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THE NEVANLINNA COUNTING FUNCTIONS FOR RUDIN'S ORTHOGONAL FUNCTIONS

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ABSTRACT. H^∞ and H^2 denote the Hardy spaces on the open unit disc D. Let ϕ be a function in H^∞ and $\|\phi\|_\infty = 1$. If ϕ is an inner function and $\phi(0) = 0$, then $\{\phi^n \ ; \ n = 0, 1, 2, \cdots\}$ is orthogonal in H^2. W.Rudin asked if the converse is true and C. Sundberg and C. Bishop showed that the converse is not true. Therefore there exists a function ϕ such that ϕ is not an inner function and $\{\phi^n\}$ is orthogonal in H^2. In this paper, the following is shown: $\{\phi^n\}$ is orthogonal in H^2 if and only if there exists a unique probability measure ν_0 on [0,1] with $1 \in \operatorname{supp} \nu_0$ such that $N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$ for nearly all z in D where N_ϕ is the Nevanlinna counting function of ϕ. If ϕ is an inner function, then ν_0 is a Dirac measure at $r = 1$.

1. Introduction

Let D be the open unit disc in the complex plane $\mathbf{C}$ and H^p, $1 \le p \le \infty$, the usual Hardy spaces on D. Any function f in H^p has a boundary value on ∂D almost everywhere with respect to the Lebesgue measure $d\theta/2\pi$. It may be assumed that H^p is a closed subspace in the usual Lebesgue space $L^p = L^p(\partial D, d\theta/2\pi)$. Hence H^2 is a Hilbert space.

If ϕ is an inner function in H^∞, that is, $|\phi| = 1$ *a.e.* $d\theta/2\pi$ on ∂D and $\phi(0) = 0$, then $\{\phi^n \ ; \ n = 0, 1, 2, \cdots\}$ is orthogonal in H^2. The following problem was posed by W.Rudin in 1988 (at an MSRI conference) (see [2]): If ϕ is a function in H^∞ with $\|\phi\|_\infty = 1$ such that $\{\phi^n \ ; \ n = 0, 1, 2, \cdots\}$ is orthogonal in H^2, must ϕ be an inner function? C.Sundberg [8] and C.Bishop [1] independently solved the problem. In fact, they showed that there exists a function ϕ such that ϕ is not an inner function and $\{\phi^n\}$ is orthogonal in H^2. We are still interested in a function ϕ in H^∞ such that $\{\phi^n\}$ is orthogonal in H^2 because such a function which is not an inner function is mysterious. For a function ϕ in H^∞ with $\|\phi\|_\infty = 1$, we call it a Rudin's orthogonal function when $\{\phi^n \ ; \ n = 0, 1, 2, \cdots\}$ is orthogonal in H^2.

For a function ϕ in H^∞ with $\|\phi\|_\infty = 1$, the Nevanlinna counting function of ϕ, N_ϕ, is defined on $D \backslash \{\phi(0)\}$ by

$$N_\phi(w) = \sum_{\phi(z)=w} \log \frac{1}{|z|}$$

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where multiplicities are counted and $N_\phi(w)$ is taken to be zero if w is not in the range of ϕ. P.S.Bourdon [2] used the Nevanlinna counting function to attack Rudin's orthogonality problem. In fact, he showed the following are equivalent: (a) $\{\phi^n\ ;\ n = 0, 1, 2, \cdots\}$ is orthogonal in H^2; (b) there is a non-negative function g on [0,1] such that for almost every $r \in [0,1]$, $N_\phi(re^{i\theta}) = g(r)$ for almost every $\theta \in [0, 2\pi]$.

In this paper, using a representing measure of a moment sequence we show the following: $\{\phi^n\ ;\ n = 0, 1, 2, \cdots\}$ is orthogonal in H^2 if and only if there exists a unique positive Borel measure ν_0 such that $1 \in \text{supp}\ \nu_0,\ \nu_0([0,1]) = 1$ and

$$N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$$

for nearly all z in D.

2. Result

Let ϕ be a function in H^∞ with $\|\phi\|_\infty = 1$. For an analytic function f on D, $C_\phi f(z) = f(\phi(z))$ $(z \in D)$. When ν is a positive Borel measure on $\bar{D}$ and $\nu(\bar{D}) = 1$, $H^2(\nu)$ denotes the closure of all analytic polynomials in $L^2(\nu)$.

