PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 4, Pages 1283–1288
S 0002-9939(02)06672-8
Article electronically published on July 26, 2002

A REMARK ON THE MAXIMUM PRINCIPLE AND STOCHASTIC COMPLETENESS

STEFANO PIGOLA, MARCO RIGOLI, AND ALBERTO G. SETTI

(Communicated by Bennett Chow)

Dedicated to the memory of Franca Burrone Rigoli

Abstract. We prove that the stochastic completeness of a Riemannian manifold $(M, \langle , \rangle)$ is equivalent to the validity of a weak form of the Omori-Yau maximum principle. Some geometric applications of this result are also presented.

1. Introduction and main result

Let $(M, \langle , \rangle)$ be a smooth, connected, non-compact, Riemannian manifold, with a fixed origin o, and denote by $r(x)$ the distance function from o. We recall that the Brownian motion (or Wiener process) on M is called recurrent if it visits every open set of M at arbitrarily large time with probability one, and transient otherwise. It is well known that recurrence is related to various geometric properties of the manifold, such as volume growth, curvature, isoperimetric inequalities, and so on. On the other hand, it is also linked to basic potential-theoretic properties of the Laplace-Beltrami operator Δ; indeed it is equivalent to parabolicity.

Another important feature of the Brownian motion is stochastic completeness. We say that the Brownian motion of M, or M for short, is stochastically complete if the probability of a particle to be found in the state space is constantly equal to one. If this is not the case, M is said to be stochastically incomplete.

The first example of a geodesically complete stochastically incomplete manifold was found by Azencott [A]. In his example, $(M, \langle , \rangle)$ has negative sectional curvature which diverges rapidly to $-\infty$ as $r(x)$ tends to infinity. In some sense the negative curvature plays the role of a drift that is strong enough to sweep the Brownian particle to infinity in a finite time.

An interesting problem is to determine which geometric properties of M ensure stochastic completeness or incompleteness. Fundamental contributions are due to Azencott [A], Gaffney [G], Yau [Y1], Li [L], and Grigor'yan [Gr1], [Gr2]. We refer the reader to the excellent survey paper by Grigor'yan [Gr3] for an exhaustive account of the theory.

Received by the editors January 3, 2001 and, in revised form, November 1, 2001.
2000 *Mathematics Subject Classification.* Primary 58J35; Secondary 58J65.
Key words and phrases. Maximum principle, stochastic completeness.

Let us now state the following seemingly unrelated question:

Given any smooth bounded above function u on M, and $\epsilon > 0$, does there exist $x_\epsilon \in M$ such that

$$\sup_M u - \epsilon \leq u(x_\epsilon), \quad |\nabla u(x_\epsilon)| \leq \epsilon \quad \text{and} \quad \Delta u(x_\epsilon) \leq \epsilon? \tag{1.1}$$

An affirmative answer to (1.1) amounts to the validity of what is known in the literature as an Omori–Yau type maximum principle. Indeed, Omori [O] first proved this result assuming that M is complete with sectional curvature bounded from below. Later, Yau [Y2] and Cheng and Yau [CY] gave a generalized and simplified version of the result relaxing the curvature assumption to Ricci bounded below. Clearly this kind of result serves as a substitute for the standard maximum principle in non-compact settings.

Recently, K. Takegoshi [T] claimed a positive answer to (1.1) under completeness and the following volume growth condition:

$$\liminf_{r\to+\infty} \frac{\log \operatorname{vol} B_r(o)}{r^2} < +\infty,$$

where $B_r = \{x \in M : r(x) < r\}$ is the geodesic ball of radius r centered at o. He also conjectured that the maximum principle holds replacing the above with the weaker assumption

$$\frac{r}{\log \operatorname{vol} B_r(o)} \notin L^1(+\infty). \tag{1.2}$$

Unfortunately, there seems to be a gap in the proof (see [T, Theorem 2.3]) that the present authors were unable to fill.

We note that Grigor'yan [Gr1] showed that, for a complete manifold, (1.2) is sufficient for stochastic completeness.

The theories of stochastically complete and parabolic manifolds display several intriguing analogies. In particular, as parabolicity may be thought of as a kind of maximum principle, in the sense that it is equivalent to the statement that if u is smooth and bounded above, then either u is constant or $inf_M \Delta u < 0$, stochastic completeness turns out to be equivalent to the validity of a weak form of the Omori-Yau maximum principle, as expressed in the next theorem.

