k} \quad \text{and} \quad -\sum_{k=1}^{\infty} \frac{\cos k(\xi - x)}{k}, \]

respectively. Hence it follows immediately that
\[ \frac{\tilde{\phi}(\cdot - x)(r, x)}{\log(1 - r)} = -\frac{1}{\log(1 - r)} \sum_{k=1}^{\infty} \frac{r^k}{k} = 1, \quad 0 < r < 1. \tag{6.4} \]

Combining (6.2)-(6.4) provides (2.3) in the general case when $\delta_x(f) \neq 0$.

**Remark 3.** In case $x := 0$, (6.4) is of the form
\[ \frac{\tilde{\phi}(r, 0)}{\log(1 - r)} = 1 = \frac{1}{\pi} d_0(\phi), \quad 0 < r < 1, \]

where $d_0(\phi)$ is defined in (1.6). This shows that relation (2.3) holds true even without “$\lim_{r \to 1^-}$” in the case of the function $\phi(t)$ defined above (observe that this time $\delta_0(\phi) = d_0(\phi)$).
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