Lemma 1. *Suppose ϕ is a function in H^∞ with $\|\phi\|_\infty = 1$. If $\{\phi^n\ ;\ n = 0, 1, 2, \cdots\}$ is orthogonal in H^2, then there exists a unique radial Borel measure ν such that C_ϕ is an isometric operator from $H^2(\nu)$ into H^2 where $d\nu = d\nu_0(r)d\theta/2\pi$ and $1 \in \text{supp}\ \nu_0$.*

Proof. Suppose $\{\phi^n\}$ is orthogonal in H^2. Let $\mathcal{H}^2_\phi$ be the closure of all polynomials of ϕ in H^2. Put

$$\beta(n) = \left(\int_0^{2\pi} |\phi|^{2n} d\theta/2\pi \right)^{1/2} \quad (n = 0, 1, 2, \cdots).$$

$\mathcal{H}^2(\beta)$ denotes the set of all power series: $f(z) = \sum_{n=0}^\infty \hat{f}(n) z^n$ with $\|f\|_\beta = \sum_{n=0}^\infty |\hat{f}(n)|^2 \beta(n)^2 < \infty$. $\mathcal{H}^2(\beta)$ is a Hilbert space with the inner product: $(f, g) = \sum_{n=0}^\infty \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^2$. Then $\mathcal{H}^2_\phi$ is isometrically isomorphic to $\mathcal{H}^2(\beta)$ where

$$F = \sum_{n=0}^\infty \left(\int_0^{2\pi} F\bar{\phi}^n d\theta/2\pi \right) \phi^n \to f = \sum_{n=0}^\infty \hat{f}(n) z^n.$$

The multiplication operator M_z on $\mathcal{H}^2(\beta)$ by z is injective and subnormal because the multiplication operator M_ϕ on $\mathcal{H}^2_\phi$ is subnormal. $\sup_n |\beta(n+1)/\beta(n)| = 1$ because $\beta(n) \geq \beta(n+1)$ and so $\|M_z\| = 1$ (see [7, p. 59]). It is known [7, Proposition 25] that $\mathcal{H}^2(\beta)$ is isometrically isomorphic to $H^2(\nu)$ where $d\nu = d\nu_0(r)d\theta/2\pi$ and ν_0 is a Borel probability measure on [0,1] with $1 \in \text{supp}\ \nu_0$. Thus $\mathcal{H}^2_\phi$ is isometrically isomorphic to $H^2(\nu)$ and so C_ϕ is an isometric operator from $H^2(\nu)$ into H^2. For the uniqueness of ν, if there exists another radial measure $d\lambda = d\lambda_0(r)d\theta/2\pi$ such that C_ϕ is isometric from $H^2(\lambda)$ into H^2, then

$$\int_{\bar{D}} |z|^{2n} d\nu = \int_{\bar{D}} |z|^{2n} d\lambda \quad (n = 0, 1, 2, \cdots)$$

and so $\int_0^1 r^{2n} d\nu_0 = \int_0^1 r^{2n} d\lambda_0$ for $n = 0, 1, 2, \cdots$. By the Műntz-Szasz theorem [6, Theorem 15.26], $\nu_0 = \mu_0$ because $\sum_{n=1}^\infty 1/2n = \infty$. Thus $\nu = \lambda$.

Lemma 2. *For nearly all w in D,*

$$N_\phi(w) = \int_0^{2\pi} \log \left| \frac{w-\phi(e^{i\theta})}{1-\bar{w}\phi(e^{i\theta})} \right| d\theta/2\pi - \log \left| \frac{w-\phi(0)}{1-\bar{w}\phi(0)} \right|.$$

Proof. By the proof of Littlewood's inequality [6, p. 187], for $w \in D \backslash \{\phi(0)\}$,

$$N_\phi(w) = \lim_{r\to 1} \int_0^{2\pi} \log \left| \frac{w-\phi(re^{i\theta})}{1-\bar{w}\phi(re^{i\theta})} \right| d\theta/2\pi - \log \left| \frac{w-\phi(0)}{1-\bar{w}\phi(0)} \right|.$$

By a generalization of a theorem of Frostman [5], the lemma follows.

Lemma 3. *Suppose ν is a radial measure, that is, $d\nu = d\nu_0(r)d\theta/2\pi$. If C_ϕ is an isometric operator from $H^2(\nu)$ into H^2, then*

$$N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$$

for nearly all z in D.

Proof. If C_ϕ is isometric, then for any $w \in D$,

$$\int_{\bar{D}} \left| \frac{w-z}{1-\bar{w}z} \right|^{2n} d\nu = \int_0^{2\pi} \left| \frac{w-\phi}{1-\bar{w}\phi} \right|^{2n} d\theta/2\pi \quad (n = 0, 1, 2, \cdots).$$

It is elementary to see that $x = \sqrt{1-(1-x^2)} = \sum_{n=0}^\infty a_n(1-x^2)^n$ and that $\sum_{n=0}^\infty |a_n|(1-x^2)^n < \infty$ $(0 \le x \le 1)$. Hence by Lebesgue's dominated convergence theorem