Theorem 1.1. *Let $(M, \langle,\rangle)$ be a smooth, connected, non-compact Riemannian manifold. Then the following are equivalent:*

1. *M is stochastically complete.*
2. *For every $\lambda > 0$, the only non-negative, bounded smooth solution u of $\Delta u = \lambda u$ is $u \equiv 0$.*
3. *For every $u \in C^2(M)$ with $\sup_M u < +\infty$, and for every $\alpha > 0$ set $\Omega_\alpha = \{x \in M : u(x) > \sup_M u - \alpha\}$. Then $\inf_{\Omega_\alpha} \Delta u \leq 0$.*
4. *For every $u \in C^2(M)$ with $\sup_M u < +\infty$ there exists a sequence $\{x_n\}$, $n = 1, 2 \ldots$, such that, for every n, $u(x_n) \geq \sup_M u - 1/n$ and $\Delta u(x_n) \leq 1/n$.*

It is an interesting problem to investigate the possible relationships between the validity of the full Omori–Yau principle and stochastic completeness, and determine if, under additional conditions on (M, g), the two are equivalent. In particular, a positive answer to the latter question could lead to a proof of the aforementioned Takegoshi's conjecture. On the other hand, the equivalence between the weak maximum principle and stochastic completeness established in Theorem 1.1 appears to be more natural. Indeed, making the additional requirement that the norm of the gradient tends to zero along the sequence selected in (4), forces (M, g) to be non-extensible, and it is well known that there exist stochastically complete, extensible Riemannian manifolds. A trivial example is provided by $(\mathbb{R}^2 \setminus \{0\}, can)$.

The equivalence between (1) and (2) is well known; see for instance Theorem 6.2 in [Gr3]. Furthermore, (3) implies (4) and (4) implies (2) in an obvious way.

We also note that, using a further characterization of stochastic completeness, it was shown by L. Karp (see the (proof of) [K, Theorem 2.3]) that, for a complete manifold, (1) implies

$$\text{for every } u \in C^\infty(M) \text{ with } \sup_M u < +\infty \text{ we have } \inf_M \Delta u \leq 0. \tag{1.3}$$

The latter is clearly implied by (3) above.

Proof of Theorem 1.1. It follows from the above discussion that it remains to prove the implication (2) $\Rightarrow$ (3). We argue by contradiction, and assume that there exists a function u satisfying the conditions in (3) and such that, for some $\alpha > 0$, $\inf_{\Omega_\alpha} \Delta u \geq 2c > 0$. We let $\Omega^* = \{x \in M : \Delta u(x) > c\}$, so that $\overline{\Omega}_\alpha$ is contained in Ω^*. Having set $\lambda = \frac{c}{\alpha}$ it is easy to see that $u + \alpha - \sup_M u$ is a C^2 subsolution of

$$\Delta u = \lambda u \tag{1.4}$$

on Ω^*. Since 0 is obviously a subsolution of equality (1.4) on M, we see that $u_\alpha = \max\{u + \alpha - \sup_M u, 0\}$ is also a subsolution on M. Since u is C^2, u_α belongs to $C^o(M) \cap H^1_{\mathrm{loc}}(M)$. Furthermore, $u_\alpha \not\equiv 0$ and $0 \leq u_\alpha \leq \alpha < +\infty$. Noting that any positive constant is a supersolution of (1.4), choosing $u_+ > \alpha$, and applying the monotone iteration scheme (see [RRV, Proposition 2.4] for the formulation needed here) yields a smooth solution v of (1.4) on M such that $u_\alpha \leq v \leq u_+$. Now, since u_α does not vanish identically, the same holds for v, and this contradicts (2). □

2. Some applications

The Omori–Yau maximum principle is closely related to several problems in differential geometry and geometric function theory. The interested reader may refer to the following (significant, but by no means exhaustive) list of papers: [CX], [K], [Ks], [MY], [O], [R], [RRS], [Y2], [Y3].

Using Theorem 1.1 a number of results can be proved under the only assumption that the underlying manifold is stochastically complete. By way of examples we describe a few typical applications.

Theorem 2.1. *Let $(M, \langle , \rangle)$ be a stochastically complete manifold of dimension $m \geq 2$ with scalar curvature $s(x) \leq -\epsilon$ on M, for some $\epsilon > 0$. Let $\varphi : M \to M$ be a conformal immersion preserving the scalar curvature, and assume that, having set $\varphi^*\langle , \rangle = v\langle , \rangle$, we have $\inf_M v > 0$. Then φ is weakly distance increasing.*

Proof. We consider the case where $m \geq 3$, the case $m = 2$ being similar. Setting $v = u^{4/(m-2)}$, it is well known that the function u is a solution of the Yamabe equation

$$\Delta u = -\frac{m-2}{4(m-1)} s(x) u \Big(u^{4/(m-2)} - 1 \Big) \quad \text{on } M,$$

where we have used the assumption that φ preserves the scalar curvature. We want to show that $\inf_M u \geq 1$. Assume the contrary. Since u is bounded below, we may use Theorem 1.1 in the form of a weak minimum principle to deduce that there exists a sequence $\{x_n\}$ such that, for every n, $u(x_n) \to \inf_M u < 1$ and $\Delta u(x_n) \geq -1/n$. The assumptions $\inf_M u > 0$ and $s(x) \leq -\epsilon$ imply that

$$\Delta u(x_n) \leq -\frac{m-2}{4(m-1)}\epsilon u(x_n)\big(1 - u(x_n)^{4/(m-2)}\big) \leq -C < 0$$

for some $C > 0$ and every sufficiently large n, contradicting $\Delta u(x_n) \geq -1/n$. □

The equivalence between stochastic completeness and the validity of the weak form of the Omori-Yau maximum principle also allows us to give extremely short proofs of several properties of stochastic complete manifolds. As an example we state the following result concerning the invariance of stochastic completeness under compact deformations, which is well known (cf. [Gr3, Corollary 6.5]).

Proposition 2.2. *Let $(M, \langle , \rangle)$ and $(N, (,))$ be noncompact Riemannian manifolds, and assume that there exist compact sets $A \subset M$ and $B \subset N$ and a Riemannian isometry $\varphi : M \setminus A \to N \setminus B$ which preserves divergent sequences in the ambient spaces, that is, $\{x_n\}$ diverges in M if and only if $\{\varphi(x_n)\}$ is divergent in N. Then M is stochastically complete if and only if N is also stochastically complete.*

We note that the situation described above occurs, for instance, in the following cases:

(a) Both (M, g) and (N, h) are geodesically complete.
(b) φ preserves precompactness in the the ambient spaces, that is, $U \subset M \setminus A$ is precompact in M if and only if $\varphi(U)$ is precompact in N.

In this respect, we observe that if (a) above fails, some further condition on the isometry must be imposed for the conclusion of the proposition to hold. Indeed, it suffices to consider $M = \mathbb{R}^2$ with the canonical metric, $A = \{x : |x|^2 \leq 1\} \cup \{(2, 0)\}$, $N = \mathbb{R}^2 \setminus \{x : |x|^2 \leq 1\}$, $B = \{(2, 0)\}$, and the isometry $id : M \setminus A \to N \setminus B$.

To describe the next application, we recall that a geodesic ball $B_R(q) \subset (N, (,))$ is said to be regular if it is contained in the complement of the cut locus of q and, denoting by ${}^N K_p$ the sectional curvature of N at p, one has $\max\{0, \sup_{B_R(q)} {}^N K_p\}^{1/2} R < \pi/2$.

Theorem 2.3. *Let $B_R(q) \subset (N, (,))$ be a regular geodesic ball with $\sup_{B_R(q)} {}^N K \leq K$ for some $K \in \mathbb{R}$, and let $\varphi : (M, \langle , \rangle) \to B_R(q)$ be a smooth map with tension field $\tau(\varphi)$ satisfying $|\tau(\varphi)| \leq \tau_o$ for some $\tau_o > 0$. Assume that $(M, \langle , \rangle)$ is complete and stochastically complete. Then, denoting by $e(\varphi)$ the energy density of φ, we have*

$$R \geq \begin{cases} K^{-1/2} \arctan\big(2K^{1/2}\tau_o^{-1} \inf_M e(\varphi)\big) & \text{if } K > 0, \\ 2\tau_o^{-1} \inf_M e(\varphi) & \text{if } K = 0, \\ (-K)^{-1/2} \text{arctanh}\big(2(-K)^{1/2}\tau_o^{-1} \inf_M e(\varphi)\big) & \text{if } K < 0. \end{cases}$$