$$\begin{aligned}
\int_0^{2\pi} \left| \frac{w-\phi}{1-\bar{w}\phi} \right| d\theta/2\pi &= \int_0^{2\pi} \sum_{n=0}^\infty a_n \left(1 - \left| \frac{w-\phi}{1-\bar{w}\phi} \right|^2 \right)^n d\theta/2\pi \\
&= \sum_{n=0}^\infty a_n \int_0^{2\pi} \left(1 - \left| \frac{w-\phi}{1-\bar{w}\phi} \right|^2 \right)^n d\theta/2\pi \\
&= \sum_{n=0}^\infty a_n \int_{\bar{D}} \left(1 - \left| \frac{w-z}{1-\bar{w}z} \right|^2 \right)^n d\nu \\
&= \int_{\bar{D}} \sum_{n=0}^\infty a_n \left(1 - \left| \frac{w-z}{1-\bar{w}z} \right|^2 \right)^n d\nu = \int_{\bar{D}} \left| \frac{w-z}{1-\bar{w}z} \right| d\nu,
\end{aligned}$$

because $|\sum_{n=0}^k a_n(1-|\frac{w-\phi}{1-\bar{w}\phi}|^2)^n| \le \sum_{n=0}^\infty |a_n| < \infty$ and $|\sum_{n=0}^k a_n(1-|\frac{w-z}{1-\bar{w}z}|^2)^n| \le \sum_{n=0}^\infty |a_n| < \infty$. Similarly, as $x^{2\ell+1} = \sqrt{1-(1-x^{4\ell+2})}$ we can show that

$$\int_0^{2\pi} |\frac{w-\phi}{1-\bar{w}\phi}|^{2n+1} d\theta/2\pi = \int_{\bar{D}} |\frac{w-z}{-\bar{w}z}|^{2n+1} d\nu \quad (n = 0, 1, 2, \cdots).$$

Thus for any $w \in D$,

$$\int_0^{2\pi} \left| \frac{w-\phi}{1-\bar{w}\phi} \right|^n d\theta/2\pi = \int_{\bar{D}} \left| \frac{w-z}{1-\bar{w}z} \right|^n d\nu \quad (n = 0, 1, 2, \cdots).$$

Since $\log x = \log(1-(1-x)) = -\sum_{n=0}^\infty \frac{(1-x)^n}{n}$ $(0 < x \le 1), \sum_{n=0}^k \frac{(1-x)^n}{n}$ is increasing when $0 \le x \le 1$ and $\lim_{k\to\infty} \sum_{n=0}^k \frac{(1-x)^n}{n} = -\log x$. By Lebesgue's monotone convergence theorem, for any $w \in D$,

$$\int_{\bar{D}} \log \left| \frac{w-z}{1-\bar{w}z} \right| d\nu = \int_0^{2\pi} \log \left| \frac{w-\phi}{1-\bar{w}\phi} \right| d\theta/2\pi.$$

Hence by Lemma 1, for nearly all $w \in D$,

$$\begin{aligned} N_\phi(w) &= \int_{\bar{D}} \log \left| \frac{w-z}{1-\bar{w}z} \right| d\nu - \log|w| \\ &= \int_0^1 d\nu_0(r) \left(\int_0^{2\pi} \log \left| \frac{w-re^{i\theta}}{1-\bar{w}re^{i\theta}} \right| d\theta/2\pi - \log|w| \right) \end{aligned}$$

because $\phi(0) = 0$ and $\nu_0([0,1]) = 1$. Then

$$N_\phi(w) = \int_{|w|}^1 \log \frac{r}{|w|} d\nu_0(r)$$

for nearly all $w \in D$ because

$$\int_0^{2\pi} \log \left| \frac{w-re^{i\theta}}{1-\bar{w}re^{i\theta}} \right| d\theta/2\pi - \log|w| = \begin{cases} \log \frac{r}{|w|} & (|w| \le r), \\ 0 & (|w| > r). \end{cases}$$

□

Theorem 1. *Suppose ϕ is a function in H^∞ with $\|\phi\|_\infty = 1$. Then the following are equivalent:*

(1) *$\{\phi^n \; ; \; n = 0, 1, 2, \cdots\}$ is orthogonal in H^2.*
(2) *$N_\phi(z) = N_\phi(|z|)$ for nearly all z in D.*
(3) *There exists a unique Borel probability measure ν_0 on $[0,1]$ with $1 \in \text{supp } \nu_0$ such that*

$$N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$$

for nearly all z in D.

Proof. (1) ⇒ (3). By Lemma 1, there exists a unique radial measure ν such that C_ϕ is an isometric operator from $H^2(\nu)$ into H^2 where $d\nu = d\nu_0(r)d\theta/2\pi$ and $1 \in \text{supp } \nu_0$. By Lemma 3,

$$N_\phi(z) = \int_{|z|}^1 \log \frac{r}{|z|} d\nu_0(r)$$

for nearly all $z \in D$. (3) ⇒ (2) is clear. (2) ⇒ (1). By the Littlewood-Paley theorem (see [2]), for $n > m$,

$$\begin{aligned} \int_0^{2\pi} \phi^n \bar{\phi}^m d\theta/2\pi &= 2nm \int_D z^{n-1} \bar{z}^{m-1} N_\phi(|z|) dA(z) \\ &= 4nm \int_0^1 r^{n+m-1} N_\phi(r) dr \int_0^{2\pi} e^{i(n-m)\theta} d\theta/2\pi = 0. \end{aligned}$$

□

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