Proof. We consider only the case $K < 0$, the other cases being similar. To simplify notation we assume that $K = -1$. Denote by ρ the distance function in $(N, (,))$ from the point q, and set $u = 1/2 \cosh(\rho \circ \varphi)$. A straightforward computation that uses the Hessian comparison theorem (see [GW]) yields

$$\Delta u \geq u\big[2e(\varphi) + \text{tahn}(\rho \circ \varphi)(\nabla\rho, \tau(\varphi))\big].$$

Since $u \geq 1/2$, and $-\tau_o \tanh R \leq \tanh(\rho \circ \varphi)(\nabla\rho, \tau(\varphi))$ on M, we deduce that

$$\Delta u \geq \inf_M e(\varphi) - 1/2\, \tau_o \tanh R.$$

Now the required conclusion (with $K = -1$) follows from (4) in the statement of Theorem 1.1. ☐

We remark that for the conclusion of Theorem 2.3 to hold it suffices to assume the validity of (1.3) in Section 1.

We also note that if $\varphi : (M, \langle, \rangle) \to (N, (,))$ is an isometric immersion and $m = \dim M$, then

$$2e(\varphi) = m \quad \text{and} \quad \tau(\varphi) = mH,$$

where H denotes the mean curvature vector. This case has been considered in [K], assuming conditions that imply stochastic completeness of the complete manifold $(M, \langle, \rangle)$, such as appropriate bounds on the volume growth or on the curvature, or the hypothesis that the $(M, \langle, \rangle)$ is a properly embedded minimal submanifold of $\mathbb{R}^n$.

Theorem 2.3 can also be applied to give, in the only assumption of stochastic completeness, a negative answer to a question of Calabi (see [EK]) on the existence of complete minimal surfaces in $\mathbb{R}^3$ with bounded image. We point out that in a recent paper N.S. Nadirashvili [N] (see also the corrections in [CR]) has exhibited an example solving in the affirmative this long-standing problem. However, the search of mild additional geometric assumptions under which the image $\varphi(M)$ is necessarily unbounded remain a challenging task. In this respect we quote the beautiful striking half-space theorem of Hoffman and Meeks [HM].

To state our last application recall that, according to the Ruh–Vilms theorem, an isometric immersion $\varphi : M \to R^n$ has parallel mean curvature if and only if its Gauss map $\gamma_\varphi : M \to G_m(\mathbb{R}^n)$ is harmonic. Further, the energy density of the Gauss map is equal to one half the square of the norm of the second fundamental form of φ (see [EL], pp. 17–19). Now, a result of Fischer–Colbrie [FC] asserts that if q is an m-decomposable vector in $G_m(\mathbb{R}^n)$, $k = \min(m, n-m)$ and $R = \pi/2$ if $k = 1$, $R = \pi/(2\sqrt{2})$ otherwise, then the set $\{p \in G_m(\mathbb{R}^n) : (p, q) > (\cos(R/\sqrt{k}))^k\}$ is contained in a regular ball in $G_m(\mathbb{R}^n)$. Thus, from Theorem 2.3 we obtain

Corollary 2.4. *Let $\varphi : M \to \mathbb{R}^n$ be a complete, stochastically complete isometric immersion with parallel mean curvature. Let $m = \dim M$, $k = \min(m, n-m)$ and let $R = \pi/2$ if $k = 1$, $R = \pi/(2\sqrt{2})$ otherwise. Finally, let $\gamma_\varphi : M \to G_m(\mathbb{R}^n)$ be the Gauss map of φ, and assume that there exists an m-decomposable vector q such that $(\gamma_\varphi, q) > (\cos(R/\sqrt{k}))^k$ on M. Then φ is minimal.*

References

[A] R. Azencott, *Behaviour of diffusion semi-groups at infinity*, Bull. Soc. Math. France **102** (1974), 193–240. MR **50:**8725

[CR] P. Collin and H. Rosenberg, *Notes sur la démonstration de Nadirashvili des conjectures de Hadamard et Calabi–Yau*, Bull. Sc. Math. **123** (1999), 563–575. MR **2000i:**53011

[CX] Q. Chen and Y.L. Xin, *A generalized maximum principle and its applications in geometry*, Amer. J. Math. **114** (1992), 355–366. MR **93g:**53054

[CY] S.Y. Cheng and S.T. Yau, *Differential equations on Riemanninan manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), 333–354. MR **52:**6608

[EK] J. Eells and S. Kobayashi, *Problems in differential geometry.* In Proc. of US-Japan Seminar on differential geometry, Kyoto (1965), 167–177.

[EL] J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*, CBMS n. **50**, AMS, Providence, 1983. MR **85g:**58030

[FC] D. Fischer-Colbrie, *Some rigidity theorems for minimal submanifolds of the sphere*, Acta Math. **145** (1980), 29–46. MR **82b:**53078

[G] M.P. Gaffney, *The conservation property of the heat equation on Riemannian manifolds*, Comm. Pure Appl. Math. **12** (1959), 1-11. MR **21:**892

[Gr1] A. Grigor'yan, *On stochastically complete manifolds*, Engl. Transl. Soviet Math. Dokl. **34** (1987), 310-313. MR **88a:**58209

[Gr2] A. Grigor'yan, *Bounded solutions of the Schrödinger equation on non-compact Riemannian manifolds*, Engl. Transl. J. of Soviet Math. **51** (1990), 2340–2349. MR **90m:**35050

[Gr3] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. **36** (1999), 135–249. MR **99k:**58195

[GW] R. Greene and H.H. Wu, *Function Theory on Manifolds Which Possess a Pole*, Lecture Notes in Math. n. 699, Springer Verlag, Berlin, 1979 MR **81a:**53002

[HM] D. Hoffman and W. Meeks, *The strong halfspace theorem for minimal surfaces,* Invent. Math. **101** (1990), 373–377. MR **92e:**53010

[K] L. Karp, *Differential inequalities on complete Riemannian manifolds and applications*, Math. Ann. **272** (1985), 449–459. MR **87g:**58119

[Ks] A. Kasue, *Estimates for solutions of Poisson equations and their applications to submanifolds*, Springer Lecture Notes in Mathematics **1090** (1984), 1-14. MR **86d:**58118

[L] P. Li, *Uniqueness of L^1 solutions for the Laplace equation and the heat equation on Riemannian manifolds*, J. Diff. Geom. **20** (1984), 447–457. MR **86h:**58133

[MY] N. Mok and S.T. Yau, *Completeness of the Kähler–Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions*, Proc. Symp. Pure Math. **39** (1983), 41–59. MR **85j:**53068

[N] N.S. Nadirashvili, *Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces*, Invent. Math. **126** (1996), 457–465. MR **98d:**53014

[O] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205–214. MR **35:**6101

[R] H.L. Royden, *The Ahlfors-Schwarz lemma in several complex variables*, Comm. Math. Helv. **55** (1980), 547–558. MR **82i:**32049

[RRS] A. Ratto, M. Rigoli and A.G. Setti, *On the Omori–Yau maximum principle and its applications to differential equations and geometry*, J. Funct. Anal. **134** (1995), 486–510. MR **96k:**53062

[RRV] A. Ratto, M. Rigoli and L. Veron, *Scalar curvature and conformal deformation of hyperbolic space*, J. Funct. Anal. **121** (1994), 15–77. MR **95a:**53062

[T] K. Takegoshi, *A volume estimate for strong subharmonicity and maximum principle on complete Riemannian manifolds*, Nagoya Math. J. **151** (1998), 25–36. MR **99i:**53042

[Y1] S.T. Yau, *On the heat kernel of a complete Riemannian manifold*, J. Math. Pures Appl. **57** (1978), 191–201. MR **81b:**58041

[Y2] S.T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228. MR **55:**4042

[Y3] S.T. Yau, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. **100** (1978), 197–203. MR **58:**6370

Dipartimento di Matematica, Università di Milano, via Saldini 50, I-20133 Milano, Italy

E-mail address: pigola@mat.unimi.it

Dipartimento di Scienze C.F.M., Università dell'Insubria - Como, via Valleggio 11, I-22100 Como, Italy

E-mail address: rigoli@matapp.unimib.it

Dipartimento di Scienze C.F.M., Università dell'Insubria - Como, via Valleggio 11. I-22100 Como, Italy

E-mail address: setti@uninsubria.it